

Am1c – Soluzioni Tutorato IX

Integrazione funzioni irrazionali

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Esercizio 1 Tramite le sostituzioni suggerite si ottiene:

$$(1) \quad x = a \sin t \Rightarrow t = \arcsin \frac{x}{a} \Rightarrow dx = a \cos t dt$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{a^2 - a^2 \sin^2 t}} a \cos t dt = \int dt = t + k = \arcsin \frac{x}{a} + k$$

$$(2) \quad x = a \sinh t \Rightarrow t = \sinh^{-1} \frac{x}{a} \Rightarrow dx = a \cosh t dt$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \int \frac{1}{\sqrt{a^2 + a^2 \sinh^2 t}} a \cosh t dt = \int dt = t + k = \sinh^{-1} \frac{x}{a} + k = \ln \left(x + \sqrt{x^2 + a^2} \right) + k$$

$$(3) \quad x = a \cosh t \Rightarrow t = \cosh^{-1} \frac{x}{a} \Rightarrow dx = a \sinh t dt$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{1}{\sqrt{a^2 \cosh^2 t - a^2}} a \sinh t dt = \int dt = t + k = \cosh^{-1} \frac{x}{a} + k = \ln \left| x + \sqrt{x^2 - a^2} \right| + k$$

$$(4) \quad x = a \sin t \Rightarrow t = \arcsin \frac{x}{a} \Rightarrow dx = a \cos t dt$$

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \int a \cos t \sqrt{a^2 - a^2 \sin^2 t} dt = a^2 \int \cos^2 t dt = \frac{a^2}{2} (t + \sin t \cos t) + k = \\ &= \frac{a^2}{2} \left(\arcsin \frac{x}{a} + \frac{x \sqrt{a^2 - x^2}}{a^2} \right) + k \end{aligned}$$

$$(5) \quad x = a \sinh t \Rightarrow t = \sinh^{-1} \frac{x}{a} \Rightarrow dx = a \cosh t dt$$

$$\int \sqrt{a^2 + x^2} dx = \int a \cosh t \sqrt{a^2 + a^2 \sinh^2 t} dt = a^2 \int \cosh^2 t dt$$

Si ricava facilmente, integrando per parti, che $\int \cosh^2 t dt = \frac{1}{2} (\cosh t \sinh t + t) + k$ e quindi

$$\int \sqrt{a^2 + x^2} dx = \frac{a^2}{2} \left(\sinh^{-1} \frac{x}{a} + \frac{x\sqrt{x^2 + a^2}}{a^2} \right) + k = \frac{a^2}{2} \ln \left(x + \sqrt{x^2 + a^2} \right) + \frac{x\sqrt{x^2 + a^2}}{2} + k$$

(6) $x = a \cosh t \Rightarrow t = \cosh^{-1} \frac{x}{a} \Rightarrow dx = a \sinh t dt$

$$\int \sqrt{x^2 - a^2} dx = \int a \sinh t \sqrt{a^2 \cosh^2 t - a^2} dt = a^2 \int \sinh^2 t dt$$

come prima, integrando per parti, si ha $\int \sinh^2 t dt = \frac{1}{2}(\cosh t \sinh t - t) + k$ e quindi

$$\int \sqrt{x^2 - a^2} dx = \frac{a^2}{2} \left(\frac{x}{a^2} \sqrt{x^2 - a^2} - \cosh^{-1} \frac{x}{a} \right) + k = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln \left| x + \sqrt{x^2 - a^2} \right| + k$$

Esercizio 2 Calcolare i seguenti integrali irrazionali:

(1) Posto $x = t^{12} \Rightarrow t = \sqrt[12]{x} \Rightarrow dx = 12t^{11} dt$ si ha

$$\begin{aligned} \int \frac{1 - \sqrt[3]{x}}{\sqrt{x + \sqrt[4]{x}}} dx &= \int \frac{1 - t^4}{t^6 + t^3} 12t^{11} dt = \\ &= -\frac{6}{5} \sqrt[6]{x^5} + \frac{12}{7} \sqrt[12]{x^7} + 2\sqrt{x} - 3\sqrt[3]{x} - 4\sqrt[4]{x} + 12\sqrt[12]{x} + 6 \ln \left(\sqrt[6]{x} - \sqrt[12]{x} + 1 \right) - 4\sqrt{3} \arctan \frac{2\sqrt[12]{x} - 1}{\sqrt{3}} + k \end{aligned}$$

