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Computing the free commutative comonoid

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Riassunto

L'argomento principale della tesi è la descrizione di una procedura per costruire un comonoide commutativo libero in una categoria simmetrica monoidale.

La prima parte consiste in una rapida introduzione alla teoria delle categorie, che intende fornire gli strumenti necessari per comprendere il resto. La seconda parte riporta le regole della logica lineare (classica e intuizionista) e una descrizione delle proprietà delle categorie che la interpretano.

La terza parte descrive nei particolari la procedura, che consiste nell'identificare gli oggetti della categoria simmetrica monoidale con dei funtori di una seconda categoria simmetrica monoidale opportunamente costruita, e poi calcolare il comonoide commutativo libero generato da un oggetto in termini di estensioni di Kan. Queste estensioni di Kan, in seguito, assumono un significato pratico perché possono essere identificate con oggetti puntati e limiti sequenziali, che è possibile costruire nella maggior parte dei modelli categorici della logica lineare.

Infine, l'ultima parte presenta due esempi della costruzione, il primo nella categoria delle relazioni e il secondo in quella degli spazi coerenti.

Résumé

Le sujet principal du mémoire est la description d'une procédure pour construire un comonoïde commutatif libre dans une catégorie symétrique monoïdale.

La première partie est une introduction rapide à la théorie des catégories, qui est pensée pour donner les instruments nécessaires à la compréhension du reste. La deuxième partie expose les règles de la logique linéaire (classique et intuitionniste) et contient une description des propriétés des catégories qui l'interprètent.

La troisième partie décrit dans les détails la procédure, qui consiste à identifier les objets de la catégorie symétrique monoïdale avec des foncteurs d'une deuxième catégorie symétrique monoïdale construite à ce but, et ensuite évaluer le comonoïde commutatif libre engendré par un objet en termes des extensions de Kan. Ces extensions de Kan, par la suite, prennent un sens concret car on peut les identifier avec des objets pointés et des limites séquentielles, qu'il est possible de construire dans la plupart des modèles catégoriques de la logique linéaire.

Enfin, la dernière partie présente deux exemples de la construction, le premier dans la catégorie des relations et le deuxième dans celle des espaces de cohérence.

Summary

For centuries, mathematicians studied abstract objects and formal theories, using as tool and as vehicle of their reasoning a highly technical jargon, which is called proof.

Intuitively, a proof can be considered as a rational argument, intended to convince logical beings of the validity of an assertion. And in practice, in mathematics it was the only acceptation of the word for a long time. Even if proofs underlay (and still underlie!) mathematical reasonings, questions about their nature, their features and their intrinsic meaning did not occur until nineteenth century.

One of the first mathematician to consider proofs *per se* as a subject of inquiry was Gottlob Frege [1] in 1879, who introduced a mathematical notation which enables to translate vernacular proofs, which are the usual informal proofs, in formal proofs, which can be studied in the same way as other mathematical objects.

Nevertheless, the birth of that mathematical branch which is now called proof theory is located in the work of David Hilbert, who took an interest in Frege's work and put a proof-theoretic problem (that is, showing that arithmetic is consistent) in his list of twenty-three open problems exposed at the International Congress of Mathematicians in Paris, in 1900.

Since then, a lot of progress has been made in this field and a deep insight on the matter has been got. One of the most important achievement is provided by the work of Gerhard Gentzen [2], whose sequent calculus provided an elegant and powerful manner of formalizing proofs.

In particular, in proof theory a distinction has been made between a proof and the validity of its conclusion (which refers to the notion of model) and this led to consider logics which are different from the classical one. Indeed, when proofs are studied as objects which can be produced through a symbolic device (for instance, the sequent calculus), it becomes perfectly legitimate to alter the symbolic device in order to produce different results.

What is most interesting is that, sometimes, changing some aspects of the system is possible to obtain an environment with more desirable properties than the properties possessed by the classical one. This is, for instance, the case of linear logic.

Linear logic was introduced by Jean-Yves Girard in 1987 [8].

In sequent calculus, proofs are obtained by applying a finite number of rules, which enables to transform formulas in a formalized way [2, 9]. Rules

can be classified in different groups, one of which is formed by structural rules.

While working on coherence spaces, Girard devised the possibility of a new type of implication, the linear implication $\multimap$, which showed to enjoy good properties [9]. But including the linear implication required a series of changes, in particular in relation to structural rules.

Limiting the use of some of them (weakening and contraction), as it happens in linear logic, enables to obtain a formal system which preserves the constructive interpretation of intuitionistic logic, without losing the symmetry of classical logic.

