UNIVERSITÀ DEGLI STUDI ROMA TRE
Facoltà di Scienze M. F. e N.

Sintesi di Laurea Magistrale in Matematica

Isometries at school

Candidato
Barbara Pacciani

Relatore
Prof. Andrea Bruno

. . . .

AMS classification: 97H40, 20D05
Key words: Geometry, didactic

Contents

Synthesis

My work is a presentation of a lesson at several levels about the isometries and their properties. I was able to try them into two fifth classes of a Primary School and in a second class of Secondary School. I analyzed the mathematic theory that is on the base of these lessons: Isometries

Definition 0.0.1. An isometry (or rigid motion) of the plan and application bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the distance between pairs of corresponding points, ie t.c.

$$d(f(x), f(y)) = d(x, y) \quad \forall (x, y) \in \mathbb{R}^2$$

Remark 0.0.2. The set of isometries of the plane form a group, respect to the composition of functions, called the **Euclidean group** E_2.

In fact, taken $f, g \in E_2$

$$d(f(g(x)), f(g(y))) = d(g(x), g(y)) = d(x, y) \in E_2 \quad \forall (x, y) \in \mathbb{R}^2$$

respectively applying the definition of isometry for f and g we have that the composition is still an element of E_2.
The composition of functions is associative and the identity of the transformation plan acts as an element of identity.
Finally, each $g \in E_2$ is a bijection and satisfies

$$d(g^{-1}(x), g^{-1}(y)) = d(g(g^{-1}(x)), g(g^{-1}(y))) = d(x, y)$$

so $g^{-1} \in E_2$. We have thus proved that $(E_2, \circ)$ is a group.
We give some examples of rigid motions of the plane and then we'll demonstrate, by Chasles theorem, which are the only existing.

1. the **translation** of a vector $\overrightarrow{v} = \overrightarrow{A,B}$ denoted by $\tau_{A,B}$ (or $\tau_{\overrightarrow{v}}$) is a transformation $\tau_{\overrightarrow{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ that associates to each point A of the plane the point $B = \tau_{\overrightarrow{v}}(A)$;

2. the **reflection** respect to an axis m, denoted by σ_m, is a transformation $\sigma_m : \mathbb{R}^2 \to \mathbb{R}^2$ that associates to each point A of the plane the point

$$\sigma_m(A) = \begin{cases} A \quad if\ A \in m \\ B \quad if\ A \notin m,\ m \perp \overline{AB}\ passing\ through\ its\ midpoint \end{cases}$$

3. the **rotation** of center point C and θ angle, denoted by $\rho_{C,\theta}$ is a transformation $\rho_{C,\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ that associates to each point A of the plane the point $B = \rho_{C,\theta}(A)$, with $\theta = A\widehat{C}B$;

4. the **glide reflection**, denoted by $\gamma_m^{\overrightarrow{v}}$, where $\overrightarrow{v} = \overrightarrow{A,B}$ is a transformation $\gamma_m^{\overrightarrow{v}} : \mathbb{R}^2 \to \mathbb{R}^2$ that associates to each point A of the plane, the point
$B = \gamma_m^{\overrightarrow{v}}(A) = \sigma_m \tau_{\overrightarrow{v}}$ with $\overrightarrow{v} \parallel m$.

The glide reflection is a composition of a translation and a reflection.
Chasles's Theorem:

Theorem 0.0.3. ***(Chasles's Theorem)***
Every motion of the plane is one of these transformation:

- *a translation*
- *a rotation*
- *a reflection or*
- *a glide reflection*

Leonardo's Theorem:

Theorem 0.0.4. ***(Leonardo's Theorem)*** *A finite group of isometries Γ is either a cyclic group C_n or a dihedral group D_n.*

Wallpaper Groups: The finite groups of isometries which can contain just rotations and reflections are called **Rosette Groups**.
Now we are interesting in discrete groups of plane isometries which are not finite and that have to contain, therefore, at least one translation (and so infinite ones obtained by repeated application of one of them). The last ones belong to translations subgroup.
If this subgroup is generated by one translation, the models of symmetry are extended, indefinitely repeated, in one direction, filling in a strip of plane and they are called **Friezes**. If translations subgroup is generated by two independent translations, we have the **Wallpaper Groups**.
There are three types of discrete groups, depending on the size of the subgroup of translations Γ'.