(lo svolgimento dell'integrale di funzione razionale viene omissso)

(2) Posto $x+1 = t^6 \Rightarrow t = \sqrt[6]{x+1} \Rightarrow dx = 6t^5 dt$ si ha

$$\int \frac{1 - \sqrt[3]{x+1}}{\sqrt{x+1} + \sqrt[3]{x+1}} dx = \int \frac{1 - t^2}{t^3 + t^2} 6t^5 dt = \frac{3}{2} \sqrt[3]{(x+1)^2} - \frac{6}{5} \sqrt[6]{(x+1)^5} + k$$

(3) $\int \frac{1}{\sqrt{x^2 - 3x + 2}} dx = \int \frac{2}{\sqrt{(2x-3)^2 - 1}} dx = \int \frac{1}{\sqrt{z^2 - 1}} dz = \ln \left| z + \sqrt{z^2 - 1} \right| + k =$
 $= \ln \left| 2x - 3 + \sqrt{(2x-3)^2 - 1} \right| + k$

(4) $\int \frac{x}{\sqrt{-x^2 + x + 2}} dx = -\frac{1}{2} \left(\int \frac{-2x+1}{\sqrt{-x^2 + x + 2}} dx - \int \frac{1}{\sqrt{-x^2 + x + 2}} dx \right) =$
 $= -\sqrt{-x^2 + x + 2} + \int \frac{1}{\sqrt{9 - (2x-1)^2}} dx = -\sqrt{-x^2 + x + 2} + \frac{1}{2} \arcsin \frac{2x-1}{3} + k$

(5) $\int \sqrt{x^2 + 4x + 13} dx = \int \sqrt{(x+2)^2 + 9} dx = \frac{1}{2} \left[9 \ln \left(x+2 + \sqrt{x^2 + 4x + 13} \right) + (x+2) \sqrt{x^2 + 4x + 13} \right] + k$

$$(6) \int \sqrt{-x^2 - x + 1} dx = \frac{1}{2} \int \sqrt{5 - (2x+1)^2} dx = \frac{1}{8} \left[5 \arcsin \frac{2x+1}{\sqrt{5}} + (2x+1) \sqrt{-x^2 - x + 1} \right] + k$$

$$(7) \int \frac{x+3}{\sqrt{x^2+2x+10}} dx = \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+10}} dx + 2 \int \frac{1}{\sqrt{x^2+2x+10}} dx =$$

$$= \sqrt{x^2+2x+10} + 2 \int \frac{1}{\sqrt{(x+1)^2+9}} dx = \sqrt{x^2+2x+10} + 2 \ln \left(\frac{x+1}{3} + \sqrt{\left(\frac{x+1}{3}\right)^2 + 1} \right) + k$$

$$(8) \int \frac{x^3+x}{\sqrt{-x^4+3x^2-2}} dx = \frac{1}{2} \int \frac{2x(x^2+1)}{\sqrt{-x^4+3x^2-2}} dx$$

ora posto $t = x^2 \Rightarrow dt = 2x dx$ si ottiene

$$\frac{1}{2} \int \frac{2x(x^2+1)}{\sqrt{-x^4+3x^2-2}} dx = \frac{1}{2} \int \frac{t+1}{\sqrt{-t^2+3t-2}} dt = -\frac{1}{4} \int \frac{-2t+3}{\sqrt{-t^2+3t-2}} dt + \frac{5}{4} \int \frac{1}{\sqrt{-t^2+3t-2}} dt =$$

$$= -\frac{1}{2} \sqrt{-t^2+3t-2} + \frac{5}{4} \int \frac{2}{\sqrt{1-(2t-3)^2}} dt =$$

$$= -\frac{1}{2} \sqrt{-x^4+3x^2-2} + \frac{5}{4} \arcsin(2x^2-3) + k$$