Yet, in order to recover the full expressive power of classical reasoning, it is necessary to introduce two dual modalities, ! (pronounced "of course") and ? (pronounced "why not"), which constitute the exponential fragment of linear logic and which define a special type of formulas over which it is possible to apply weakening and contraction [9, 10].

The special attention paid to structural rules is also suggested by the remark that Gentzen's sequent calculus provides intuitionistic logic simply restricting admissible sequents to sequents with only one formula on the right side. This can also be interpreted thinking that the use of weakening and contraction rules is only admissed on the left side of a sequent.

This apparently harmless limitation provides a logic with desirable constructive features, but, as it is plain, suffering from a conspicuous asymmetry. Linear logic was developed in order to compensate for this flaw.

One of the most important features of Gentzen's sequent calculus, which still holds in linear logic and preserves its original importance, is the cut-elimination theorem, that is the possibility of transforming any proof into a proof which does not use a specific rule: the cut. This ensures a series of nice and useful properties, which are highly desirable for a logic to have [11].

The cut-elimination theorem is constructive, that means that it provides a mechanical procedure that transforms a generic proof in a cut-free proof (which is sometimes called in normal form). In order to better understand this procedure, denotational semantics studies mathematical invariants of proofs under cut-elimination, that is to say functions which associate with every proof π a mathematical object $[\pi]$, such that if π' is obtained from π by cut-elimination, then $[\pi] = [\pi']$.

The design of linear logic and of its cut-elimination procedure entails that such invariants have a categorical structure [15]. Robert Seely showed in 1989 that the multiplicative fragment of linear logic is naturally interpreted in a $*$-autonomous category, while the category requires finite products to interpret the additive fragment [18]. On the other hand, Yves Lafont pointed out that the exponential fragment can be interpreted by categories that have a free commutative comonoid $!A$ for every object A [14].

The aim of this thesis is describing a procedure to construct the free commutative comonoid $!A$ in a generic symmetric monoidal category $\mathbb{C}$, provided some additional hypothesis are satisfied. The original work is due to Melliès, Tabareau and Tasson [17].

This procedure is a generalization of the obvious procedure that consists in defining $!A$ as the infinite cartesian product:

$$\mathop{\&}_{n\in\mathbb{N}} A^n \tag{1}$$

where A^n is the equaliser of

$$\begin{tikzcd} A^{\otimes n} \arrow[r, shift left=2, "symmetry"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "symmetry"'] & A^{\otimes n} \end{tikzcd}$$

Indeed, this formula works when the infinite product commutes with the tensor product, as it happens in the relational model (but not in the category of coherence spaces, for instance).

To obtain a higher level of applicability (that is unfortunately far from being general [17]), it is necessary to introduce more refined tools.

Symmetric monoidal categories, free commutative comonoids, free pointed objects

The notion of monoidal category provides a theoretical background to consider monoid objects in categories, generalizing the well-known algebraic concept.

The formal definition is the following:

Definition 1 (Symmetric monoidal category)**.** A monoidal category is a category $\mathbb{C}$ equipped with a bifunctor $\_ \otimes \_ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and with natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\lambda_A : I \otimes A \to A$ and $\varrho_A : A \otimes I \to A$ for every $A, B, C \in \mathbb{C}$ (where I is a fixed object of $\mathbb{C}$ called unit), such that

$$
\begin{tikzcd}
& ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\alpha \otimes D"'] \arrow[dr, "\alpha"] & \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "A \otimes \alpha"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}
\tag{2}
$$

and

$$
\begin{tikzcd}
& (A \otimes I) \otimes B \arrow[dl, "\varrho \otimes B"'] \arrow[dr, "\alpha"] & \\
A \otimes B & & A \otimes (I \otimes B) \arrow[ll, "A \otimes \lambda"']
\end{tikzcd}
\tag{3}
$$

commute for every $A, B, C, D \in \mathbb{C}$.