A discrete group Γ is classified by its subgroup of translations as follows:

1. it does not contain any translation, but it contains rotations and reflections
   $\Gamma =< E_2 >$ **Rosette Groups**
2. it contains translations of collinear vectors $\Gamma' = \{\tau_{\overrightarrow{v}}; \overrightarrow{v} = n\overrightarrow{a}, n \in \mathbb{Z}\}$
   $\Gamma =< \tau_{\overrightarrow{a}} >$ **Friezes Groups**
3. it contains translations of non collinear vectors
   $\Gamma' = \{\tau_{\overrightarrow{v}}; \overrightarrow{v} = n\overrightarrow{a} + m\overrightarrow{a}, a \nparallel b\ m, n \in \mathbb{Z}\}$
   $\Gamma =< \tau_a, \tau_b >$ **Wallpaper Groups**

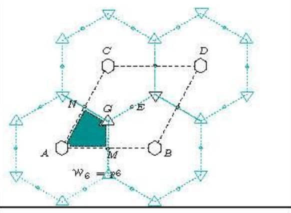
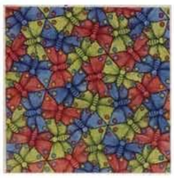
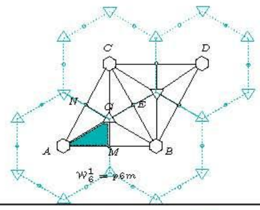
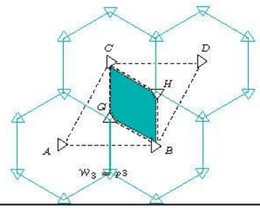
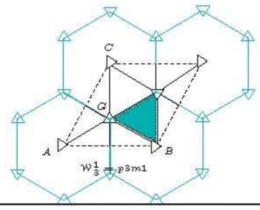
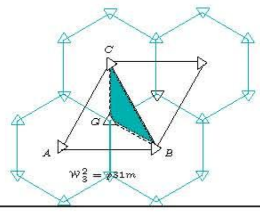
We have already faced the first point, so we'll be interested in the subsequent point. We are enunciating, but we aren't giving a rigorous proof, the theorem of friezes groups classification. In mathematical speech the word frieze means a plane shape whose symmetry's group (i.e. the set of transformation of the plane that leave unchanged the distances and change the figure in itself) contains the translations, but only unidirectional ones and alls multiple of a basic-translation. Such a figure is necessarily unlimited.
For the friezes, in addition to translations, there are others symmetries that are added for composition. We consider symmetries under which such models

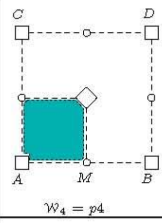
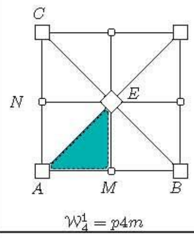
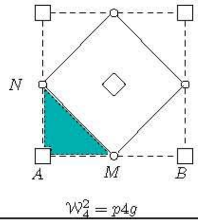
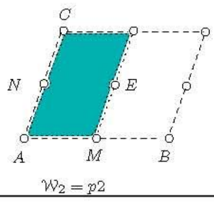
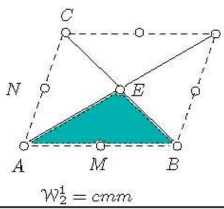
are invariant by translation, we will see that there are seven possible types of ornamental friezes.

