AM5: Tracce delle lezioni- IX Settimana

CONVOLUZIONE CON NUCLEI SINGOLARI
e
DISEGUAGLIANZA HARDY-LITTLEWOOD-SOBOLEV

Se $0 < \lambda < N$ e $G_\lambda(x) := \frac{1}{|x|^\lambda}$, $x \in \mathbf{R}^N$, G_λ non é sommabile, ma lo é localmente: $\int\limits_{|x|\le R} \frac{dx}{|x|^\lambda} = \frac{N\,vol(B_1)}{N-\lambda}\, R^{N-\lambda}$. In particolare

$$(\varphi * G_\lambda)(x) := \int\limits_{\mathbf{R}^N} \frac{\varphi(y)}{|x-y|^\lambda}\, dy = \int\limits_{\mathbf{R}^N} \frac{\varphi(x-y)}{|y|^\lambda}\, dy$$

é definita per ogni $x \in \mathbf{R}^N$ ed é infatti una funzione $C^\infty(\mathbf{R}^N)$. Ad esempio, se $\Delta = \Delta_y := \sum\limits_{j=1}^N \frac{\partial^2}{\partial y_j^2}$, risulta $\Delta\,(\varphi * G_\lambda) = (\Delta\varphi) * G_\lambda$.

Proposizione. Sia $\mathcal{N} = \frac{G_{N-2}}{c_N}$, ove $c_N := N(N-2) \int\limits_{\mathbf{R}^N} \frac{dx}{(1+|y|^2)^{\frac{N+2}{2}}}\, dy.$ Allora $\varphi \in C_0^\infty(\mathbf{R}^N) \Rightarrow -\Delta\,(\varphi * \mathcal{N}) = \varphi$ in $\mathbf{R}^N$.

Tale formula si basa sulla **formula di integrazione per parti**

$$\int\limits_{\mathbf{R}^N} \frac{\partial u}{\partial x_j}\, v = -\int\limits_{\mathbf{R}^N} u\, \frac{\partial v}{\partial x_j} \qquad \forall u \in C^\infty, \quad \forall v \in C_0^\infty(\mathbf{R}^N)$$

che é a sua volta conseguenza del Teorema Fondamentale del Calcolo. Ad esempio,

$$\int\limits_{\mathbf{R}^N} \frac{\partial(u\,v)}{\partial x_1} = \int\limits_{\mathbf{R}^{N-1}} \left(\int\limits_{-\infty}^{+\infty} \frac{\partial(u\,v)}{\partial x_1} \right) dx_2 \dots dx_N = 0$$

Prova della Proposizione. É $\Delta\,(\varphi * G_{N-2})(x) =$

$$\int \frac{(\Delta\,\varphi)(x-y)}{|y|^{N-2}}\, dy = \lim_{\epsilon\to 0} \int \frac{\Delta_y[\varphi(x-y)]}{(\epsilon^2+|y|^2)^{\frac{N-2}{2}}}\, dy = \lim_{\epsilon\to 0} \int \varphi(x-y)\, \Delta_y \frac{1}{(\epsilon^2+|y|^2)^{\frac{N-2}{2}}}\, dy$$

$$= \lim_{\epsilon\to 0} \int \varphi(x-y) \sum_{j=1}^N \frac{\partial}{\partial y_j} [-(N-2)\, \frac{y_j}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\ ]\, dy =$$

$$\lim_{\epsilon\to 0} \int \varphi(x-y) \sum_{j=1}^N [N(N-2) \frac{y_j^2}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}} - (N-2) \frac{1}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\ ]\, dy =$$

$$\lim_{\epsilon\to 0}\int \varphi(x-y)\ \frac{-N(N-2)\epsilon^2}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}}\,dy\ =\ \lim_{\epsilon\to 0}\int \varphi(x-\epsilon\xi)\ \frac{-N(N-2)}{(1+|\xi|^2)^{\frac{N+2}{2}}}\,d\xi\ =$$