A symmetric monoidal category is a monoidal category $\mathbb{C}$ equipped with a symmetry, that is a natural isomoprhism $\gamma_{A,B} : A \otimes B \to B \otimes A$, for every pair of objects A, B, such that $\gamma_{B,A} = \gamma_{A,B}^{-1}$ and the following diagram

$$
\begin{tikzcd}
& A \otimes (B \otimes C) \arrow[r, "\gamma"] & (B \otimes C) \otimes A \arrow[dr, "\alpha"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha"] \arrow[dr, "\gamma \otimes C"'] & & & B \otimes (C \otimes A) \\
& (B \otimes A) \otimes C \arrow[r, "\alpha"] & B \otimes (A \otimes C) \arrow[ur, "B \otimes \gamma"'] &
\end{tikzcd}
\tag{4}
$$

commutes for every $A, B, C \in \mathbb{C}$. ♦

A comonoid is simply a monoid in the dual category $\mathbb{C}^{op}$:

Definition 2 (Free commutative comonoid)**.** A comonoid (C, d, u) in a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ is given by an object C and morphisms $d : C \to C \otimes C$ and $u : C \to I$ (called multiplication and unit of the comonoid), such that the diagrams

$$
\begin{tikzcd}
 & C \arrow[dl, "d"'] \arrow[dr, "d"] & \\
C \otimes C \arrow[dr, "C \otimes d"'] & & C \otimes C \arrow[dl, "d \otimes C"] \\
 & C \otimes C \otimes C &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \arrow[dl, "d"'] \arrow[dr, "d"] \arrow[dd, "id"] & \\
C \otimes C \arrow[dr, "C \otimes u"'] & & C \otimes C \arrow[dl, "u \otimes C"] \\
 & C &
\end{tikzcd}
$$

commute (they are called associativity and unitality properties of a comonoid). The comonoid is commutative if and only if

$$
\begin{tikzcd}
C \arrow[r, "d"] \arrow[dr, "d"'] & C \otimes C \arrow[d, "\gamma_{C,C}"] \\
 & C \otimes C
\end{tikzcd}
$$

commutes.

A comonoid is freely generated by an object A if there exists a morphism $\varepsilon : C \to A$ and for every morphism $f : D \to A$, where D is a commutative comonoid, there exists a unique comonoid morphism $\widetilde{f} : D \to C$ such that

$$
\begin{tikzcd}
 & C \arrow[dd, "\varepsilon"] \\
D \arrow[ur, "\widetilde{f}"] \arrow[dr, "f"'] & \\
 & A
\end{tikzcd}
\tag{5}
$$

commutes. ♦

Theorem 1. *The category $\mathbb{C}^{co}$ denotes the subcategory of $\mathbb{C}$ which has comonoids as objects and comonoid morphisms as morphisms. It is a symmetric monoidal category, with symmetric monoidal structure inherited from the symmetric monoidal structure of $\mathbb{C}$.*

The notion of pointed object is closely related to that of comonoid: a pointed object could be thought of as a comonoid without multiplication.

Definition 3 (Free pointed object)**.** A pointed object in a monoidal category $(\mathbb{C}, \otimes, I)$ is a pair (B, u), where B is an object of $\mathbb{C}$ and $u : B \to I$ is a morphism.

Given an object $A \in \mathbb{C}$, (B, u) is the free pointed object generated by A if and only if there exists a morphism $\varepsilon_A : B \to A$ and for every other pointed object (C, v) such that there exists a morphism $g : C \to A$, there exists a unique pointed morphism $h : C \to B$ such that the following diagram

$$\begin{tikzcd} C \arrow[rr, "h"] \arrow[dr, "g"'] & & B \arrow[dl, "\varepsilon_A"] \\ & A & \end{tikzcd} \tag{6}$$

commutes. ♦

Theorem 2. *The category $\mathbb{C}_\bullet$ denotes the subcategory of $\mathbb{C}$ which has pointed objects as objects and pointed morphisms as morphisms. It is a symmetric monoidal category, with symmetric monoidal structure inherited from the symmetric monoidal structure of $\mathbb{C}$.*

Commutative comonoids and exponential modality

The idea which usually underlies a denotation is that a formula A should be associated with an "object" $[A]$ and a proof π of the sequent $\Gamma \vdash \Delta$ with a "morphism" $[\pi] : [\Gamma] \to [\Delta]$.

Yves Lafont showed that to interpret the exponential modality it is enough considering symmetric monoidal closed categories (or $*$-autonomous categories) where for every object $[A]$ there exists the free commutative comonoid $![A]$ generated by $[A]$. Obviously, if this holds the interpretation of formula $!A$ will be $![A]$.