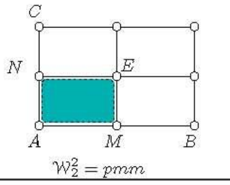
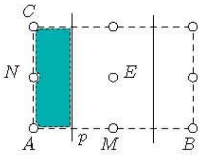
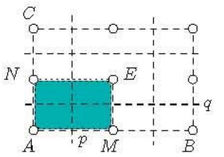
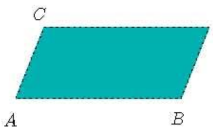
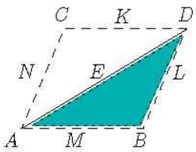
Definition 0.0.5. A frieze group with center c is an isometries group that fixes a a given line c and whose translations form a infinite cyclic group.

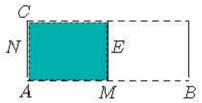
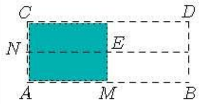
and Escher and the Tessellations of the plane: Now we have a look at the symmetry of Escher art.
Escher's works, gathered from the book entitled *Visions of simmetry* by Doris Schattschneider, and here presented, suggest us a visual idea of what is a tessellation of the plane and what are the 'elements' of symmetry that part of it. We therefore proposed to examine the artist's works by identifying, through their careful analysis, which and what order are the pivot points and which symmetries appear thus ascending to the type of unit cell used to produce printing (the tessellation of the plane) and then to which crystallographic group it is classified. In the space group $p1$, we have already seen that there is just the operation identity and therefore a material point is only repeated by the translational lattice carriers. I already finished this chapter by a roundup of Escher's works, where, for the most of them, I illustrated the wallpaper

group which them belong.

$\mathcal{W}_6 = \mathbf{p6}$	$p6$		
$\mathcal{W}_6^1 = \mathbf{p6m}$	**p6mm**		
$\mathcal{W}_3 = \mathbf{p3}$	**p3**		
$\mathcal{W}_3^1 = \mathbf{p3m1}$	**p3m1**		
$\mathcal{W}_3^2 = \mathbf{p31m}$	**p31m**		

$\mathcal{W}_4 = \mathbf{p4}$	$p4$		
$\mathcal{W}_4^1 = \mathbf{p4m}$	**p4mm**		
$\mathcal{W}_4^2 = \mathbf{p4g}$	**p4gm**		
$\mathcal{W}2 = \mathbf{p2}$	**p211**		
$\mathcal{W}_2^1 = \mathbf{cmm}$	**c2mm**		

$\mathcal{W}_2^2 = \mathbf{pmm}$	$p2mm$	 $\mathcal{W}_2^2 = pmm$	
$\mathcal{W}_2^3 = \mathbf{pmg}$	**p2mg**	 $\mathcal{W}_2^3 = pmg$	
$\mathcal{W}_2^4 = \mathbf{pgg}$	**p2gg**	 $\mathcal{W}_2^4 = pgg$	
$\mathcal{W}_1 = \mathbf{p1}$	**p1**	 $\mathcal{W}_1 = p1$	
$\mathcal{W}_1^1 = \mathbf{cm}$	**c1m1**	 $\mathcal{W}_1^1 = cm$	

$\mathcal{W}_1^2 = \mathbf{pm}$	$p1m1$	 $\mathcal{W}_1^2 = pm$	
$\mathcal{W}_1^3 = \mathbf{pg}$	**p1g1**	 $\mathcal{W}_1^3 = pg$	

In the second chapter I dealt some considerations about the teaching of mathematic in the compulsory education. These considerations are the result of a selection of speeches by several authors and important people which work in instruction's field also as educator of teachers of the compulsory school. Actually on the base of these considerations there are also several different readings of didactic experience of various kind of important mathematicians which, in a certain way,revolutionized the teaching of mathematic; two names of all: Emma Castelnuovo and Serge Lang.