$$=\ -\varphi(x)N(N-2)\int \frac{d\xi}{(1+|\xi|^2)^{\frac{N+2}{2}}}$$

H-L-S Se $\lambda\in(0,N)$, $p,r>1$, $\frac{1}{p}+\frac{\lambda}{N}+\frac{1}{r}=2$, esiste $c=c(\lambda,N,p)$:

$$\Big|\int_{\mathbf{R}^N\times\mathbf{R}^N}\frac{h(x)\ f(y)}{|x-y|^\lambda}\,dxdy\Big|\ \le\ c\quad \|h\|_r\ \|f\|_p\qquad \forall f\in L^p(\mathbf{R}^N),\quad h\in L^r(\mathbf{R}^N)$$

In particolare, posto $\frac{1}{s}=\frac{\lambda}{N}+\frac{1}{p}-1$, (ovvero s é l'esponente coniugato di r), allora

$$\exists c>0:\qquad \|G_\lambda * f\|_s\ \le\ c\,\|f\|_p\qquad \forall f\in L^p(\mathbf{R}^N)$$

NOTA La relazione sopra indicata tra i parametri λ,p,r,N é necessaria perché una siffatta diseguaglianza possa valere, e ció per il suo carattere di invarianza rispetto ai cambi di scala.

Alla dimostrazione premettiamo alcune notazioni ed utili formule. Data $f\ge 0$ misurabile in $\mathbf{R}^N$, sia

$$\chi_f:=\chi_{\Gamma_f},\quad \Gamma_f:=\{(x,t)\in\mathbf{R}^N\times[0,+\infty]:\quad 0\le t<f(x)\}$$

la funzione caratteristica del sottografico di f. Chiaramente Γ_f e quindi χ_f sono misurabili e

$$f(x)=\int_0^{+\infty}\chi_f(x,t)dt\quad \forall x\in\mathbf{R}^N,\qquad \int_{\mathbf{R}^N}f=\int_0^{+\infty}|(f>t)|dt$$

ove abbiamo indicato con $|(f>t)|$ la misura dell'insieme $(f>t):=\{x\in\mathbf{R}^N: f(x)>t\}$ (la seconda uguaglianza deriva da Fubini). Analogamente

$$f^p(x)=p\int_0^{f^p(x)}s^{p-1}ds=p\int_0^{+\infty}\chi_f(x,s)s^{p-1}ds\ ,\qquad \int_{\mathbf{R}^N}f^p=p\int_0^{+\infty}|(f>s)|s^{p-1}ds$$

Infine, effettuando il cambio di variabile $t=\frac{1}{\tau^\lambda}$, vediamo che

$$\frac{1}{|x|^\lambda}=\int_0^{\frac{1}{|x|^\lambda}}dt=\lambda\int_{|x|}^{+\infty}\tau^{-\lambda-1}ds=\lambda\int_0^{+\infty}\chi_{\{|x|<\tau\}}\tau^{-\lambda-1}d\tau\quad \forall x\in\mathbf{R}^N$$

Prova di (HLS). Dividendo per $\|f\|_p\ \|h\|_r$, (HLS) si riscrive

$$c(N,\lambda,p) \ := \ \sup\{\int_{\mathbf{R}^N\times\mathbf{R}^N} \frac{h(x)\ f(y)}{|x-y|^\lambda}\,dx\,dy : \quad f,h\geq 0, \|f\|_p = 1 = \|h\|_r\} < +\infty$$

