Universitá degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di AM210

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1. (a) Essendo
$$\frac{\partial}{\partial x}\frac{\arctan(xt)}{t^3+t}=\frac{1}{(t^2+1)(1+x^2t^2)}$$
e
$$\int_0^{+\infty}\frac{dt}{(t^2+1)(1+x^2t^2)}\leq\int_0^{+\infty}\frac{dt}{t^2+1}<+\infty$$
si ha che $f(x)$ é $\mathcal{C}^1((R))$;

(b) Per il punto precedente
$$f'(x)=\int_0^{+\infty}\frac{dt}{(t^2+1)(1+x^2t^2)}=_{(x\neq\pm1)}\int_0^{+\infty}\left(\frac{-\frac{1}{x^2-1}}{t^2+1}+\frac{\frac{x^2}{x^2-1}}{1+x^2t^2}\right)dt=$$
$$=-\frac{1}{x^2-1}\left[\arctan(t)\right]_0^{+\infty}+\frac{x}{x^2-1}\left[\arctan(xt)\right]_0^{+\infty}=\frac{\pi}{2}\left(\frac{1}{x+1}\right);$$
Notiamo che $f'(x)$ é una funzione pari dalla sua forma integrale, quindi
$$f'(x)=\frac{\pi}{2(|x|+1)}\quad\forall x\neq\pm1;$$
Essendo
$$f'(\pm1)=\int_0^{+\infty}\frac{dt}{(t^2+1)^2}=_{(t=\tan(y))}\int_0^{\frac{\pi}{2}}\cos^2(y)dy=\frac{\pi}{4}.$$
abbiamo che
$$f'(x)=\frac{\pi}{2(|x|+1)}\quad\forall x.$$

(c) Essendo $f'(x)=\frac{\pi}{2(|x|+1)}\Longrightarrow f(x)=\frac{\pi}{2}\log(|x|+1)\ \mathrm{sign}(x)$.

(**NB.**$f(x)$ é una funzione dispari, cosí come si poteva notare dalla sua forma integrale essendo $\arctan(xt)$ dispari.)

2. Secondo la teoria delle serie di Fourier
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]$$
ove
$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx\ ;$$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx\ ;$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx\ .$$
NB. Vista la struttura dei coefficienti di Fourier a_n e b_n é chiaro che:

- se $f(x)$ é una funzione pari, allora $b_n = 0\ \forall n$;
- se $f(x)$ é una funzione dispari, allora $a_n = 0\ \forall n$.

(a) $f_1(x) = |x|$.
Essendo $f_1(x)$ una funzione pari ne consegue che $b_n = 0\ \forall n$.
Procediamo con il calcolo dei coefficienti a_n :

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\ dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi\ ;$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\ dx = \frac{2}{\pi}\left(\left[\frac{x\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\ dx\right) =$$

$$= \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{2[(-1)^n - 1]}{\pi n^2} = \begin{cases} 0 & \text{se } n = 2k \\ \frac{-4}{\pi(2k+1)^2} & \text{se } n = 2k+1 \end{cases}.$$

Dunque

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\cos[(2k+1)x]}{(2k+1)^2}.$$

(b) $f_2(x) = e^x$. Calcoliamo i coefficienti di Fourier di $f_2(x)$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\ dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi}) = \frac{2\sinh(\pi)}{\pi}\ ;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(nx)\ dx = \frac{1}{\pi}\left(\left[\frac{e^x\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} e^x\sin(nx)\ dx\right) =$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi} e^x\sin(nx)\ dx = -\frac{1}{n\pi}\left(\left[-\frac{e^x\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} e^x\cos(nx)\ dx\right) =$$

$$= \frac{2(-1)^n\sinh(\pi)}{n^2\pi} - \frac{1}{n^2}a_n \Longrightarrow a_n = \frac{2(-1)^n\sinh(\pi)}{(n^2+1)\pi}\ ;$$

$$b_n = -na_n = \frac{2n(-1)^{n+1}\sinh(\pi)}{(n^2+1)\pi}\ .$$

Dunque

$$e^x = \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+1}\left(\cos(nx) - n\sin(nx)\right).$$

(c) $f_3(x) = (\pi - |x|)^2$.
Essendo $f_3(x)$ una funzione pari ne consegue che $b_n = 0\ \forall n$.
Procediamo con il calcolo dei coefficienti a_n :