Indeed, given a proof π of the sequent $\Gamma, !A, !A, \Delta \vdash B$ and an interpretation $[\pi]$ of this proof, the application of a contraction rule, that is, the following proof

$$\dfrac{\dfrac{\begin{matrix}\pi \\ \vdots\end{matrix}}{\Gamma, !A, !A, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\,\text{Contraction}$$

is interpreted by pre-composing $[\pi]$ with d, where d is the multiplication of the comonoid:

$$[\Gamma] \otimes ![A] \otimes [\Delta] \xrightarrow{id_{[\Gamma]} \otimes d \otimes id_{[\Delta]}} [\Gamma] \otimes ![A] \otimes ![A] \otimes [\Delta] \xrightarrow{\quad [\pi] \quad} [B]$$

Similarly, the application of a weakening rule to the proof π of the sequent $\Gamma, !A, \Delta \vdash B$, that is

$$\cfrac{\cfrac{\pi \\ \vdots}{\Gamma, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B} \text{Weakening}$$

is interpreted by pre-composing with the unit u:

$$[\Gamma] \otimes ! [A] \otimes [\Delta] \xrightarrow{id_{[\Gamma]} \otimes u \otimes id_{[\Delta]}} [\Gamma] \otimes [\Delta] \xrightarrow{\quad [\pi] \quad} [B]$$

Concerning the promotion rule, given a proof π of the sequent $!\Gamma \vdash A$, the following proof

$$\cfrac{\cfrac{\pi \\ \vdots}{!\Gamma \vdash A}}{!\Gamma \vdash !A} \text{Promotion}$$

is interpreted by the unique comonoid morphism

$$! [\Gamma] \xrightarrow{\quad \widetilde{[\pi]} \quad} ! [A]$$

which exists because there is a morphism $[\pi]$ from the commutative comonoid $![\Gamma]$ to $[A]$.

Finally, given a proof π of the sequent $\Gamma, A, \Delta \vdash B$, the following proof

$$\cfrac{\cfrac{\pi \\ \vdots}{\Gamma, A, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B} \text{Dereliction}$$

is interpreted by pre-composing with the morphism $\varepsilon : ![A] \to [A]$

$$[\Gamma] \otimes ! [A] \otimes [\Delta] \xrightarrow{id_{[\Gamma]} \otimes \varepsilon \otimes id_{[\Delta]}} [\Gamma] \otimes [A] \otimes [\Delta] \xrightarrow{\quad [\pi] \quad} [B]$$

Construction of the free commutative comonoid

Basically, the construction starts with a preliminary stage, the evaluation of a free pointed object (which, as it has been said, can be thought as a comonoid without multiplication), and then enables to identify the free commutative comonoid with a sequential limit, evaluated on a family of equalisers.

This approach provides a unit for the comonoid in the first place, then it "extends" the result to a proper comonoid: this can be seen as a construction "little by little". It is not misleading thinking that it succeeds where the brute construction as infinite product does not thanks to the "little by little" strategy.

However, this should not make lose sight of the theoretical depth of the result, since the justification of the "practical" steps of the construction involves refined notions of category theory. Therefore, it is more correct saying that what makes the approach work is the powerful tool of Kan extensions, which can be used because there is the possibility of "translating" objects of the category into functors of another and vice versa.

The triplet $(\mathbb{C}, \otimes, I)$ denotes hereinafter a symmetric monoidal category.

Definition 4 (Symmetric monoidal theory)**.** A symmetric monoidal theory $\mathbb{T}$ (or PROP) is defined as a symmetric monoidal category whose objects are natural numbers and whose tensor product is the ordinary sum. ♦

Definition 5 (Model of $\mathbb{T}$)**.** A model of a symmetric monoidal theory $\mathbb{T}$ in $\mathbb{C}$ is a symmetric (strong) monoidal functor $\mathcal{F} : \mathbb{T} \to \mathbb{C}$. ♦

Definition 6 ($Mod(\mathbb{T}, \mathbb{C})$)**.** $Mod(\mathbb{T}, \mathbb{C})$ is the category which has models of $\mathbb{T}$ as objects and monoidal natural transformations as morphisms. ♦

It is possible to find an equivalence between $\mathbb{C}$, $\mathbb{C}_\bullet$ and $\mathbb{C}^{co}$ and three categories of models:

Theorem 3. *Consider the symmetric monoidal theory $\mathbb{B}$, which has bijections as morphisms between finite ordinals $[n] = 0, 1, \ldots, n-1$. $\mathbb{C}$ is equivalent to $Mod(\mathbb{B}, \mathbb{C})$.*

Theorem 4. *The symmetric monoidal theory $\mathbb{I}$ has injections as morphisms between finite ordinals $[n] = [0, \ldots, n-1]$. $Mod(\mathbb{I}^{op}, \mathbb{C})$ is equivalent to $\mathbb{C}_\bullet$.*