Si tratta cioé di provare che esiste $c = c(N,\lambda,p) > 0$ tale che

$$p\int_0^{+\infty} |(f>t)|t^{p-1}ds = \int_{\mathbf{R}^N} f^p = 1 = \int_{\mathbf{R}^N} h^r = r\int_0^{+\infty} |(h>s)|s^{r-1}ds \quad \Rightarrow$$

$$\int_{\mathbf{R}^N\times\mathbf{R}^N}\left[\left(\int_0^{+\infty}\chi_f(y,t)dt\right)\left(\int_0^{+\infty}\chi_h(x,s)ds\right)\left(\int_0^{+\infty}\chi_{\{|x-y|<\tau\}}\tau^{-\lambda-1}d\tau\right)\right]dx\,dy \leq c$$

ovvero, usando Fubini, che $$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{I(t,s,\tau)}{\tau^{\lambda+1}}dtdsd\tau \leq c$$

ove si é posto

$$I(t,s,\tau) := \int_{\mathbf{R}^N\times\mathbf{R}^N} \chi_f(x,t)\ \chi_h(y,s)\ \chi_{\{|x-y|<\tau\}}\,dx\,dy$$

Osserviamo che

$$\chi_{\{|x-y|<\tau\}} \leq 1 \quad \Rightarrow \quad I \ \leq \ |(f>t)|\ |(h>s)|$$

$$\chi_h \leq 1 \quad \Rightarrow \quad I \ \leq \ volB_\tau\,|(f>t)| \ = \ c_N\ \tau^N|(f>t)|$$

$$\chi_f \leq 1 \quad \Rightarrow \quad I \ \leq \ volB_\tau\,|(h>s)| = c_N\ \tau^N\,|(h>s)|$$

$$\Rightarrow \quad I \leq \frac{c_N\,\tau^N\,|(f>t)|\ |(h>s)|}{\max\{c_N\,\tau^N,\ \ |(f>t)|,\ \ |(h>s)|\}}$$

Sostituendo τ con $c_N^{\frac{1}{N}}\ \tau$, otteniamo

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{I(t,s,\tau)}{\tau^{\lambda+1}}\,dt\,ds\,d\tau \ \leq$$

$$\leq\ c_N^{\frac{\lambda}{N}}\int_0^{+\infty}\int_0^{+\infty}\left(\int_0^{+\infty}\frac{1}{\tau^{\lambda+1}}\frac{\tau^N\,|(f>t)|\ |(h>s)|}{\max\{\ \tau^N,\ \ |(f>t)|,\ \ |(h>s)|\}}\,d\tau\right)\ ds\,dt$$

Passo 1 Per ogni s,t si ha

$$\int_0^{+\infty}\frac{I(t,s,\tau)}{\tau^{\lambda+1}}\,d\tau \ \leq\ \frac{N\,c_N^{\frac{\lambda}{N}}}{\lambda(N-\lambda)}\ \min\{|(h>s)|\ |(f>t)|^{\frac{N-\lambda}{N}},\ |(f>t)|\ |(h>s)|^{\frac{N-\lambda}{N}}\}$$

Infatti, se $\quad |(h>s)| \leq |(f>t)|, \quad$ allora

$$\frac{\tau^N\,|(f>t)|\;|(h>s)|}{\max\{\,\tau^N,\;|(f>t)|,\;|(h>s)|\}} \leq \frac{\tau^N\,|(f>t)|\;|(h>s)|}{\max\{\tau^N,\;|(f>t)|\}}$$

e quindi

$$\int_0^{+\infty}\frac{I(t,s,\tau)}{\tau^{\lambda+1}}d\tau \leq c_N^{\frac{\lambda}{N}}\left[\int_0^{|(f>t)|^{\frac{1}{N}}}\frac{\tau^N\;|(h>s)|}{\tau^{\lambda+1}}\,d\tau \;+\; \int_{|(f>t)|^{\frac{1}{N}}}^{\infty}\frac{|(f>t)|\;|(h>s)|}{\tau^{\lambda+1}}\,d\tau\right]$$

$$= \frac{c_N^{\frac{\lambda}{N}}}{N-\lambda}\,|(h>s)|\;|(f>t)|^{\frac{N-\lambda}{N}} + \frac{c_N^{\frac{\lambda}{N}}}{\lambda}\,|(h>s)|\;|(f>t)|^{1-\frac{\lambda}{N}} =$$

$$= \frac{Nc_N^{\frac{\lambda}{N}}}{\lambda(N-\lambda)}|(h>s)|\;|(f>t)|^{\frac{N-\lambda}{N}} \;\leq\; \frac{Nc_N^{\frac{\lambda}{N}}}{\lambda(N-\lambda)}|(f>t)|\;|(h>s)|^{\frac{N-\lambda}{N}}$$

