AM120 Settimana 8

SERIE DI POTENZE NEL CAMPO COMPLESSO

1. Definizione $z_n \in \mathbf{C}$ converge a z (e scriveremo $z_n \to_n z$) se e solo se $|z_n - z| \to_n 0$, ovvero, se $\forall \epsilon > 0 \;\; \exists n_\epsilon : \;\; |z_n - z| \le \epsilon \;\; \forall n \ge n_\epsilon$.

Siccome $|z_n - z|^2 = |Rez_n - Rez|^2 + |Imz_n - Imz|^2$, si ha che:

$$z_n \to_n z \qquad \Leftrightarrow \qquad Rez_n \to_n Rez \quad \text{e} \quad Imz_n \to_n Imz$$

La condizione necessaria e sufficente di Cauchy

$$z_n \to_n z \quad \Leftrightarrow \quad \forall \epsilon > 0, \exists n_\epsilon : \quad n, m \ge n_\epsilon \quad \Rightarrow \quad |z_n - z_m| \le \epsilon$$

2. Definizione $\sum\limits_{n=1}^{\infty} z_n$ converge sse $S_N := \sum\limits_{n=1}^{N} z_n$ converge.
$\sum_n z_n$ si dice assolutamente convergente se $\sum_n |z_n| < +\infty$.

(**Cauchy**) $\sum_n z_n$ converge $\Leftrightarrow$ $\forall \epsilon > 0, \; \exists N_\epsilon : \; |\sum\limits_{n=N}^{N+p} z_n| \le \epsilon \quad \forall N \ge N_\epsilon, \; \forall \, p.$

In particolare, $\sum_n |z_n| < +\infty \quad \Rightarrow \quad \sum_n z_n$ converge e in particolare,

$\limsup_n |z_n|^{\frac{1}{n}} < 1 \;\; \Rightarrow \;\; \sum_n |z_n| < +\infty \quad \Rightarrow \quad \sum z_n$ converge. Si ha cosí

3. Cauchy-Hadamard Sia $a_n \in \mathbf{C}$, $r := \limsup_n |a_n|^{-\frac{1}{n}}$. Allora

$$z \in \mathbf{C}, \quad |z| < r \quad \Rightarrow \quad \sum_{n=0}^{\infty} |a_n z^n| \; < \; +\infty \quad , \qquad |z| > r \quad \Rightarrow \quad \sum_{n=0}^{\infty} |a_n z^n| \; = \; +\infty$$

r:= raggio di convergenza, $D_r := \{z : \; |z| < r\}$:= disco di convergenza .

ESEMPIO. $\sum\limits_{n=0}^{\infty} z^n$ converge in $|z| < 1$ e $\sum\limits_{n=0}^{\infty} z^n \; = \; \frac{1}{1-z}$.

4. La funzione esponenziale nel campo complesso

$$\exp\, z \;\; := \;\; \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{converge} \qquad \forall z \in \mathbf{C}$$

5. Proposizione $\exp\,(z + w) \; = \; \exp z \; + \; \exp w \qquad \forall \, z, \; w \; \in \mathbf{C}.$

Segue da

6. Lemma (Prodotto secondo Cauchy). $\sum_{n=0}^{\infty} |z_n| + \sum_{n=0}^{\infty} |w_n| < +\infty \Rightarrow$

$$\sum_{n=0}^{\infty} | \sum_{j+k=n} z_j\, w_k| < +\infty \qquad e \qquad \sum_{n=0}^{\infty}(\sum_{j+k=n} z_j\, w_k) = (\sum_{n=0}^{\infty} z_n)\ (\sum_{n=0}^{\infty} w_n)$$

Da 6. segue che $\quad \exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} (\frac{1}{n!} \sum_{j+k=n} \frac{n!}{j!\,k!} z^j\, w^k) =$

$$\sum_{n=0}^{\infty}(\sum_{j+k=n} \frac{z^j}{j!}\, \frac{w^k}{k!}) = (\sum_{n=0}^{\infty} \frac{z^j}{j!})\, (\sum_{n=0}^{\infty} \frac{w^k}{k!}) = \exp z\, \exp w$$

In particolare, $(\exp z)^p = \exp(pz)\ \ \forall p \in \mathbf{N},\ z \in \mathbf{C}$,exp $p = (\exp 1)^p =: e^p$, $(\exp(\frac{1}{p}))^p = e$. Dunque $\exp(\frac{1}{p}) = e^{\frac{1}{p}}$ e quindi $\exp(\frac{p}{q}) = (\exp \frac{1}{q})^p = (e^{\frac{1}{q}})^p = e^{\frac{p}{q}}$: $x \to \exp x,\ \ x \in \mathbf{R}$ é prolungamento continuo di $r \to e^r, r \in \mathbf{Q}$.

