Formal Birkhoff normal form for water waves problem with infinite depth

Andrea Mignucci

In this thesis we will study the water waves, namely the motions of oceans, seas, rivers and canals.

One can consider the two-dimensional case, namely the water waves of a river or a canal (when width is negligible with respect to length), or the three-dimensional case, that is the ocean waves.

The physical problems correlated to this subject range from the off-shore wave motion (for example the phenomenon of the "tsunami", namely the solitary wave) to the periodic breaking down of the waves near the sea-side, the flood waves in rivers, the motion of a ship in a seaway, to mention just a few.

As it is natural, the water waves problem interested and still interests a remarkable number of mathematicians (as well as physicists and engineers), beginning from Euler and Bernoulli, discoverers of two basic hydrodynamics laws, and continuing with Lagrange, Cauchy and Poisson.

In the XIX century the British school gave notable contributions especially with Stokes, Kelvin and Rayleigh. Afterwards, Poincaré studied the equilibrium of rotating and gravitating liquids.

In the last century another fundamental result in the field, namely the existence of periodic progressing waves, was obtained by Nekrassov ([N21], [N]) in 1921 and (independently) by Levi-Civita ([L25]) in 1925. The existence of solitary waves was finally proved by Lavrentieff ([La47]) and Friedrichs and Hyers ([FH54]).

The interest on the water waves problem, and, in general on Fluid Mechanics of so many considerable mathematicians is an evident sign of the importance of this problem from a purely mathematical point of view. Moreover, one of the five "Millennium Problems" (namely, unsolved problems whose solution is recompensed with a prize amounting to one million of dollars) concerns the existence and oneness of the solution of the Navier-Stokes equation, which describes the motion of a fluid in presence of viscosity.

On the other hand, it is clear that a greater understanding of this subject would be very useful for other sciences for the study of problems related to the water waves phenomenon: the tsunami in geology, the construction of the dikes in building engineering, the seaways in aeronautics, and many others.

Let us now pass to a more mathematical description of the water waves problem.

By physical observations, we will make the assumptions that the water is an incompressible fluid (its volume does not change in the time), with no viscosity (perfect fluid), with no vortex during the motion (irrotational regime) and that is subjected to the gravitational force only.

Given a perfect fluid of volume V, with pressure $p = p(x,t)$, density $\rho = \rho(x,t)$ and velocity field $u = u(x,t)$, subjected to conservative external forces, then, the fundamental fluid-dynamics law, namely the Euler equation, is given by

$$\rho(\partial_t u + (u \cdot \nabla)u) = -\nabla p - \rho \nabla v,$$

where v is such that the external force $f = -\int_V (\rho \nabla v dx)$. Since we will suppose that the only external force is the gravitational one, we obtain that $\nabla v = \vec{g}$, where $\vec{g}$ just denotes the gravity acceleration.

If the perfect fluid is also in irrotational regime, that is, if $\operatorname{rot} u = 0$, there will be Φ (called stream function) such that $u = \nabla\Phi$: therefore making use of the Euler equation, we obtain another basic identity of the fluid-dynamics, known as Bernoulli equation:

$$\rho(\partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + v) + p = 0.$$

Since the incompressibility hypothesis is equivalent to the condition $\operatorname{div} u = 0$, the stream function of an irrotational and incompressible fluid will satisfy the Laplace equation, namely $\triangle\Phi = 0$.

In the analysis of the water waves motion, generally, it is opportune supposing these two conditions:

(i) Kinematic condition: if a particle is on the free surface of the water, it will remain forever on this surface; the same if it is on the bottom.

(ii) Dynamic condition: the pressure on the free surface is equal to the atmospheric one $p_0 \equiv$const.

If one assumes that the free surface is the graph of a function $\eta(x,t)$ (no surf waves), and the same for the ground with a function $\beta(x)$, then, one can prove that the equations which govern the motion of the water waves are

$$\begin{cases} \triangle\Phi = 0 & \text{in } \Omega_\eta := \{\beta(x) < y < \eta(x,t)\} \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gy = 0 & \text{on } y = \eta(x,t) \\ \nabla\Phi \cdot N = 0 & \text{on } y = \beta(x) \\ \partial_t\eta = \partial_y\Phi - \partial_x\eta \cdot \partial_x\Phi & \text{on } y = \eta(x,t), \end{cases} \tag{1}$$

where N denotes the external normal to the bottom, and $g = |\vec{g}|$.

