Asymptotic invariants of line bundles, semiampleness and finite generation

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As the history of algebraic geometry shows, a mainstream problem has always been the classification of algebraic varieties. A projective algebraic variety is, in the first place, a subset of the projective space $\mathbb{P}^n$ defined as the locus of zeros of an homogeneous prime ideal in $\mathbb{C}[x_0, \ldots, x_n]$.
One of the most common ways to classify varieties is through studying algebraic objects associated to each variety.
Maybe the simplest example of this kind is given by the *coordinate ring*. If $X \subseteq \mathbb{P}^n$ is the projective variety defined by the ideal $I(X)$, the coordinate ring of X is $S(X) = \mathbb{C}[x_0, \ldots, x_n]/I(X)$. Considering the elements of degree zero in the quotient field of the coordinate ring we also have the *function field of* X, or *field of rational functions on* X, denoted by $K(X)$.
In the ring $S(X)$ are contained basic geometrical properties of X such as its dimension and its degree. In particular, denoting by H a hyperplane section on X and by $H^0(X, mH)$ the group of rational functions on X having, on every subvariety of codimension one, poles of degree bounded by the multiplicity of mH, we see that the coordinate ring is strictly related to

$$R(X, H) = \bigoplus_{m \geq 0} H^0(X, mH),$$

and in fact they have the same Hilbert polynomial.
Since H is a divisor (see below), looking for other geometrical properties of X, this suggests us the idea of considering an arbitrary divisor on X.
If X is a smooth variety, we can view a divisor D on X as a finite $\mathbb{Z}$-linear combination of irreducible subvarieties of X of codimension 1:

$$D = \sum n_i Y_i.$$

In particular D is effective if $n_i \geq 0$ for all i.
We define the *graded ring associated* to D as

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, mD),$$

where the structure of graded ring is given by the multiplication map

$$H^0(X, aD) \otimes H^0(X, bD) \longrightarrow H^0(X, (a+b)D)$$
$$s \otimes t \longmapsto s \cdot t.$$

One of the purposes of this thesis is to study the relation between the geometrical characteristics of the couple (X, D) and the algebraic properties of the ring $R(X, D)$. For example, if X is an n-dimensional variety and K_X is the canonical divisor on X, that is the divisor locally defined by the nth exterior power of the module of differential forms, we have that X is of general type if and only if, asymptotically, $dim\ R(X, K_X)_m = O(m^n)$.
The most significant property that $R(X, D)$ might have is the finite generation as a $\mathbb{C}$-algebra and the divisor D itself is said to be finitely generated if its graded ring $R(X, D)$ is such.
In particular the finite generation of $R(X, K_X)$ has been considered for a long time, especially because it holds a particular role in the theory of minimal models. Given an algebraic variety X the idea is that a minimal model of X is the "simplest" variety birational to X. The existence of minimal models for surfaces was shown by the Italian school (Castelnuovo, Enriques, etc.) in the 30's. Passing to dimension n things get more complicated. However an important very recent theorem of Birkar, Cascini, Hacon and McKernan proves the existence of minimal models for smooth n-dimensional varieties of general type and the finite generation of the graded ring $R(X, K_X)$ (see [BCHM07]). Moreover the importance of considering arbitrary divisors and not just the canonical one is given by the tight link between the finite generation of $R(X, K_X)$ and the properties of rings of kind $R(X, K_X + \Delta)$, for suitable divisors Δ.
Another remarkable historical reason that led to consider the rings $R(X, D)$ is the connection between their finite generation and Hilbert's fourteenth problem. We can formulate Hilbert's 14th problem in these terms: given a field k, a polynomial algebra $k[x_1, \ldots, x_n]$ and a subfield of the rational functions in n-variables over k, $K \subseteq k(x_1, \ldots, x_n)$, is the ring $K \cap k[x_1, \ldots, x_n]$ finitely generated as a k-algebra?
The answer is negative and, in a famous counter-example given in 1959, Nagata considers the surface X obtained by blowing up $\mathbb{P}^2$ in r general points $p_1, \ldots, p_r$ and finds a suitable divisor D on X such that $R(X, D)$ is not finitely generated and it is of kind $K \cap k[x_1, \ldots, x_n]$.