7. Formule di Eulero $\qquad \exp(\pm\, it) = \cos t \pm i \sin t \qquad \forall\, t \in \mathbf{R}$

$$\sin t = \frac{\exp(it) - \exp(-it)}{2i}, \qquad \cos t = \frac{\exp(it) + \exp(-it)}{2} \qquad \forall\, t \in \mathbf{R}$$

Da $\exp(it) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$ segue $Re(\exp it) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = \cos t$, $Im(\exp it) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sin t$. In particolare, $\exp(2k\pi i) = 1\ \ \forall k \in \mathbf{Z}$, cioé $t \to \exp(it)\ \ t \in \mathbf{R}$ é 2π-periodica.

Infine, da 5., $\exp(x+iy) = \exp x \exp(iy) = e^x(\cos t + i \sin t)$. In particolare, $\exp(z + 2\pi i) = \exp z,\ \ \forall z$.

8. $\qquad (i)\ \ \exp(-z) = (\exp z)^{-1} \qquad (ii)\ \ \overline{\exp z} = \exp \overline{z}$

$(iii)\ |\exp(it)| = 1\ \ \forall t \in \mathbf{R}$ e $|z| = 1 \Rightarrow \exists\ !\ t \in (-\pi, \pi] :\ z = \exp(it)$

$(i)\ \exp z\, \exp(-z) = 1 \quad (ii)\ \exp \overline{z} = \lim_{N \to +\infty} \sum_{n=0}^{N} \frac{\overline{z}^n}{n!} = \lim_{N \to +\infty} \overline{\sum_{n=0}^{N} \frac{z^n}{n!}} = \overline{\exp z}$

$(iii)\ |\exp(it)|^2 = \exp(it)\ \overline{\exp(it)} = \exp(it)\ \exp(-it) = 1.$ Poi,

$z = x + iy,\ x^2 + y^2 = 1 \Rightarrow \exists\ !\ t \in [0, \pi] :\ x = \cos t$ e $y = \sin t$ se $y \geq 0,\ x = \cos(-t),\ y = \sin(-t)$ se $y < 0$.

9. Funzioni circolari ed iperboliche sui complessi

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \forall z \in \mathbf{C}$$

$$\sinh z := \frac{1}{2}(\exp z - \exp(-z)) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \forall z \in \mathbf{C}$$

$$\cosh z := \frac{1}{2}(\exp z + \exp(-z)) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \forall z \in \mathbf{C}$$

10. (i) $\exp(iz) \equiv \cos z + i \sin z, \qquad \exp(-iz) \equiv \cos z - i \sin z$

(ii) $\cos z \equiv \dfrac{\exp(iz) + \exp(-iz)}{2} \equiv \cosh iz \qquad \sin z \equiv \dfrac{\exp(iz) - \exp(-iz)}{2i} \equiv \dfrac{\sinh(iz)}{i}$

Da (ii) segue: $\sin z, \cos z$ sono funzioni 2π-periodiche mentre $\sinh z, \cosh z$ sono $2\pi i$-periodiche. Inoltre, $\sin^2 z + \cos^2 z \equiv 1$, $\cosh^2 z - \sinh^2 z \equiv 1$

11. Definizione di $\arg z$, $\log z$, $z \in \mathbf{C}$

Dato $z \in \mathbf{C}$, $\arg z$ (**argomento di** z) é l'unico reale in $(-\pi, \pi]$ tale che

$$z = |z| \exp(i \arg z)$$

Notiamo che, per periodicitá, $z = |z| \exp(i(\arg z + 2k\pi))$ $\forall k \in \mathbf{Z}$. Scriveremo

$$Arg\, z := \{\arg z + 2k\pi, \quad k \in \mathbf{Z}\}$$

Ora, dato $w \in \mathbf{C}$, $w \neq 0$

$$\exp z = w \quad \Leftrightarrow \quad \exp(Re\, z) \exp(i\, Im\, z) = |w| \exp(i \arg w) \quad \Leftrightarrow$$

$$\exp Re\, z = |w| \quad \text{e} \quad Im\, z - \arg w \in 2\pi \mathbf{Z} \qquad \text{cioé}$$

$$\exp z = w \quad \Leftrightarrow \quad z \in \{ \log |w| + i\, Arg\, w\}$$

Porremo $\qquad Log\, w := \{ \log |w| + i\, Arg\, w\} \quad \forall w \in \mathbf{C}, \quad w \neq 0$

La funzione $log\, w := \log |w| + i \arg w$ si chiama valore principale del logaritmo.