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - 2\pi x + x^2)dx = \frac{2}{\pi}\left[\pi^2 x - \pi x^2 + \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2\ ;$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - 2\pi x + x^2)\cos(nx)\ dx = 2\pi\int_0^{\pi} cos(nx)dx - 4\int_0^{\pi} x\cos(nx)dx +$$

$$+\frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx = -4\left(\left[\frac{x\sin(nx)}{n}\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)dx\right) - \frac{2}{\pi}\left(\left[\frac{x^2\sin(nx)}{n}\right]_0^\pi\right) +$$

$$+\frac{2}{\pi}\left(-\frac{2}{n}\int_0^\pi x\sin(nx)dx\right) = \frac{4}{n}\left[-\frac{\cos(nx)}{n}\right]_0^\pi - \frac{4}{n\pi}\int_0^\pi x\sin(nx)dx =$$

$$= \frac{4(1-(-1)^n)}{n^2} - \frac{4}{n\pi}\left(\left[-\frac{x\cos(nx)}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)dx\right) =$$

$$= \frac{4(1-(-1)^n)}{n^2} + \frac{4(-1)^n}{n^2} = \frac{4}{n^2}.$$

Dunque

$$(\pi - |x|)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty}\frac{\cos(nx)}{n^2}.$$

(d) Essendo $f_4(x) = \sin(x)\cos(x) = \frac{\sin(2x)}{2}$ abbiamo che $f_4(x)$ é giá sviluppata in serie di Fourier con $a_n = 0 \quad \forall n, \quad b_n = 0 \quad \forall n \neq 2$, $b_2 = \frac{1}{2}$.

(e) $f_5(x) = x^3$.
Essendo $f_5(x)$ una funzione dispari ne consegue che $a_n = 0\ \forall n$.
Procediamo con il calcolo dei coefficienti b_n :

$$b_n = \frac{2}{\pi}\int_0^\pi x^3\sin(nx)dx = \frac{2}{\pi}\left(\left[-\frac{x^3\cos(nx)}{n}\right]_0^\pi + \frac{3}{n}\int_0^\pi x^2\cos(nx)dx\right) =$$

$$= -\frac{2\pi^2(-1)^n}{n} + \frac{6}{n\pi}\left(-\frac{2}{n}\int_0^\pi x\sin(nx)dx\right) = -\frac{2\pi^2(-1)^n}{n} - \frac{12}{n^2\pi}\left[-\frac{x\cos(nx)}{n}\right]_0^\pi =$$

$$= -\frac{2\pi^2(-1)^n}{n} + \frac{12(-1)^n}{n^3} = \frac{2(-1)^n}{n}\left(\frac{6}{n^2} - \pi^2\right).$$

Dunque

$$x^3 = 2\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}\left(\frac{6}{n^2} - \pi^2\right)\sin(nx).$$

(f) Essendo $f_6(x) = \sin^2(x) = \frac{1-\cos(2x)}{2}$ abbiamo che $f_6(x)$ é giá sviluppata in serie di Fourier con $b_n = 0 \quad \forall n, \quad a_n = 0 \quad \forall n \neq 0, 2$, $a_0 = 1, \quad a_2 = -\frac{1}{2}$.

3. Essendo $f(x)$ una funzione pari abbiamo che $b_n = 0 \quad \forall n$.
Calcoliamo i coefficienti a_n :

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\ dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2\pi^2}{3}\ ;$$

$$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx = \frac{2}{\pi}\left(-\frac{2}{n}\int_0^\pi x\sin(nx)dx\right) =$$

$$= -\frac{4}{n\pi}\left[-\frac{x\cos(nx)}{n}\right]_0^\pi = \frac{4(-1)^n}{n^2}.$$

Dunque

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}\cos(nx).$$

Se consideriamo l'uguaglianza in $x = 0$ otteniamo che

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} \Longrightarrow \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Se consideriamo l'uguaglianza in $x = \pi$ otteniamo che

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{1}{n^2} \Longrightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

4.
$$f(x) = \begin{cases} \sin(x) & \text{se } x \in (0,\pi) \\ 0 & \text{altrimenti} \end{cases} \quad \text{in } [-\pi,\pi].$$