We now come to the description of the work of this thesis.
We study how the notion of finite generation can be related to different geometrical properties of the couple (X, D). Our approach will often be

asymptotic, in the sense that we study a divisor looking at all its sufficiently large multiples and the graded ring $R(X, D)$ helps us in this sense, since, by definition, it depends on all the multiples of D.
A first step consists in analyzing the geometrical behaviour of (X, D) and observing how it reflects in the graded ring associated to D. With this goal in mind we first study divisors.
While in the first chapter we essentially follow [Har77] (except for the definition of intersection numbers, for whom we refer to [Kle66]), from the second one our point of view is that of [Laz04]. The reader will find in these books lacking proofs of some results presented and more explanations.
We begin by recalling some basic notions such as linear series, base loci and base ideals: The *linear series* associated to the divisor D is the set of all the effective divisors linearly equivalent to D and it is denoted by $|D|$.
The *base locus* of $|D|$, denoted by $Bs(|D|)$, is the closed subset of X obtained as the intersection of the supports of all the divisors in $|D|$, while the *base ideal* $b(|D|)$ is the ideal defining the base locus.
Then we outline the classic theory of ample divisors before passing to the more recent one of nefness and we generalize the concept of *divisor* introducing $\mathbb{Q}$ and $\mathbb{R}$-divisors:
A divisor D on X is *ample* if it admits a multiple mD that is a hyperplane section of X in a suitable projective embedding, while it is *nef* if it has "non-negative intersection" with every irreducible curve on X. $\mathbb{Q}$ and $\mathbb{R}$-divisors are $\mathbb{Q}$ (resp. $\mathbb{R}$)-linear combinations of irreducible subvarieties of codimension 1.
In the third chapter we pass to the description of some important properties and notions, such as bigness, mainly developed in the last 30 years, closely linked to base loci and projective morphisms defined by divisors.
We begin by defining the stable base locus:

Definition 1. Let X be a variety and let D be a divisor on X.
The *stable base locus* of D is the closed subset

$$\mathbb{B}(D) = \bigcap_{m \geq 1} Bs(|mD|).$$

Then we pass to the definition of semiampleness:

Definition 2. Let X be a variety. A divisor D on X is *semiample* if there exists an integer $m > 0$ such that $Bs(|mD|) = \emptyset$.

Finally we introduce the notion of bigness:

Definition 3. Let X be a variety. A divisor D on X is *big* if there exists an ample divisor A, together with an effective divisor F, such that $mD = A+F$ for some $m \in \mathbb{N}$.

Note that if D is semiample then it is finitely generated and it is easy to check that every divisor on a curve is finitely generated. On the contrary, when passing to surfaces, a famous example of Zariski provides a big and nef divisor D whose graded ring is not finitely generated. Already in this example one can observe how the fundamental reason not allowing finite generation is that the multiplicity of a curve in the base locus of $|mD|$ is bounded. This drives us to the following definition:

Definition 4. Let X be a variety, let $|D|$ be the linear series associated to the Cartier divisor D and let $x \in X$. We define the *multiplicity* of $|D|$ at x, denoted by $mult_x|D|$, as the multiplicity at x of a general divisor in $|D|$. Equivalently

$$mult_x|D| = \min_{D' \in |D|} \{mult_x D'\}.$$

Using this definition we prove the following original characterization of semiampleness in terms of finite generation and boundedness of the multiplicity at every point:

Theorem 5. Let X be a normal variety and let D be a Cartier divisor on X.
Then D is semiample if and only if D satisfies the following three conditions:

1. D is finitely generated.
2. There exists an integer $m > 0$ such that $|mD| \neq \emptyset$.
3. There exists a constant $C > 0$ such that for all $m > 0$, with $|mD| \neq \emptyset$, and for all $x \in X$, we have

$$mult_x|mD| \leq C.$$

Then we note that the multiplicity of a linear series is nothing but a particular discrete valuation on the function field $K(X)$. Hence the idea developed in the fifth chapter is to generalize the previous result in terms of valuations, along the lines of recent works of Ein, Lazarsfeld, Mustaţă, Nakamaye and Popa:

Definition 6. Let K be a field. A *discrete valuation* on K is an application $v : K^* \to \mathbb{Z}$ such that

- $v(xy) = v(x) + v(y) \quad \forall x, y \in K^*$,
- $v(x+y) \geq \min\{v(x), v(y)\} \quad \forall x, y \in K^*$ such that $x + y \neq 0$.

If $k \subseteq K$ is a subfield, a discrete valuation on K/k is a discrete valuation v on K such that $v(x) = 0$ for all $x \in k$.
The ring

$$R_v = \{f \in K^* : v(f) \geq 0\} \cup \{0\}$$

is called the *valuation ring* of v.
Moreover, given a discrete valuation v on $K(X)/\mathbb{C}$, the *center* of v, denoted by Z_v, is the subvariety of X defined by the unique maximal ideal of R_v, that is the ideal made up by the rational functions of positive valuation.

In these matters a relevant role is played by the restricted base locus:

Definition 7. Let X be a normal variety and let D be an $\mathbb{R}$-divisor on X. The *restricted base locus* of D is

$$\mathbb{B}_-(D) = \bigcup_A \mathbb{B}(D + A),$$

where the union is taken over all ample $\mathbb{R}$-divisors A such that $D + A$ is a $\mathbb{Q}$-divisor.

Then, following [ELMNP06], we adopt the language of valuations to generalize the concept of multiplicity at a point of a linear series:

Definition 8. Let X be a variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ with valuation ring R_v and center $\xi \in X$.
If D is a Cartier divisor on X, with $|D| \neq \emptyset$, we put

$$v(|D|) = v((b(|D|)_\xi)_{R_v}),$$

where $b(|D|)$ is the base ideal of D and $((b(|D|)_\xi)_{R_v}$ is the ideal generated by the stalk $(b(|D|)_\xi$ in the ring R_v.

In analogy with Theorem 5 we define a v-bounded divisor, that is a divisor D such that the center of the valuation v is asymptotically contained in the base locus of mD with bounded multiplicity. In other words:

Definition 9. Let X be a normal variety and let v be a discrete valuation on $K(X)/\mathbb{C}$. If D is a Cartier divisor on X with Iitaka dimension $k(X, D) \geq 0$, that is there exists an integer $m > 0$ such that $|mD| \neq \emptyset$, then we say that D is *v-bounded* if there is a constant $C > 0$ such that

$$v(|pD|) \leq C$$

for every $p > 0$ such that $|pD| \neq \emptyset$.

Moreover we generalize the concept of semiample divisor with the weaker one of v-semiample, that is a divisor not containing the center of v in its stable base locus:

Definition 10. Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$, having center $Z_v \subseteq X$, and let D be a Cartier divisor on X with $k(X, D) \geq 0$.
Then D is *v-semiample* if there exists a positive integer l_0 such that $|l_0 D| \neq \emptyset$ and $v(|l_0 D|) = 0$, or equivalently if $Z_v \nsubseteq \mathbb{B}(D)$.