Esempi. $Log\, x = \log x + 2k\pi i$, $\forall x > 0$, $Log\, x = \log |x| + (2k+1)\pi i$, $\forall x < 0$.
$log\,(-1) = \pi i$, $log\, i = \frac{\pi}{2} i$, $Log\,(1-i) = \log \sqrt{2} + (2k - \frac{1}{4})\pi i$.

12. Potenze in C Se $w, z \in C, w \neq 0$

$$w^z := \exp(z\, Log\, w) = \exp\{z\, [\log |w| + i(arg\, w + 2k\pi)]\} \qquad k \in \mathbf{Z}$$

Esempi. Sia $z = n \in \mathbf{N}$; $\qquad w^n = \exp\{n\, [\log |w| + i(arg\, w + 2k\pi)]\} =$

$\exp\{n \log|w|\} \exp\{n i(arg\, w + 2k\pi)\} = |w|^n [\exp\{ i(arg\, w + 2k\pi)\}]^n = w \times \ldots \times w$ (n volte).

Se $z = \frac{1}{n}$, $n \in \mathbf{N}$, $a^{\frac{1}{n}} = \{|a|^{\frac{1}{n}} \exp\, i\, \frac{arg\, a + 2k\pi}{n},\ k = 0, \ldots, n-1\}$ (le n radici complesse di a). Se $z \notin \mathbf{Q}$, a^z é un insieme infinito. In particolare, $e^z = \exp z$ se e solo se $z \in \mathbf{Z}$.

APPENDICE

A1. Funzioni complesse di variabile complessa.

Sia $O \subset \mathbf{C}$ aperto, ovvero, se $z_0 \in O, z_n \to_n z_0 \Rightarrow z_n \in O$ definitivamente (ovvero $z_0 \in O \Rightarrow \exists D_r(z_0) := \{z \in \mathbf{C} : |z - z_0| < r\} \subset O$).
Sia $f : O \to \mathbf{C}$.

f é **continua** in $z_0 \in O \Leftrightarrow \forall \epsilon > 0\, \exists \delta > 0 : |z - z_0| \le \delta \Rightarrow |f(z) - f(z_0)| \le \epsilon$

f é **derivabile** in $z_0 \in O$ con derivata $f'(z_0)$ $\Leftrightarrow$

$$\forall\, \epsilon > 0\, \exists\, \delta > 0 : |z - z_0| \le \delta \Rightarrow |\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)| \le \epsilon$$

Anche qui, come nel caso reale: f é derivabile in $z_0 \Rightarrow f$ é continua in z_0.

Esercizio Sia $f(z) := \sum_{n=0}^{\infty} a_n z^n$ somma di una serie di potenze avente raggio di convergenza $r > 0$. Allora $f \in C^\infty(D_r)$.

Proviamo che $\frac{d^n f}{dz^n}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k} z^k \quad \forall z \in D_r$. Come nel caso reale, basta provare la formula per $n = 1$. Siano z, $z_0 \in D_\rho$, $\rho < r$. É

$$|\frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n a_n z_0^{n-1}| \le \sum_{n=2}^{\infty} |a_n|\, |\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1}|$$