Theorem 5. *The symmetric monoidal theory $\mathbb{F}$ has functions as morphisms between finite ordinals $[n] = [0, \ldots, n-1]$. $Mod(\mathbb{F}^{op}, \mathbb{C})$ is equivalent to $\mathbb{C}^{co}$.*

Using the previous categories equivalences, it is possible to see objects as functors and this enables to consider Kan extensions of them:

Definition 7 (Kan extensions)**.** Given a 2-category $\mathbb{C}$, 0-cells X, Y, Z and 1-cells $F : X \to Z$ and $G : X \to Y$, a right Kan extension of F along G in $\mathbb{C}$ (if it exists) consists of a pair $(\widetilde{F}, \eta)$ where $\widetilde{F} : Y \to Z$ is a 1-cell and $\eta : G; \widetilde{F} \Rightarrow F$ is a 2-cell. Moreover, for every other pair $(H : Y \to Z, \mu :$

$G; H \Rightarrow F$), it is required the existence of a unique 2-cell $\delta : H \Rightarrow \widetilde{F}$ such that the following diagram commutes:

$$\begin{tikzcd} & G; \widetilde{F} \arrow[dl, "\eta"'] & \\ F & & G; H \arrow[ll, "\mu"] \arrow[ul, "\delta \diamond 1^G"'] \end{tikzcd} \tag{7}$$

♦

Normally, Kan extensions are calculated in the 2-category Cat, which has categories as 0-cells, functors as 1-cells and natural transformations as 2-cells.

Nevertheless, in this case it is necessary to consider Kan extensions in the 2-category $SymMonCat$, which has symmetric monoidal categories as 0-cells, symmetric monoidal functors as 1-cells and monoidal natural transformations as 2-cells.

Remark 1. The forgetful functor from $\mathbb{C}^{co}$ to $\mathbb{C}$ corresponds to the functor $U : Mod(\mathbb{F}^{op}, \mathbb{C}) \to Mod(\mathbb{B}, \mathbb{C})$ which sends every model M to $i; M : \mathbb{B} \to \mathbb{C}$ and every model morphism $\theta : M \to N$ to $i; \theta : i; M \to i; N$ (where $i : \mathbb{B} \to \mathbb{F}^{op}$ is the inclusion functor defined as the identity on objects and morphisms).

Notice that i is obviously a symmetric monoidal functor.

Given an object A in $\mathbb{C}$, call simply $A : \mathbb{B} \to \mathbb{C}$ the model associated with it by the categories equivalence defined previously. So A is a symmetric monoidal functor.

The following lemma is the core issue of the construction:

Lemma 1. *For every $A \in \mathbb{C}$, the right Kan extension $(Ran_i A : \mathbb{F}^{op} \to \mathbb{C}, \eta : i; Ran_i A \to A)$ of A along i in the 2-category $SymMonCat$ induces the free commutative comonoid $!A$, which corresponds to $(Ran_i A)([1])$.*

Proof. $Ran_i A$ is a symmetric monoidal functor from $\mathbb{F}^{op}$ to $\mathbb{C}$, so it is an object of $Mod(\mathbb{F}^{op}, \mathbb{C})$, which is equivalent to the category of commutative comonoids $\mathbb{C}^{co}$. Therefore $(Ran_i A)([1])$ is a commutative comonoid. To show that it is freely generated by A, it is necessary to find a morphism $\varepsilon : (Ran_i A)([1]) \to A$ which satisfies the universal property expressed by diagram (5).

Notice that the monoidal natural transformation η is a morphism in $Mod(\mathbb{B}, \mathbb{C})$ from $U(Ran_i A)$ to A (where U is the forgetful functor). Therefore $\eta([1])$ is a morphism in $\mathbb{C}$ from $(Ran_i A)([1])$ to $A([1]) = A$ and it is possible to define $\varepsilon \equiv \eta([1])$.

Now suppose there is a commutative comonoid C and a morphism $f : C \to A$. This is equivalent to having a model $C \in Mod(\mathbb{F}^{op}, \mathbb{C})$ and a monoidal natural transformation $(f : i; C \to A) \in Mod(\mathbb{B}, \mathbb{C})$. Hence, by definition of Kan extension, there is a unique monoidal natural transformation $\delta : C \Rightarrow Ran_i A$ such that the diagram (7) commutes. Since $\delta([1])$ is a comonoid morphism in $\mathbb{C}^{co}$ (thanks to the categories equivalence), it is possible to define $\widetilde{f} \equiv \delta([1])$.