After these consideration I illustrated the change in the didactic planning happened by the approval of the law to reform the education system (Law No. 53 of March 28, 2003), the subsequent enactment of the implementing decree (Decree No. 59 of 19 February 2004) and other legislations, the Middle School has undergone to a transformation beginning from its same name: Secondary School of First Grade, that with the Primary School, complete the first cycle of studies. In this perspective it is interesting to see the news that such innovation introduced that actually required significant changes in the way of organizing the educational work and that reminded to ideas and key words by then familiar in the Italian school, such as autonomy, flexibility, planning, centrality of the student and learning. The background and the frame of reference for the educational institutions continued to be represented by the autonomy, meaning above all as the planning autonomy and the teaching research, ie autonomy of those teachers who work to promote the integral education of each student. This implies a challenging job, actually, the development of the training objectives and their implementation. In fact, while the objectives are centrally set and compete to the Ministry, it is up to the individual schools to develop appropriate strategies and build educational pathways in which specific learning objectives (knowledge and skills) can be transformed into personal skills of the students.

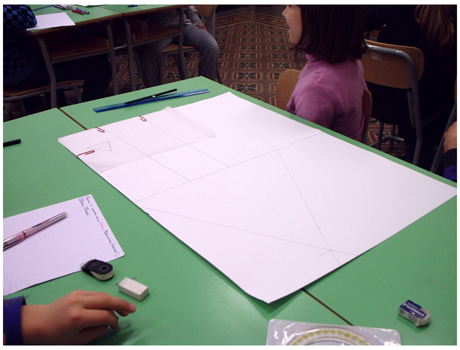
I had an experience in two fifth classes of a Primary School, with my advisor. This presentation has the following order: first of all I illustrated the planning of the lessons, where I so listed the Specific Objectives of Learning concerning these lessons:

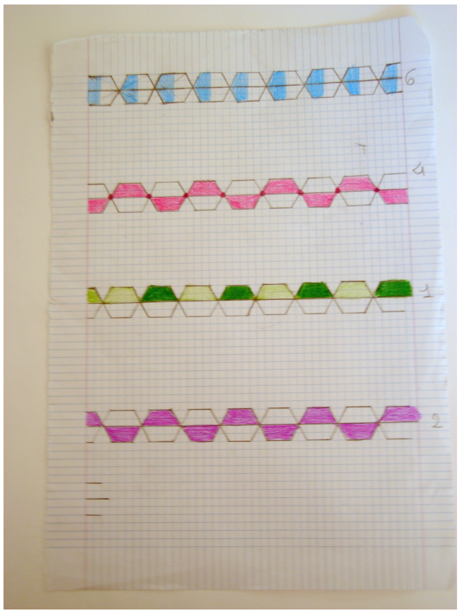
Specific learning objectives FIFTH CLASSES that I

MATHEMATICS

- exploring models of geometric figures by recognizing the properties, building and drawing the main geometric figures explored; working effectively with the figures by processing them;
- in simple cases determining perimeters, areas and volumes of the known geometric figures;
- classifying objects, pictures, numbers, making appropriate representations;
- identifying, describing and building meaningful relationships in different contexts;
- verifying by example a formulated hypothesis;
- strengthening the capacity of collecting data and distinguishing the qualitative from the quantitative;
- classifying uncertain situations, quantifying in simple contexts by using the information held, in particular the possible outcomes of the symmetry and the relative frequency of similar situations;
- identifying the information necessary to achieve a goal by analyzing the text of a problem , organizing and achieving a solution path;
- considering the followed resolution procedure and comparing it with other possible solutions.

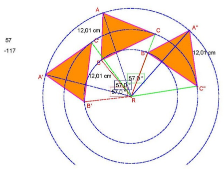
We did several activities with the students, and I like to show just some of works of them.

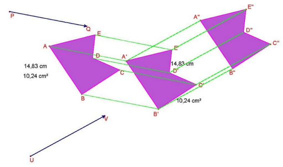




Then, analogy with the Primary School, I had also an experience in a second class of a Secondary School. We work also with the software Cabri, in fact, I used this software with the second class of the Secondary School, to show them the composition of the isometries.