The unknown quantities of this system are the wave profile η and the stream function Φ.

The system (1) consists of the Laplace equation in the open (unknown!) domain Ω_η and of three boundary conditions.

The first boundary condition derives from the dynamic condition and the Bernoulli equation, the other two are consequence of the kinematic condition.

There are two reasons for which (1) is very difficult to be solved:

1) It is a "free boundary problem", namely, the boundary $\partial\Omega_\eta$ is an unknown quantity (changing in time);

2) The second and the last equation are nonlinear.

In this thesis, we will study the problem with depth $\beta(x) = -h$, where h is a positive constant, and, mainly, the case with unbounded ground ($h = +\infty$); the equations which describe the last one are

$$\begin{cases} \triangle\Phi = 0 & \text{in } \{-\infty < y < \eta(x,t)\} \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gy = 0 & \text{on } y = \eta(x,t) \\ \partial_y\Phi \to 0 & y \to -\infty \\ \partial_t\eta = \partial_y\Phi - \partial_x\eta \cdot \partial_x\Phi & \text{on } y = \eta(x,t). \end{cases} \tag{2}$$

Moreover, we will suppose periodic conditions with period 2π, and thus we will study the problem in the domain $[0, 2\pi] \times \{-h < y < \eta(x,t)\}$ $(0 \le h \le +\infty)$.

V. E. Zakharov, quite recently (1968), discovered that the water waves problem, with infinite depth or not, can be written in Hamiltonian form (see [Z68]), with canonical variables (η, ξ), where this last quantity is defined by

$$\xi(x,t) := \Phi(x, \eta(x,t), t),$$

and the Hamiltonian is given by the mechanical energy

$$H = \frac{1}{2}\int(|\nabla\Phi|^2 + g\eta^2).$$

Many problems of physical significance are described by partial differential equations; some of these (the nonlinear wave equation, the nonlinear Schrödinger equation, the Korteweg–de Vries equation (KdV) and the Euler equation) can be posed as Hamiltonian systems. The PDE formulation and the Hamiltonian one give rise to two complementary ways to analyze the same phenomenon: the first focus mainly the attention on the existence of solutions for the initial value problem, while the second poses the question of stability of solutions for all times.

Main contribution

We note that $(\eta, \xi) = (0, 0)$ is an elliptic equilibrium point for the Hamiltonian H. Hence, as for the finite dimensional case, it is natural to consider the Birkhoff normal form: by a series of canonical transformations one would remove all the *nonresonant terms*, namely all those terms that, introducing action-angle variables, do not depend on the angle.

We remark that for the water waves problem *the convergence of the above procedure (in some suitable Banach space) is an open problem.*

Actually it was proved in [C96] and [Ba05] that the remainder is small when considered as an operator from a Sobolev space to a Sobolev space of much small order. As a consequence the remainder is a small, but very singular perturbation.

So we focus on the formal aspects of these transformations, without taking care of questions of convergence.

Our purpose is to write the Birkhoff normal form up to the fourth order for H, proving that it is integrable through the use of suitable action-angle variables.

We will follow the work of W. Craig and P. A. Worfolk [CW95], who, in their turn, follow [DZ94].

The most important part of our work is the analysis and the removing of some particular resonances, called Benjamin-Feir resonances.

Consequence of this "natural elimination" (not just immediate), is the integrability of the resulting truncated system of equations.

In section 4.6, *we give the complete (and cumbersome!) calculations.* This evaluation is not explicitly written in [CW95] and in [DZ94], where, using Maple or Mathematica, it is only stated the final form of the fourth order.