To conclude, notice that the diagram (5) is a special case of the diagram (7) (it corresponds to that diagram evaluated in [1]), so it commutes. The uniqueness of $\widetilde{f}$ follows from that of δ. □

Unfortunately, $Ran_i A$ is not computable in most cases.

The solution is to split i in two functors, $j : \mathbb{B} \to \mathbb{I}^{op}$ and $k : \mathbb{I}^{op} \to \mathbb{F}^{op}$, such that $i = j; k$ (notice that i and j are still defined as the identity on objects and morphisms). Then, it is possible to compute $Ran_i A$ as $Ran_k(Ran_j A)$.

Concerning the first Kan extension that is necessary to compute, it turns out that $Ran_j A$ is equivalent to finding a free pointed object $(A_\bullet, u)$ for every object $A \in \mathbb{C}$.

Considering the other Kan extension, it is folklore that $Ran_k A_\bullet$ is computed in the 2-category Cat as the end formula

$$\int_{[n] \in \mathbb{I}^{op}} (A_\bullet)([n], [n]) \tag{8}$$

where $A_\bullet$ is the functor from $\mathbb{I} \times \mathbb{I}^{op}$ to $\mathbb{C}$ defined as follows:

- $A_\bullet([n], [m]) \equiv \mathcal{F}_\bullet(A_\bullet)([m])$.
- Consider a morphism $f : [n] \to [n']$ in $\mathbb{I}$ and a morphism $g : [m] \to [m']$ in $\mathbb{I}^{op}$;
$$A_\bullet(f, g) \equiv \mathcal{F}_\bullet(A_\bullet)(g).$$

Call $(A^{\leq n}, e_n)$ the equaliser of the $n!$ symmetries:

$$A_\bullet^{\otimes n} \underset{symmetry}{\overset{symmetry}{\underset{\vdots}{\rightrightarrows}}} A_\bullet^{\otimes n}$$

Notice that in this case the fact that $(A^{\leq n}, e_n)$ is the equaliser ensures that the following diagram

$$\begin{tikzcd}
A^{\leq n} \arrow[r, "e_n"] \arrow[dr, "e_n"'] & A_\bullet^{\otimes n} \arrow[d, "s"] \\
& A_\bullet^{\otimes n}
\end{tikzcd}$$

commutes for every symmetry s (because $id_{A_\bullet^{\otimes n}}$ is a symmetry).

Lemma 2. *There exists a canonical morphism* $A^{\leq n+1} \longrightarrow A^{\leq n}$.

Proof. Notice that

$$A^{\leq n+1} \xrightarrow{e_{n+1}} A_\bullet^{\otimes n+1} \xrightarrow{A_\bullet^{\otimes n} \otimes u} A_\bullet^{\otimes n} \otimes I \xrightarrow{\varrho} A_\bullet^{\otimes n}$$

is a morphism from $A^{\leq n+1}$ and $A_\bullet^{\otimes n}$ (call it d_{n+1}). If $d_{n+1}; s_n = d_{n+1}$ for every symmetry $s_n : A_\bullet^{\otimes n} \to A_\bullet^{\otimes n}$, it is possible to find a morphism from $A^{\leq n+1}$ to $A^{\leq n}$ using the properties of the equaliser (and the morphism is canonical because it is the unique morphism $\widetilde{d}_{n+1}$ such that $d_{n+1} = \widetilde{d}_{n+1}; e_n$).

Consider the following diagram

$$\begin{tikzcd}
A^{\leq n+1} \arrow[r, "e_{n+1}"] & A_\bullet^{\otimes n+1} \arrow[r, "\widetilde{s}_n"] \arrow[d, "A_\bullet^{\otimes n} \otimes u"] & A_\bullet^{\otimes n+1} \arrow[r, "A_\bullet^{\otimes n} \otimes u"] & A_\bullet^{\otimes n} \otimes I \arrow[d, "\varrho"'] \\
& A_\bullet^{\otimes n} \otimes I \arrow[r, "\varrho"] & A_\bullet^{\otimes n} \arrow[r, "s_n"] & A_\bullet^{\otimes n}
\end{tikzcd}$$