Scambiando h ed f, si ottiene

$$|(h>s)| \geq |(f>t)| \quad\Rightarrow\quad \int_0^{+\infty}\frac{I(t,s,\tau)}{\tau^{\lambda+1}}d\tau \leq \frac{Nc_N^{\frac{\lambda}{N}}}{\lambda(N-\lambda)}|(f>t)|\,|(h>s)|^{\frac{N-\lambda}{N}} \leq$$

$$\leq \frac{Nc_N^{\frac{\lambda}{N}}}{\lambda(N-\lambda)}\;|(h>s)|\;|(f>t)|^{\frac{N-\lambda}{N}}$$

Dal Passo 1 otteniamo

$$\forall T>0: \qquad \frac{\lambda(N-\lambda)}{Nc_N^{\frac{\lambda}{N}}}\int_{\mathbf{R}^N\times\mathbf{R}^N}\frac{h(x)\;f(y)}{|x-y|^\lambda}dxdy \leq$$

$$\leq \int_0^\infty\left(|(h>s)|\int_0^T\;|(f>t)|^{\frac{N-\lambda}{N}}\,dt\right)ds + \int_0^\infty\left(\;|(h>s)|^{\frac{N-\lambda}{N}}\int_T^\infty|(f>t)|\,dt\right)ds$$

Ora,

$$\int_0^T\;|(f>t)|^{\frac{N-\lambda}{N}}\,dt = \int_0^T\;|(f>t)|^{\frac{N-\lambda}{N}}t^{(p-1)\frac{N-\lambda}{N}}\;t^{-(p-1)\frac{N-\lambda}{N}}\,dt \;\leq$$

$$\leq \left(\int_0^\infty|(f>t)|t^{p-1}\,dt\right)^{\frac{N-\lambda}{N}}\;\left(\int_0^T t^{-(p-1)\frac{N-\lambda}{\lambda}}\,dt\right)^{\frac{\lambda}{N}} =$$

$$= \left(\frac{1}{p}\right)^{\frac{N-\lambda}{N}}\left[\frac{T^{[1-(p-1)\frac{N-\lambda}{\lambda}]}}{1-(p-1)\frac{N-\lambda}{\lambda}}\right]^{\frac{\lambda}{N}} \;=\; c(\lambda,N,p)T^{(r-1)\frac{p}{r}}$$

perché $\quad \frac{1}{p}+\frac{\lambda}{N}+\frac{1}{r}=2 \quad \Rightarrow$

$$\frac{\lambda}{N}-(p-1)\frac{N-\lambda}{N}=1-p+\frac{p\lambda}{N}=2p-\frac{p}{r}-p=(r-1)\,\frac{p}{r}$$

Dunque, prendendo $\quad T=s^{\frac{r}{p}}, \quad$ vediamo che

$$p\int_0^{+\infty}|(f>t)|t^{p-1}ds=1=r\int_0^{+\infty}|(h>s)|s^{r-1}ds \quad \Rightarrow$$

$$\int_0^{\infty}\left(|(h>s)|\int_0^{s^{\frac{r}{p}}}|(f>t)|^{\frac{N-\lambda}{N}}\,dt\right)ds\ \leq$$

$$\leq\ c(N,\lambda,p)\int_0^{\infty}|(h>s)|\ s^{r-1}\,ds\ =\ \frac{c(N,\lambda,p)}{r}$$