At the end I presented some final works of the students.





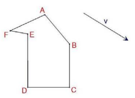
<u>**V COMPITO DI MATEMATICA –A.S. '10-'11- (II Quadr.) Classe II**</u>

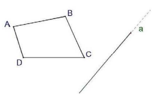
<u>**Fila 1**</u>

1) Esegui, sul foglio, le seguenti trasformazioni:

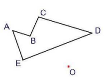
a) Traslazione del vettore $\vec{v}$: **[6]**

b) Simmetria assiale di asse a: **[6]**





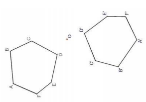
d) Composizione di rotazioni di centro O e angoli: $\vec{\alpha} = 75°$ e $\vec{\beta} = 165°$ **[5 o 6]**



2) Data la seguente trasformazione (una rotazione di centro O) trova la sua ampiezza e verso: **[4]**

α = ...

verso = ...



3) Cancella, se sono presenti, il termine o i termini intrusi: **[4]**

a) Un glissoriflessione resta definito da: asse di simmetria / vettore di traslazione / punto di applicazione

b) Una rotazione attorno ad un punto resta definita da: verso / ampiezza / centro di rotazione

4) Vero o Falso? **[3]**

	V	F
a)La rotazione è definita unicamente da un centro di rotazione.	☐	☐
b)Le figure trasformate in una isometria non sempre mantengono la forma	☐	☐
c)La composizione di due traslazioni è una rotazione.	☐	☐

5) Dai la definizione di: **[6]**

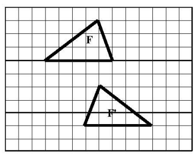
*isometria:*____

*traslazione:*____

6) Dato il seguente fregio disegna la sua glissoriflessione orizzontale: **[5]**

ECCELLENZA

7) Individua quali isometrie sono state applicate alla figura F per ottenere la figura F'



Punteggio totale: 40 + Eccellenza

VOCI di VALUTAZIONE:

- **Conoscenze:**
- **Linguaggio:**
- **Applicazioni:**
- **Ordine:**

Alunno/a:……………………………………………………………………………

Bibliography

[1] Roberto Magari *Le basi della matematica* ed.Universale Sansoni. (1989) Sansono Editore S.p.A., Firenze ISBN 88-383-1052-1 .
[2] Bruno Spotorno Vinicio Villani *Mondo reale e modelli matematici* ed. La nuova Italia Editrice, Firenze (1976)
[3] Serge Lang *The Beauty of Doing Mathematics. Three Public Dialogues Math! Encounters with High School Students* (1985) Springer- Verlag New York Inc. Traduzione it. di Maria Gherardelli *La bellezza della matematica* Torino Bollati Boringhieri, (1991) ISBN 88-339-0606-X
[4] C. Sitia *La didattica della matematica oggi- Problemi, ricerche, orientamenti* Pitagora Editrice - Bologna (1979)
[5] V.V. Nikulin - I.R. Shafarevich *Geometries and Groups* Springer-Verlag (1994) New York Berlin Heidelberg ISBN 0-387-15281-4
[6] George Edward Martin. *Transformation Geometry - An introduction to Symmetry* Springer-Verlag, (1982) New York Heidelberg Berlin ISBN 0-387-90636-3
[7] Doris Schattschneider. *Visioni della Simmetria* Zanichelli, (1996)
[8] Marina Cazzola *L'insegnamento della matematica: una didattica metacognitiva* tratto da "Metacognizione ed educazione" a cura di O. Albanese,P.A. Doudin, D. Martin (edizione riveduta e ampliata da F. Angeli) (1995-2003)
[9] Emma Castelnuovo *Insegnare Matematica- Lectio Magistralis* tenuta il 15 marzo 2007 Auditorium - Parco della Musica di Roma dvd Iacobelli (2008) ISBN 978-88-6252-008-9 [10] [11][12][13][14][15]