We conclude with a final remark. Due to the unexpected cancellations in the resonant coefficients which result in an integrable fourth order Birkhoff normal form, it is natural to conjecture (as Dyachenko and Zakharov do) that resonant term cancellations occur for all order normal forms, which would be strong evidence for the integrability of the full deep water wave problem. This conjecture was negative answered by Craig &Worfolk in [CW95], where they exhibit certain fifth order resonant terms whose coefficients after the fifth order normal forms transformations do not vanish. The conclusion is that the degree five truncation of the water wave Hamiltonian in normal form is not integrable, at least not with respect to the natural action-angle variables adapted to the quadratic part of the Hamiltonian.

Scheme of the thesis

We will divide our work in four chapters, with the addition of an appendix about the Birkhoff normal form.

CHAPTER I.
We will introduce the Dirichlet-Neumann operator $G(\eta)[\xi]$, defined by

$$G(\eta)[\xi] := \nabla\Phi(x, \eta(x,t), t) \cdot N(x,t)\sqrt{1 + (\partial_x \eta(x,t))^2},$$

where Φ denotes the only solution of the elliptic problem with mixed conditions on the boundary(of Dirichlet and Neumann)

$$\begin{cases} \triangle\Phi = 0 & \text{in } S_\eta^h \\ \nabla\Phi \cdot N = 0 & \text{on } y = -h \\ \Phi = \xi & \text{on } y = \eta(x,t); \end{cases}$$

then, we will write the mechanical energy in this way

$$H(\eta, \xi) = \frac{1}{2}\int_0^{2\pi} (\xi G(\eta)[\xi] + g\eta^2)dx, \tag{3}$$

and we will prove the Hamiltonian formulation for the systems (1) (with flat bottom, namely $\beta(x) \equiv -h$) and (2).

CHAPTER II.
Our goal is to write the first few terms of the series expansion of H.

Since the Dirichlet-Neumann operator is analytic in $\{\eta : |\eta|_{C^1} < R_0, \ \ |\eta|_{C^{s+1}} < \infty\}$ as a mapping in the space of bounded linear operator from H^{s+1} to H^s (see [CSSc97]), we know that H has a Taylor series expansion about the equilibrium point $(\eta, \xi) = (0, 0)$.

By the recursively formula of $G_l(\eta)$ for the water waves with depth $-h$ (where $G_l(\eta)$ is the homogenous term of degree l in η in the expansion of $G(\eta)$), we achieve that the terms G_0, G_1, G_2, of the expansion of the Dirichlet-Neumann operator with infinite depth are given by

$$\begin{aligned} &G_0[\xi] = (|k|\xi_k)_k; \\ &G_1(\eta)[\xi] = \frac{1}{\sqrt{2\pi}}\left(k\sum_l \eta_l \xi_{k-l}(k-l) - |k|\sum_l \eta_l \xi_{k-l}|k-l|\right)_k; \\ &G_2(\eta)[\xi] = -\frac{1}{4\pi}\Bigg(|k|\sum_m \eta_m \sum_l \eta_l \xi_{k-l-m}(k-l-m)^2 + k^2\sum_m \eta_m \sum_l \eta_l \xi_{k-l-m}|k-l-m| \\ &\qquad\qquad - 2|k|\sum_m \eta_m |k-m| \sum_l \eta_l \xi_{k-l-m}|k-l-m|\Bigg)_k, \end{aligned}$$

where $(f_k)_k$ denotes the succession of the Fourier coefficients of f (with $f \in L^2((0, 2\pi))$).

Therefore, the expansion of H is given by

$$H(\eta, \xi) = H_2(\eta, \xi) + H_3(\eta, \xi) + H_4(\eta, \xi) + R_5(\eta, \xi),$$

where

$$
\begin{aligned}
H_2(\eta,\xi) =& \frac{1}{2}\sum_k (|k|\xi_k\xi_{-k} + g\eta_k\eta_{-k});\\
H_3(\eta,\xi) =& \frac{1}{2\sqrt{2\pi}}\sum_{k+l+m=0}(-kl-|kl|)\xi_k\xi_l\eta_m;\\
H_4(\eta,\xi) =& -\frac{1}{8\pi}\sum_{k+l+m+n=0}|kl|(|k|+|l|-2|k+m|)\xi_k\xi_l\eta_m\eta_n,
\end{aligned}
$$