Ora, $|z^n - z_0^n| = |(z - z_0)(z^{n-1} + z_0 z^{n-2} + \ldots + z_0^{n-2} z + z_0^{n-1})| \le |z - z_0| n \rho^{n-1} \Rightarrow$

$$|\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1}| = |z^{n-1} - z_0^{n-1} + z_0(z^{n-2} - z_0^{n-2}) + \ldots + z_0^{n-2}(z - z_0)| \le$$

$$\le |z^{n-1} - z_0^{n-1}| + |z_0|\, |z^{n-2} - z_0^{n-2}| + \ldots + |z_0|^{n-2} |z - z_0| \le$$

$$\le |z - z_0| \left[(n-1)\rho^{n-2} + (n-2)|z_0|\rho^{n-3} + \ldots + |z_0|^{n-2}\right] \le \frac{n(n-1)}{2} |z - z_0|\, \rho^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} |a_n|\, |\frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1}| \le |z - z_0| \sum_{n=2}^{\infty} |a_n|\, \frac{n(n-1)}{2}\, \rho^{n-2} \to_{z \to z_0} 0$$

perché $\rho < r \Rightarrow \sum_{n=2}^{\infty} |a_n| \frac{n(n-1)}{2} \rho^{n-2} < +\infty$.

A1: Prova di 6. Siano $s_N := \sum_{n=0}^{N} z_n$, $\sigma_N := \sum_{n=0}^{N} w_n$

$$p_N := \sum_{n=0}^{N}(\sum_{j+k=n} z_j w_k) = z_0w_0+(z_0w_1+z_1w_0)+\ldots+(z_0w_N+z_1w_{N-1}+\ldots+z_{N-1}w_1+z_Nw_0)$$

$= z_0(w_0+w_1+\ldots+w_N)+z_1(w_0+\ldots+w_{N-1})+\ldots+z_Nw_0$. Dunque

$$|s_N\,\sigma_N - p_N| =$$

$$|z_0(w_0+\ldots+w_N)+z_1(w_0+\ldots+w_N)+\ldots+z_{N-1}(w_0+\ldots+w_N)+z_N(w_0+\ldots+w_N)-$$

$$[z_0\,(w_0+w_1+\ldots+w_N)+z_1\,(w_0+\ldots+w_{N-1})+\ldots+z_{N-1}(w_0+w_1)+z_N\,w_0]| =$$

$$|z_1\,w_N + z_2(w_{N-1}+w_N)+\ldots+z_{N-1}(w_2+\ldots+w_N)+z_N\,(w_1+\ldots+w_N)| \le$$

$$\le \sum_{j=1}^{n}\left[|z_j|\,|\sum_{i=1}^{j} w_{N-j+i}|\right] + \sum_{j=n+1}^{N}\left[|z_j|\,|\sum_{i=1}^{j} w_{N-j+i}|\right] \le$$

$$\le \left[\sum_{j=1}^{n}|z_j|\right]\left[\sum_{k=N-n+1}^{\infty}|w_k|\right]+\left[\sum_{j\ge n+1}|z_j|\right]\left[\sum_{k=1}^{\infty}|w_k|\right] \qquad n := [\frac{N}{2}].$$ Da

$\sum_{k=N-[\frac{N}{2}]+1}^{\infty} |w_k| \to_{N\to+\infty} 0$, $\sum_{j\ge[\frac{N}{2}]+1} |z_j| \to_{N\to+\infty} 0$, $\sum_j |z_j| < +\infty$, $\sum_{k=1}^{\infty} |w_k| < \infty$
segue $|s_N\,\sigma_N - p_N| \to_{N\to+\infty} 0$ e quindi $\lim_N p_N = \lim_N s_N\,\sigma_N$.

ESERCIZIO. $e := \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to+\infty}(1+\frac{1}{n})^n$ É

$(1+\frac{1}{n})^n = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\frac{1}{n^k} < \sum_{k=0}^{n}\frac{1}{k!}$ perché $\frac{n!}{n^k\,(n-k)!} = \frac{(n-k)!(n-k-1)\ldots n}{(n-k)!n\,n\ldots n} < 1$

e quindi $$\limsup_n(1+\frac{1}{n})^n \le \sum_{n=0}^{\infty}\frac{1}{n!}$$

Viceversa, $n > n_0 \Rightarrow (1+\frac{1}{n})^n > \sum_{k=0}^{n_0} \frac{n!}{k!\,(n-k)!}\frac{1}{n^k} \Rightarrow \liminf_n(1+\frac{1}{n})^n \ge \sum_{k=0}^{n_0}\frac{1}{k!},\ \forall n_0$

perché $\frac{n!}{n^k\,(n-k)!} = (1-\frac{1}{n})\,(1-\frac{2}{n})\,\ldots(1-\frac{k-1}{n}) \to_n 1$. Quindi

$$\liminf_n(1+\frac{1}{n})^n \ge \sum_{k=0}^{\infty}\frac{1}{k!}$$