where s_n represents a symmetry of $A_\bullet^{\otimes n}$ and $\widetilde{s}_n$ is defined as $s_n \otimes A_\bullet$ and is a symmetry of $A_\bullet^{\otimes n+1}$. The lower path corresponds to $d_{n+1}; s_n$, while the upper one is equal to d_{n+1} thanks to the properties of e_{n+1}. The commutativity of this diagram ensures the result. Concerning this, notice that $s_n \otimes A; A_\bullet^{\otimes n} \otimes u; \varrho = s_n \otimes u; \varrho$ (because $\otimes$ is a bifunctor), which is equal to $A_\bullet^{\otimes n} \otimes u; \varrho; s_n$ if and only if $\varrho; s_n = s_n \otimes id_I; \varrho$, that is if and only if the following diagram commutes:

$$\begin{tikzcd}
A_\bullet^{\otimes n} \otimes I \arrow[r, "\varrho"] \arrow[d, "s_n \otimes I"] & A_\bullet^{\otimes n} \arrow[d, "s_n"'] \\
A_\bullet^{\otimes n} \otimes I \arrow[r, "\varrho"] & A_\bullet^{\otimes n}
\end{tikzcd}$$

And this is true by naturality of ϱ. □

Therefore, it is possible to define an object A^∞ as the sequential limit of

$$I \longleftarrow A^{\leq 1} \longleftarrow A^{\leq 2} \longleftarrow \ldots \longleftarrow A^{\leq n} \longleftarrow A^{\leq n+1} \longleftarrow \ldots \tag{9}$$

whent it exists, with limiting cone given by projection maps p_n.

Lemma 3. *If the equalisers $(A^{\leq n}, e_n)$ exist and if the sequential limit (9) exists, then the end formula (8) exists and corresponds to the sequential limit.*

Proof. It is enough to show that (A^∞, ω) is an end of the functor $A_\bullet$, where $\omega_{[n]} \equiv p_n; e_n$. In order to do this, it is necessary to show that ω makes this diagram

$$\begin{tikzcd}
 & A^\infty \arrow[r, "p_n; e_n"] & A_\bullet^{\otimes n} \arrow[dr, "A_\bullet({[n]}, f)"] & \\
A^\infty \arrow[ur, "id"] \arrow[dr, "id"'] & & & A_\bullet^{\otimes n'} \\
 & A^\infty \arrow[r, "p_{n'}; e_{n'}"] & A_\bullet^{\otimes n'} \arrow[ur, "id"'] &
\end{tikzcd}$$

commute for every $f : [n] \to [n']$ in $\mathbb{I}^{op}$ (notice that it ensures $n' \leq n$). That happens if and only if

$$\begin{tikzcd}
 & A^\infty \arrow[dl, "p_n; e_n"'] \arrow[dr, "p_{n'}; e_{n'}"] & \\
A_\bullet^{\otimes n} \arrow[rr, "A_\bullet({[n]}, f)"] & & A_\bullet^{\otimes n'}
\end{tikzcd}$$

commutes for every $n' \leq n$. Suppose $n = n' + 1$. The commutativity of the last diagram is equivalent to that of

$$\begin{tikzcd}
 & A^\infty \arrow[dl, "p_{n'+1}"'] \arrow[dr, "p_{n'}"] & \\
A^{\leq n'+1} \arrow[rr, "\widetilde{d}_{n'+1}"] \arrow[d, "e_{n'+1}"'] & & A^{\leq n'} \arrow[d, "e_{n'}"] \\
A_\bullet^{\otimes n'+1} \arrow[rr, "A_\bullet({[n'+1]}, f)"] & & A_\bullet^{\otimes n'}
\end{tikzcd}$$

Now, the triangle commutes thanks to the properties of limit and the square commutes because

$$
\begin{aligned}
e_{n'+1}; A_\bullet([n'+1], f) \quad &= \quad e_{n'+1}; s_{n'+1}; id_{A_\bullet^{\otimes n'}} \otimes u; \varrho_{A_\bullet^{\otimes n'}} = \\
= e_{n'+1}; id_{A_\bullet^{\otimes n'}} \otimes u; \varrho_{A_\bullet^{\otimes n'}} \quad &= \quad d_{n'+1} = \widetilde{d}_{n'+1}; e_{n'}
\end{aligned}
$$