CHAPTER III.
In this chapter, we will introduce and describe tools which will be useful in the next chapter. First of all we will prove that we can restrict our attention to the subspace $\{(\eta_0,\xi_0) = (0,0)\}$; in the second section we will introduce complex coordinates (z,w) and we will write the Hamiltonian H in these new variables:

$$
H(z,w) = H_2(z,w) + H_3(z,w) + H_4(z,w) + R_5(z,w)\,,
$$

where H_2, H_3, H_4 are given by

$$
\begin{aligned}
H_2(z,w) =& \frac{1}{2}\sum_{k>0}\sqrt{g|k|}(z_k\bar{z}_k + w_k\bar{w}_k);\\
H_3(z,w) =& \frac{\sqrt[4]{g}}{32\sqrt{\pi}}\sum_{k,l>0}\sqrt[4]{k^3l^3(k+l)}(\alpha_k\alpha_l\beta_{-k-l} + \alpha_{-k}\alpha_{-l}\beta_{k+l});\\
H_4(z,w) =& \frac{1}{512\pi}\sum_{k,l,m}\sqrt[4]{|k^3l^3(m-k)(m-l)|}(|k|+|l|-2|m|)\alpha_k\alpha_{-l}\beta_{m-k}\beta_{l-m},
\end{aligned}
$$

with $\alpha_k := z_k - \bar{z}_k + i(w_k - \bar{w}_k),\ \beta_k := z_k + \bar{z}_k + i(w_k + \bar{w}_k)$.

Then, we will define the space of formal series, and we will write the Poisson bracket; finally, we will discuss the Lie's method used to write the Birkhoff normal form of H.

CHAPTER IV.
This is the main part of our work.

We start the chapter with the Birkhoff normal form in the space of the formal series, showing that through the Lie's method we are able to eliminate the nonresonant terms whose degree is equal or lower than l, for all $l \geq 3$.

Then, we will prove that there are no resonant terms in H_3: indeed they should have wave numbers k, l, m such that

$$\begin{cases} k \pm l \pm m = 0 \\ \sqrt{k} \pm \sqrt{l} \pm \sqrt{m} = 0, \end{cases}$$

but it is easy to see that this system has not positive integer solutions.

Therefore, we will eliminate all the cubic terms from H: if $\mathrm{ad}_{H_2}(\cdot) := [H_2, \cdot\,]$ ($[\,\cdot\,,\cdot\,]$ denotes the Poisson bracket), then the Hamiltonian in the new canonical variables will be

$$\widehat{H} = \widehat{H}_2 + \widehat{H}_4 + \widehat{R}_5,$$

where

$$\widehat{H}_2 = H_2, \qquad \widehat{H}_4 = H_4 + \frac{1}{2}[K_3, H_3],$$

and $K_3 := \mathrm{ad}_{H_2}^{-1}(H_3)$.

Since we want to find all the resonant quartic terms of $\widehat{H}$, we will compute explicitly K_3 and $[K_3, H_3]$ in the third and fourth section respectively.

After these calculations, we will study the nature of the wave numbers of the resonant quartic terms which occur in $\widehat{H}$: these numbers are the solutions of

$$\begin{cases} k \pm l \pm m \pm n = 0 \\ \sqrt{k} \pm \sqrt{l} \pm \sqrt{m} \pm \sqrt{n} = 0, \end{cases} \tag{4}$$

where $(k, l, m, n) \in \mathbb{N}^4$, and $k \geq l \geq m \geq n$. The system (4) has two sets of solutions: the trivial one is parameterized by

$$\begin{cases} k = p \\ l = p \\ m = q \\ n = q, \end{cases}$$

with $p, q \in \mathbb{N}$, $p \geq q$, while, the non trivial one is described by

$$\begin{cases} k = h\dfrac{1}{q^2}(p^2 + q^2 + pq)^2 \\ \\ l = \dfrac{p^2}{q^2}(p+q)^2 \\ \\ m = (p+q)^2 \\ \\ n = p^2, \end{cases} \tag{5}$$

where $h, p, q \in \mathbb{N}$, $q|p^2, q \leq p$.

The terms of the first kind are called *generic resonances*, those of the second kind *Benjamin-Feir resonances.*

Then, we will make all the calculation, "reproving" the following theorem stated in [DZ94] and [CW95], about the non-generic resonances.