Analoga limitazione per il secondo integrale: usando Fubini e poi Holder,

$$\int_0^{\infty}\left(|(h>s)|^{\frac{N-\lambda}{N}}\int_{s^{\frac{r}{p}}}^{\infty}|(f>t)|\,dt\right)ds=\int_0^{\infty}\left(|(f>t)|\int_0^{t^{\frac{p}{r}}}|(h>s)|^{\frac{N-\lambda}{N}}\,ds\right)dt=$$

$$=\int_0^{\infty}\left(|(f>t)|\int_0^{t^{\frac{p}{r}}}|(h>s)|^{\frac{N-\lambda}{N}}s^{(r-1)\frac{N-\lambda}{N}}\ s^{-(r-1)\frac{N-\lambda}{N}}\,ds\right)dt\leq$$

$$\leq\int_0^{\infty}|(f>t)|\left(\int_0^{\infty}|(h>s)|s^{r-1}\,ds\right)^{\frac{N-\lambda}{N}}\left(\int_0^{t^{\frac{p}{r}}}s^{-(r-1)\frac{N-\lambda}{\lambda}}\,ds\right)^{\frac{\lambda}{N}}dt\ =$$

$$c(\lambda,N,p,r)\int_0^{\infty}|(f>t)|\ t^{[\frac{\lambda}{N}-(r-1)\frac{N-\lambda}{N}]\frac{p}{r}}\,dt\ =$$

$$=\ c(\lambda,N,p,r)\int_0^{\infty}|(f>t)|\ t^{p-1}\,dt$$

DUE CASI IMPORTANTI.

$\lambda=N-2, \quad \frac{N}{2}>\ p\ >\ 1 \quad \Rightarrow \quad \frac{1}{s}=\frac{N-2p}{Np} \quad (=\frac{N-2}{2N} \quad$ se $\quad p=\frac{2N}{N+2}) \quad \Rightarrow$

$$\|G_{N-2}*f\|_{\frac{Np}{N-2p}}\ \leq\ c(N)\ \|f\|_p \qquad \forall f\in L^p(\mathbf{R}^N)$$

$\lambda=N-1, \quad N>\ p\ >\ 1, \quad \Rightarrow \quad \frac{1}{s}=\frac{N-p}{Np} \quad \Rightarrow$

$$\|G_{N-1}*f\|_{\frac{Np}{N-p}}\ \leq\ c(N,p)\ \|f\|_p \qquad \forall f\in L^p(\mathbf{R}^N)$$

LA DISEGUAGLIANZA DI SOBOLEV

$$\forall p \in (1,N), \exists c = c(N,p) : \quad \left(\int_{\mathbf{R}^N} |u|^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}} \leq c \int_{\mathbf{R}^N} |\nabla u|^p \qquad \forall u \in C_0^\infty(\mathbf{R}^N)$$

Una formula di rappresentazione. Sia $c_N := N \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}$. É

$$u(x) = \frac{1}{c_N} \int_{\mathbf{R}^N} \frac{< \nabla u(y)\ ,\ x-y >}{|x-y|^N} dy = \frac{1}{c_N} \int_{\mathbf{R}^N} \frac{< \nabla u(x-y)\ ,\ y >}{|y|^N} dy \qquad \forall u \in C_0^\infty(\mathbf{R}^N)$$