These last two notions are not equivalent but they are linked by the following result:

Theorem 11. *Let X be a normal variety with function field $K = K(X)$, let v be a discrete valuation on $K/\mathbb{C}$ and let D be a Cartier divisor on X with $k(X, D) \geq 0$.*

- *If D is v-semiample, then D is v-bounded.*
- *If D is finitely generated and v-bounded, then D is v-semiample.*

Actually for the second statement it is not necessary to take D v-bounded, but it suffices the hypothesis of "v-sublinearity", that is $v(|mD|)$ can go to infinity but in a "sublinear" way with respect to m.
As a corollary, considering all the discrete valuations on $K(X)/\mathbb{C}$ together and noting that D is semiample if and only if D is v-semiample for every valuation v, we can generalize Theorem 5 with a different characterization of semiampleness involving v-boundedness:

Theorem 12. *Let X be a normal variety with function field $K = K(X)$ and let D be a Cartier divisor on X with $k(X, D) \geq 0$.*
Then the following statements are equivalent:

1. *D is semiample*
2. *D is finitely generated and there exists a constant $C > 0$ such that*

$$mult_x |mD| \leq C$$

for every $m > 0$ such that $|mD| \neq \emptyset$, for every $x \in X$.

3. *D is finitely generated and, for every discrete valuation v on $K/\mathbb{C}$, D is v-bounded.*

On the other hand we extend the notion of asymptotic order of vanishing of a divisor along a valuation v, defined in [ELMNP06] only in the big case, to non-big divisors:

Definition 13. Let X be a normal variety with function field $K = K(X)$, let D be a Cartier divisor on X with $k(X, D) \geq 0$ and let v be a discrete valuation on $K/\mathbb{C}$.
We define the *exponent* of D as

$$e(D) = \text{g.c.d.}\{m \in \mathbb{N} \ : \ |mD| \neq \emptyset\}.$$

If $e = e(D)$ is the exponent of D, the *asymptotic order of vanishing* of D along v is

$$v(||D||) = \lim_{m\to\infty} \frac{v(|meD|)}{me}.$$

We can simply extend this definition to $\mathbb{Q}$-divisors. In particular we fix as the right work environment the set of $\mathbb{Q}$-linearly effective $\mathbb{Q}$-divisors:

Definition 14. Let X be a variety and let D be a $\mathbb{Q}$-divisor on X.
A $\mathbb{Q}$-divisor F is $\mathbb{Q}$-*linearly equivalent* to D if there exists an integer $r > 0$ such that rD and rF are linearly equivalent integral divisors.
We define the $\mathbb{Q}$-*linear series* $|D|_{\mathbb{Q}}$ as the set of all the effective $\mathbb{Q}$-divisors that are $\mathbb{Q}$-linearly equivalent to D.
D is $\mathbb{Q}$-*linearly-effective* if $|D|_{\mathbb{Q}} \neq \emptyset$.

Finally, we present the main result in [ELMNP06, §2] about the asymptotic order of vanishing of big $\mathbb{Q}$-divisors.

Theorem 15. Let X be a smooth variety with function field $K = K(X)$ and let v be a discrete valuation on $K/\mathbb{C}$, having center Z_v on X.
If D is a big $\mathbb{Q}$-divisor on X, then the following conditions are equivalent:

1. D is v-bounded;
2. $v(||D||) = 0$;
3. $Z_v \not\subseteq \mathbb{B}_-(D)$.

One of the aims of this thesis is, using Definition 13, to study which implications of the above theorem are still true in the non-big case. In particular, if D is a $\mathbb{Q}$-linearly effective $\mathbb{Q}$-divisor, we first observe that

(i) D is v-bounded $\Rightarrow v(||D||) = 0$;

(ii) $v(||D||) = 0 \Rightarrow Z_v \not\subseteq \mathbb{B}_-(D)$

are still true, using the same argument of [ELMNP06].
On the other hand the reverse implication of **(ii)** does not hold and we provide a counter-example in Section 5.6.
Finally we outline the questions that, at the moment, remain unsolved:

Open questions:

- We do not know whether D is v-bounded whenever the asymptotic order of vanishing of D along v is zero. However it is true if D is a divisor on a curve or a normal surface.

- It remains an open problem how we can weaken the hypothesis of the second statement of Theorem 11 in order to have the viceversa.

Remark 16. Besides Theorem 5, the original results in this thesis are Theorem 11 and Theorem 12 (which are inspired by Theorem 5) and the example in Section 5.6.

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