Theorem 1 *The normal form transformation inducted by K_3 removes all the Benjamin-Feir resonances, namely, $\widehat{H}_4$ does not contain nonzero monomials with wave numbers satisfying (5).*

Thus, after a new canonical transformation, able to remove the nonresonant term of fourth degree, we can write the Birkhoff normal form to the fourth order:

$$\widehat{\widehat{H}} = \widehat{\widehat{H}}_2 + \widehat{\widehat{H}}_4 + \widehat{\widehat{R}}_5$$

where

$$\widehat{\widehat{H}}_2 = \widehat{H}_2 = H_2 = \sum_{k>0} \sqrt{gk} I_1(k), \qquad \widehat{\widehat{H}}_4 = \sum_{k>0} \frac{k^3}{8\pi}(3I_2^2(k) - I_1^2(k)),$$

and the action variables $I_1(k), I_2(k)$ are defined by

$$I_1(k) := \frac{z_k \bar{z}_k + w_k \bar{w}_k}{2}, \qquad I_2(k) := \frac{z_k \bar{w}_k - \bar{z}_k w_k}{2j}.$$

As a consequence, the truncation of $\widehat{\widehat{H}}$ to degree four is integrable.

References

[AP] A. Ambrosetti, G. Prodi: *A Primer of Nonlinear Analysis*, Cambridge University Press, 1993.

[Ba05] D. Bambusi: *Galerkin averaging method and Poincaré normal form for some quasilinear PDEs*, Mathematics Subject Classification (2000): 35B20, 37K55, 37L10, 2005.

[B] H. Brezis: *Analyse Functionnelle-Théorie et applications*, Masson Editeur, Paris, 1983.

[C96] W. Craig: *Birkhoff normal forms for water waves*, In Mathematical problems in the theory of water waves (Luminy, 1995), volume 200 of Contemp. Math., pages 57-74, Amer. Math. Soc., Providence, RI, 1996.

[CS92] W. Craig, C. Sulem: *Numerical simulation of gravity waves*, journal of computational physics 108, (1993), 73-83.

[CSSc97] W. Craig, U. Schanz, C. Sulem: *The modulational regime of three-dimensional water waves and the Davey-Stewartson system*, Annales de l'I.H.P., section C, tome 14, no.5 (1997), p.615-667.

[CW95] W. Craig, P. A. Worfolk: *An integrable normal form for water waves in infinite depth*, Physica D 84 (1995) 513-531.

[DZ94] A. I. Dyachenko, V. E. Zakharov: *Is free surface hydrodynamics an integrable system?*, Physics Letters A 190 (1994) 144-148.

[FH54] K.O. Friedrichs, D. H. Hyers: *The existence of solitary waves*, Comm. Pure Appl. Math. 7, 1954, 517–550.

[GT] D. Gilbarg, N. Trudinger: *Ellipptic partial differential equations of second order*, Springer (1977.)

[L25] T. Levi-Civita: *Détermination rigoureuse des ondes permanentes d'ampleur finie.* Mathematische Annalen, Vol. 93, 1925, 264-314.

[La47] M. Lavrentieff: *Sur la théorie exacte des ondes longues*, Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1947, no. 8, 13–69.

[N] A. I. Nekrassov: *The exact theory of steady waves on the surface of a heavy fluid*, Izdat. Akad. Nauk, SSSR, Moscow, 1951.

[N21] A. I. Nekrassov: *On steady waves*, Izv. Ivanovo-Voznesensk. Politekhn. In-ta, 3, 1921.

[St] J. J. Stoker: *Water Waves*, Wiley Classics Library, John Wiley & Sons Inc., New York 1992, the mathematical theory with applications, reprint of the 1957 original, a Wiley-Interscience Publication. MR MR1153414 (92m:76029)

[Wi36] J. Williamson: *On the algebraic problem concerning the normal forms of linear dynamical systems*, Am. J. Math. 58, No. 1, (1936), 141–163.

[Z68] V. E. Zakharov, *Stabiltiy of periodic waves of infinite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys. 9, (1968), 190-194.