Dunque :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)dx = \left[-\frac{\cos(x)}{\pi}\right]_0^{\pi} = \frac{2}{\pi}\ ;$$

$$\begin{aligned} a_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx &= \frac{1}{2\pi}\int_0^{\pi} (\sin[(n+1)x] - \sin[(n-1)x])dx = \\ &= \frac{1}{2\pi}\left[-\frac{\cos[(n+1)x]}{n+1} + \frac{\cos[(n-1)x]}{n-1}\right]_0^{\pi} = \\ &= \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1} - \frac{\cos[(n+1)\pi]}{n+1} + \frac{\cos[(n-1)\pi]}{n-1}\right) = \\ &= \begin{cases} -\frac{2}{\pi(4k^2-1)} & \text{se } n = 2k \\ 0 & \text{se } n = 2k+1 \end{cases}\ ; \end{aligned}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)dx = \frac{1}{2\pi}\int_0^{\pi} (\cos[(n-1)x] - \cos[(n+1)x])dx = \begin{cases} 0 & \text{se} \quad n \neq 1 \\ \frac{1}{2} & \text{se} \quad n = 1 \end{cases}.$$

Dunque

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{k=1}^{+\infty} \frac{1}{4k^2-1}\cos(2kx).$$

Se consideriamo l'uguaglianza in $x = 0$ otteniamo che

$$0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{k=1}^{+\infty} \frac{1}{4k^2-1} \Longrightarrow \sum_{k=1}^{+\infty} \frac{1}{4k^2-1} = \frac{1}{2}.$$

Se consideriamo l'uguaglianza in $x = \frac{\pi}{2}$ otteniamo che

$$1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{+\infty} \frac{(-1)^k}{4k^2-1} \Longrightarrow \sum_{k=1}^{+\infty} \frac{(-1)^k}{4k^2-1} = \frac{\pi}{2}\left(\frac{1}{\pi} - \frac{1}{2}\right) = \frac{1}{2} - \frac{\pi}{4} = \frac{2-\pi}{4}.$$

5. Procedendo come nell'esercizio 3. sviluppiamo in serie di Fourier $f(x) = x^4$:

$$a_0 = \frac{2}{\pi}\int_0^\pi x^4 dx = \frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^\pi = \frac{2\pi^4}{5} \ ;$$

$$a_n = \frac{2}{\pi}\int_0^\pi x^4\cos(nx)dx = \frac{2}{\pi}\left(-\frac{4}{n}\int_0^\pi x^3\sin(nx)dx\right) =$$

$$= -\frac{8}{n\pi}\left(\left[-\frac{x^3\cos(nx)}{n}\right]_0^\pi + \frac{3}{n}\int_0^\pi x^2\cos(nx)dx\right) =$$

$$= \frac{8\pi^2(-1)^n}{n^2} - \frac{24}{n^2\pi}\left(\frac{2\pi(-1)^n}{n^2}\right) = \frac{8\pi^2(-1)^n}{n^2} - \frac{48(-1)^n}{n^4} \ ;$$

$$b_n = 0 \ \ \forall n \ .$$

Dunque

$$x^4 = \frac{\pi^4}{5} + 8\pi^2\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos(nx) - 48\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^4}\cos(nx).$$

Se consideriamo l'uguaglianza in $x = \pi$ otteniamo che

$$\pi^4 = \frac{\pi^4}{5} + 8\pi^2\left(\frac{\pi^2}{6}\right) - 48\sum_{n=1}^{+\infty}\frac{1}{n^4} \Longrightarrow \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

6. Calcoliamo i coefficienti di Fourier di $f(x)$:

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^0 -dx + \int_0^\pi dx\right) = \frac{1}{\pi}\left([-x]_{-\pi}^0 + [x]_0^\pi\right) = \frac{1}{\pi}(-\pi+\pi) = 0 \ ;$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^0 -\cos(nx)dx + \int_0^\pi \cos(nx)dx\right) = 0 \ ;$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^0 -\sin(nx)dx + \int_0^\pi \sin(nx)dx\right) = \frac{1}{\pi}\left(\left[\frac{\cos(nx)}{n}\right]_{-\pi}^0 - \left[\frac{\cos(nx)}{n}\right]_0^\pi\right) =$$

$$= \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} \frac{4}{(2k+1)\pi} & \text{se } \ n = 2k+1 \\ 0 & \text{se } \ n = 2k \end{cases} .$$

Dunque

$$f(x) = \frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\sin[(2k+1)x]}{2k+1}.$$

Se consideriamo l'uguaglianza in $x = \frac{\pi}{2}$ otteniamo che

$$1 = \frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1} \Longrightarrow \sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$