Given another dinatural transformation β from an object x to $A_\bullet$, consider the following diagram:

$$
\begin{tikzcd}
 & & A^\infty \arrow[dll, "p_0"'] \arrow[dl, "p_1"'] \arrow[d] \arrow[dr, "p_n"] \arrow[drr, "p_{n+1}"] \arrow[drrr] & & & \\
I \arrow[d, "id_I"] & A^{\leq 1} \arrow[l, "\widetilde{d}_1"] \arrow[d, "e_1"] & \dots \arrow[l, "\widetilde{d}_2"] \arrow[d] & A^{\leq n} \arrow[l, "\widetilde{d}_n"] \arrow[d, "e_n"] & A^{\leq n+1} \arrow[l, "\widetilde{d}_{n+1}"] \arrow[d, "e_{n+1}"] & \dots \arrow[l, "\widetilde{d}_{n+2}"] \arrow[d] \\
I & A_\bullet^{\otimes 1} \arrow[l, "u"'] & \dots \arrow[l, "id \otimes u; \varrho"'] & A_\bullet^{\otimes n} \arrow[l, "id \otimes u; \varrho"'] & A_\bullet^{\otimes n+1} \arrow[l, "id \otimes u; \varrho"'] & \dots \arrow[l, "id \otimes u; \varrho"'] \\
 & & x \arrow[ull, "\beta_{[0]}"] \arrow[ul, "\beta_{[1]}"] \arrow[u] \arrow[ur, "\beta_{[n]}"'] \arrow[urr, "\beta_{[n+1]}"'] \arrow[urrr] & & &
\end{tikzcd}
\tag{10}
$$

Notice that for every n and for every symmetry s_n, it holds that $\beta_{[n]}; s_n = \beta_{[n]}$ thanks to the properties of dinatural transformation. So for every n there exists a unique morphism $h_n : x \to A^{\leq n}$ such that $\beta_{[n]} = h_n; e_n$ (thanks to the definition of equaliser).

Now for every n,

$$
h_n; e_n = \beta_{[n]} = \beta_{[n+1]}; id \otimes u; \varrho = h_{n+1}; e_{n+1}; id \otimes u; \varrho = h_{n+1}; \widetilde{d}_{n+1}; e_n
$$

that is to say $h_n = h_{n+1}; \widetilde{d}_{n+1}$ (because e_n is a monomorphism). This means that by definition of limit, there exists a unique morphism h such that $h; p_n = h_n$. This ensures that $h; p_n; e_n = h_n; e_n = \beta_{[n]}$ (see above), that is $h; \omega_{[n]} = \beta_{[n]}$ for every n and this concludes the proof. $\square$

To sum up, it is possible to evaluate the right Kan extension of $A_\bullet$ along k using the sequential limit (9). Nevertheless, the implied claim that the end formula (8) provides the right Kan extension in the 2-category $SymMonCat$ (and not simply in Cat) remains to be proved.

To solve this, it is possible to apply a result proven in [16], which states that the Kan extension in $SymMonCat$ coincides with that in Cat, provided that the canonical morphism

$$X \otimes \int_{[n]\in\mathbb{I}^{op}} A_\bullet^{\otimes n} \longrightarrow \int_{[n]\in\mathbb{I}^{op}} (X \otimes A_\bullet^{\otimes n}) \tag{11}$$

is an isomorphism for every $X \in \mathbb{C}$.

Lemma 4. *If the tensor product commutes with the equalisers and with the sequential limit, then the morphism (11) is an isomorphism for every $X \in \mathbb{C}$.*

Remark 2. Commutativity between tensor product and equalisers means that for every $X \in \mathbb{C}$, $(X \otimes A^{\leq n}, id_X \otimes e_n)$ corresponds to the equaliser of

$$X \otimes A_\bullet^{\otimes n} \underset{X \otimes symmetry}{\overset{X \otimes symmetry}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}} X \otimes A_\bullet^{\otimes n}$$

Similarly, commutativity between tensor product and sequential limit means that for every $X \in \mathbb{C}$, $(X \otimes A^\infty, \{id_X \otimes p_n\}_n)$ corresponds to the sequential limit of

$$X \otimes I \xleftarrow{X \otimes \widetilde{d}_1} X \otimes A^{\leq 1} \xleftarrow{X \otimes \widetilde{d}_2} \dots \xleftarrow{X \otimes \widetilde{d}_n} X \otimes A^{\leq n} \xleftarrow{X \otimes \widetilde{d}_{n+1}} \dots \tag{12}$$

The result obtained is summarized in the following theorem:

Theorem 6. *Consider a symmetric monoidal category $\mathbb{C}$. If every object $A \in \mathbb{C}$ generates a free pointed object $A_\bullet$ and if the equalisers $(A^{\leq n}, e_n)$ (for every n) and the sequential limit (9) exist, then this sequential limit corresponds to the free commutative comonoid generated by A, provided that the tensor product commutes with the equalisers and with the sequential limit.*

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