Prova . Per ogni fissato x,

$$\int_{\mathbf{R}^N} \frac{< \nabla u(x-y)\ ,\ y >}{|y|^N}\, dy = \lim_{\epsilon \to 0} \sum_{j=1}^N \int_{\mathbf{R}^N} -\frac{\partial}{\partial y_j} [u(x-y)] \quad \frac{y_j}{(\epsilon^2+|y|^2)^{\frac{N}{2}}}\, dy =$$

$$\lim_{\epsilon \to 0} \sum_{j=1}^N \int_{\mathbf{R}^N} u(x-y) \left(\frac{1}{(\epsilon^2+|y|^2)^{\frac{N}{2}}} - N \frac{y_j^2}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}} \right) dy =$$

$$N \lim_{\epsilon \to 0} \epsilon^2 \int_{\mathbf{R}^N} \left[\frac{u(x-y)}{(\epsilon^2+|y|^2)^{\frac{N+2}{2}}} \right] dy =$$

$$= N \lim_{\epsilon \to 0} \int_{\mathbf{R}^N} \left[\frac{u(x-\epsilon z))}{(1+|z|^2)^{\frac{N+2}{2}}} \right] dz = N u(x) \int_{\mathbf{R}^N} \frac{dz}{(1+|z|^2)^{\frac{N+2}{2}}}$$

Prova della diseguaglianza di Sobolev. $u \in C_0^\infty(\mathbf{R}^N) \ \Rightarrow$

$$|u(x)| \leq c \int_{\mathbf{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\, dy = c\, (|\nabla u| * G_{N-1})(x) \quad \forall x \in \mathbf{R}^N \ \Rightarrow$$

$$\|u\|_{\frac{Np}{N-p}} \leq c\, \|G_{N-1} * |\nabla u|\|_{\frac{Np}{N-p}} \leq c\ \|\nabla u\|_p$$

Diseguaglianza di POINCARÉ. Sia $1 < p < N$, $\Omega \subset \mathbf{R}^N$ aperto limitato.

Allora $\exists\, c = c(\Omega) > 0 :$ $\displaystyle\int_\Omega |\nabla u|^p \geq c \int_\Omega |u|^p \quad \forall u \in C_0^\infty(\Omega)$

Infatti, da $\frac{p}{N} + \frac{N-p}{N} = 1$, usando Holder e quindi Sobolev, segue

$$\int_{\mathbf{R}^N} |u|^p \leq \left(\int_{\mathbf{R}^N} |u|^{\frac{Np}{N-p}} \right)^{\frac{N-p}{N}} vol(\Omega)^{\frac{p}{N}} \leq M(\Omega) \int_{\mathbf{R}^N} |\nabla u|^p \quad \forall u \in C_0^\infty(\Omega)$$

Poincaré non vale in $\mathbf{R}^N$: $\displaystyle\inf_{u \in C_0^\infty(\mathbf{R}^N),\, u \neq 0} \frac{\int_{\mathbf{R}^N} |\nabla u|^p}{\int_{\mathbf{R}^N} |u|^p} = 0$

Se $u_\epsilon(x) := u(\epsilon x)$, é $\int_{\mathbf{R}^N} |u_\epsilon|^p = \epsilon^{-N} \int_{\mathbf{R}^N} |u|^p$, $\int_{\mathbf{R}^N} |\nabla u_\epsilon|^p = \epsilon^{p-N} \int_{\mathbf{R}^N} |\nabla u|^p$

e quindi $\displaystyle\frac{\int_{\mathbf{R}^N} |\nabla_\epsilon u|^p}{\int_{\mathbf{R}^N} |u_\epsilon|^p} = \epsilon^p \frac{\int_{\mathbf{R}^N} |\nabla u|^p}{\int_{\mathbf{R}^N} |u|^p} \to_\epsilon 0$ Allo stesso modo

si vede che $\displaystyle\lambda_1(\Omega) := \inf_{u \in C_0^\infty(\Omega),\, u \neq 0} \frac{\int_{\mathbf{R}^N} |\nabla u|^2}{\int_{\mathbf{R}^N} |u|^2} < \frac{\int_{\mathbf{R}^N} |\nabla u|^2}{\int_{\mathbf{R}^N} |u|^2} \quad \forall u \in C_0^\infty(\Omega)$

[$l'\inf$ non é realizzato in $C_0^\infty(\Omega)$: $\forall u \in C_0^\infty(\Omega)$ $\exists \epsilon < 1 :$ $u_\epsilon \in C_0^\infty(\Omega)$]

Diseguaglianze di MORREY. Sia $p > N$.

(i) $\forall\, R > 0\ \exists\ c = c(N,p,R)\ :\ \displaystyle\|u\|_\infty \leq c \left(\int_{\mathbf{R}^N} |\nabla u|^p \right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(B_R)$

(ii) $\exists c = c(p,N) :$ $\displaystyle|u(x) - u(y)| \leq c|x-y|^{\frac{p-N}{p}} \left(\int_{\mathbf{R}^N} |\nabla u|^p \right)^{\frac{1}{p}} \quad \forall u \in C_0^\infty(\mathbf{R}^N)$

(i) Utilizzando la formula di rappresentazione e quindi Holder, ed usando il fatto

che $\displaystyle p > N \Rightarrow \frac{1}{q} = 1 - \frac{1}{p} > \frac{N-1}{N} \Rightarrow q(N-1) < N$ vediamo che

$$u \in C_0^\infty(B_R),\ \ x \in \mathbf{R}^N \qquad \Rightarrow \qquad |u(x)| \leq c \int_{\mathbf{R}^N} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\, dy \leq$$

$$c\left(\int_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int_{B_R} \frac{1}{|x-y|^{q(N-1)}}\, dy\right)^{\frac{1}{q}} \leq c\left(\int_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}} \left(\int_{B_{2R}} \frac{dz}{|z|^{q(N-1)}}\right)^{\frac{1}{q}}$$

(ii) Sia $Q_r := \{x : \ |x_i| \leq r \ \forall i\}$ (cubo di lato $2r$ centrato nell'origine). Fissato $\overline{x}$, sia $\overline{u} = \frac{1}{2^N r^N} \int_{Q_r+\overline{x}} u$ la media di u su $Q := Q_r + \overline{x}$. Per ogni $x \in Q$ risulta

$$|\overline{u} - u(x)| = |\frac{1}{(2r)^N} \int_Q [u(y) - u(x)]\, dy| \leq \int_Q \left[\frac{|y-x|}{(2r)^N} \int_0^1 |\nabla u(ty + (1-t)x)|\, dt\right] dy$$

$$\leq \frac{\sqrt{N}}{(2r)^{N-1}} \int_0^1 \left(\int_{(1-t)x+tQ} \frac{|\nabla u(z)|}{t^N}\, dz\right) dt \leq \frac{\sqrt{N}}{(2r)^{N-1}} \left(\int_Q |\nabla u|^p\right)^{\frac{1}{p}} \int_0^1 vol(tQ)^{1-\frac{1}{p}} \frac{dt}{t^N} =$$

$$\sqrt{N}\, (2r)^{1-\frac{N}{p}} \left(\int_{Q_{2r}+\overline{x}} |\nabla u|^p\right)^{\frac{1}{p}} \int_0^1 t^{-\frac{N}{p}}\, dt = c(N,p)\, r^{1-\frac{N}{p}} \left(\int_{Q_{2r}+\overline{x}} |\nabla u|^p\right)^{\frac{1}{p}}$$

Dunque, fissati x, y e posto $r = |x-y|$, $\overline{x} = \frac{x+y}{2}$, per cui $x, y \in Q_r + \overline{x}$, si ha

$$|u(x) - u(y)| \leq 2c(N,p)\, r^{1-\frac{N}{p}} \left(\int_{Q_{2r}+\overline{x}} |\nabla u|^p\right)^{\frac{1}{p}} = 2c(N,p)|x-y|^{1-\frac{N}{p}} \left(\int_{\mathbf{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}$$

Morrey (i) non vale in $\mathbf{R}^N$. Se $u_\epsilon(x) := u(\epsilon x)$, é

$$\int_{\mathbf{R}^N} |\nabla u_\epsilon|^p = \epsilon^{p-N} \int_{\mathbf{R}^N} |\nabla u|^p \qquad \text{mentre} \qquad \|u_\epsilon\|_\infty = \|u\|_\infty$$

Il Teorema di compattezza di RELLICH.

Sia $u_n \in C_0^\infty(B_R)$, con $\sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^p\right)^{\frac{1}{p}} < +\infty$. Allora

(i) se $1 < p < N$, u_n ha una sottosuccessione convergente in $L^r(B_R)$ $\forall r < \frac{Np}{N-p}$.

(ii) se $p = N$, u_n ha una sottosuccessione convergente in $L^r(B_R)$ $\forall r$.

(iii) se $p > N$, u_n ha una sottosuccessione uniformemente convergente in B_R

Prova. (i) Sia $1 \le r \le \frac{Np}{N-p}$. Da Holder e quindi Sobolev segue che

$$\sup_n \left(\int_{B_R} |u_n|^r \right)^{\frac{1}{r}} \le c(R) \sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^p \right)^{\frac{1}{p}} < +\infty$$

Poi, la diseguaglianza di interpolazione con $\alpha \in [0,1),\ \alpha + (1-\alpha)\frac{N-p}{Np} = \frac{1}{r}$ dá

$$\left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|^r \, dx \right)^{\frac{1}{r}} \le$$

$$\left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)| \right)^{\alpha} \left(\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)|^{\frac{Np}{N-p}} \right)^{\frac{(1-\alpha)(N-p)}{Np}}$$

Il secondo fattore, grazie a Sobolev, resta, nelle nostre ipotesi, limitato e

$$\int_{\mathbf{R}^N} |u_n(x+h) - u_n(x)| \, dx \le \int_{\mathbf{R}^N} \left(\int_0^1 | < \nabla u_n(x+th), h > | dt \right) dx$$

$$\le vol(B_R)^{1-\frac{1}{p}} \, |h| \int_0^1 \left(\int_{\mathbf{R}^N} |\nabla u_n(x+th)|^p \, dx \right)^{\frac{1}{p}} dt \le c|h| \sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n(x)|^p \, dx \right)^{\frac{1}{p}}$$

La compattezza di u_n in $L^r(\mathbf{R}^N)$ segue quindi da Frechet-Kolmogoroff.

(ii) In tal caso $\sup_n \left(\int_{\mathbf{R}^N} |\nabla u_n|^r \right)^{\frac{1}{r}} < +\infty \ \ \forall r$, e quindi, come in (i), otteniamo la compattezza di u_n in ogni L^r.

(iii) La (i) nel Teorema di Morrey dice $\sup_n \|u_n\|_\infty < +\infty$ mentre la (ii) assicura la equicontinuitá delle u_n. La conclusione segue quindi dal Teorema di Ascoli-Arzelá.

Nota. Rellich non vale in tutto $\mathbf{R}^N$ né fino all'esponente limite $p^* := \frac{Np}{N-p}$.

(i) Se $f \in C_0^\infty(\mathbf{R}^N)$, $f \ne 0$, $h \in \mathbf{R}^N, h \ne 0$, $f_n(x) := f(x+nh)$, allora $\|\nabla f_n\|_2 \equiv \|\nabla f\|_2$, ma f_n non ha estratte convergenti in alcun L^p

(i) Se $f \in C_0^\infty(B_1)$, $f \ne 0$, $\epsilon_n \to_n 0$, $f_n(x) := \epsilon_n^{\frac{N-2}{2}} f(\frac{x}{\epsilon_n})$ allora $\|\nabla f_n\|_2 \equiv \|\nabla f\|_2$ e $\|f_n\|_{\frac{2N}{N-2}} \equiv \|f_n\|_{\frac{2N}{N-2}}$ e quindi f_n non ha estratte convergenti in $L^{\frac{2N}{N-2}}$ (mentre converge a zero in L^p per $1 \le p < \frac{2N}{N-2}$).