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# Closure operations and star operations in commutative rings

Sintesi

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A *closure operation* is a map $f$ from a partial ordered set to itself that verifies three axioms: extension ($x \leq f(x)$), order-preservation ($x \leq y$ implies that $f(x) \leq f(y)$) and idempotence ($f(f(x)) = f(x)$ for every $x$). The most known closure operation is perhaps the operation of closure between sets of a topological space; another common operation is the one that assigns to the subsets of an algebraic structure (a group, a ring, a vector space...) the smallest substructure containing the subset, that is, the substructure spanned by the subset. Moreover, almost every field of mathematics has some construction that can be seen as a closure operation between structures, where the order is given by the inclusion: algebra has integral closure of rings and ring completion (of Noetherian local rings, to guarantee idempotence), topology has compactifications and completion of metric spaces, analysis has completion of measure spaces.

This thesis is about closure operation in the partial ordered set of ideals of a commutative unitary ring $R$.

Due to their generality, these closure operations does not satisfy many ring-theoretic properties, and thus they have rarely been the subject of a general theory: however, single closures (like integral closure or tight closure) and smaller sets of closures (such as star operations) have been studied in detail. The case of star operations is somewhat emblematic: many of their basic properties are in fact valid for wider class of closures, that of semi-prime closure operations (neither requiring different proofs), but the concept was instead generalized to semistar operations.

Recently, some authors have found useful to consider closure operations as an autonomous subject; new definitions has been given, pursuing generalizations of properties of previously known closures, or trying to understand the structure of some subsets of the set of closure operations. However, the various fields of study are still very far, partly because of different assumptions and problems, partly because of very different techniques.

The thesis is divided into four chapters, each one narrowing down the subject: the first is dedicated to arbitrary closure operation, trying to identify some general properties; the second to a special class of closures, star operations, and the third to closures arising from overrings of the original ring, focusing on those induces by localizations. The last one deals with one specific closure, integral closure, and with its links to two other operations, which can be seen as its variants: complete integral closure and tight closure.

All rings will be assumed to be commutative and with unity; in Chapters 2 and 3, and in most of Chapter 4, we will consider only integral domains. We will not assume (if not for specific results) that the rings are Noetherian, as many definitions and theorems become trivial in the Noetherian context; in fact, Sections 1.6 and 3.5 study to what extent some properties of Noetherian rings can be transferred to some classes of non-Noetherian ones.

Numbers of equations and theorems in this synthesis match the numbers in the full version of the thesis.

Chapter 1 is dedicated to general closure operations, to their set and to three properties that they can have: finite type, semi-primality, and $c$-finiteness of ideals.

The set $C(R)$ of closure operations is a very big set, due to the very low requirement for a map to be a closure; it can be naturally identified as a subset of the power set of $\mathcal{I}(R)$ (the set of ideals of $R$) by the application that send a closure $c$ to the set of $c$-closed ideals ($I$ is said to be $c$-closed, or a $c$-ideal, if $I = I^c$); this map is injective (i.e., the set of $c$-ideals determines the closure) because the closure of an ideal $I$ is intersection of all the $c$-closed ideals containing $I$, and its image is constituted by the sets of ideals closed by arbitrary intersections. From this, we can construct a closure operation by any set $\mathcal{A}$ of ideals of $R$ by $I^c := \bigcap\{J \in \mathcal{A} \mid I \subseteq J\}$.

This correspondence also gives a natural partial ordering on $C(R)$, in which smaller closures have less closed ideals; however, since having few closed ideals means that the closure of an ideal $I$ is usually big (with respect to $I$), we reverse the ordering, and say that a closure $c$ is smaller than $d$ if the set of $c$-ideals contains the set of $d$-ideals or, in another way, if $I^c \subseteq I^d$ for every ideal $I$. This order makes $C(R)$ into a complete lattice: given a set $\{c_\lambda\}_{\lambda\in\Lambda}$ of closure operation, its supremum is the closure whose closed ideals are those that are closed for every $c_\lambda$, while the infimum is the closure operation generated by the union of all the sets of $c_\lambda$-closed ideals or, more explicitly,

$$I^c := \bigcap_{\lambda\in\Lambda} I^{c_\lambda}. \tag{1.8}$$

On the other hand, $C(R)$ fails to be a monoid, because the composition of two closure operation is not always idempotent, even on very simple rings.

Finite type closure operations are closures whose behaviour is determined by the finitely generated ideals: more precisely, $c$ is of finite type if, for every ideal $I$,

$$I^c = \bigcup\{J^c \mid J \subseteq I \text{ and } J \text{ is finitely generated}\}. \tag{1.28}$$

If $c$ is of finite type, all the information about the closure of an ideal $I$ can be recovered by the closures of finitely generated ideals: there is no "jump" between finitely and non-finitely generated ideals. To every closure operation $c$ is canonically associated a closure operation of finite type $c_f$, defined by $I^{c_f} := \bigcup\{J^c \mid J \subseteq I \text{ and } J \text{ is finitely generated}\}$, which agrees with the original one on finitely generated ideals and is the biggest closure of finite type smaller than $c$.

Semi-primality is probably the most general property that uses effectively the ring structure of $R$: a closure $c$ is *semi-prime* if $x \cdot I^c \subseteq (xI)^c$ for every $x \in R$ and every ideal $I \trianglelefteq R$. For example, this property allows the study of the $c$-spectrum of $R$, that is, the set of prime ideals that are also $c$-ideals: if $c$ is also of finite type, many classical results (existence of maximal ideals, primality of maximal ideals, representation of an ideal as intersection of its extensions in localizations) have analogues for $c$-ideals: we obtain that every

$c$-ideal is contained in a $c$-maximal ideal (i.e., maximal elements of the set of proper $c$-ideals), that every $c$-maximal ideal is prime, and that every $c$-ideal can be represented as intersection of its extensions in the localizations at $c$-maximal ideals (i.e., $I^c = \bigcap I^c R_M$ where $M$ varies among the $c$-maximal ideals).

Semi-primality is a very natural concept: many constructions (analysed in more detail in Chapter 3, Section 3.1) yield naturally semi-prime operations. On the other hand, semi-prime operation are a natural generalization of star operations (which are defined by the equality $x \cdot I^c = (xI)^c$), and in fact the properties cited above are usually proved in the star operation setting, although the proofs need almost no change to adapt them to the semi-prime case. Moreover, some closures (for example integral closure or tight closure) happens to be star operations only in certain rings (typically integrally closed ones), but they are always semi-prime: thus studying semi-prime operations permits to recover some results also in "bad" cases.

$c$-finiteness of an ideal is a more strict form of the finite type property, although it is local (on a single ideal) rather than global (on all ideals). An ideal $I$ is *$c$-finite* if its closure $I^c$ is also the closure of a finitely generated ideal, and is *strictly $c$-finite* if this ideal can be taken to be contained in $I$; the two concepts are equivalent for finite type closure operations, but not in the general case. In both cases (but especially if $I$ is strictly $c$-finite), for the study of $c$ we have that $I$ can be (almost) considered finitely generated because we can replace $I^c$ with the closure $H^c$ of a finitely generated ideal $H$. The condition that all the ideals are $c$-finite is a much more strict condition than $c$ being of finite type: for example, the identity is always of finite type, but an ideal is $c$-finite only if it is finitely generated. In fact, the rings where all ideals are $c$-finite (or rather strictly $c$-finite) are somewhat "close" to being Noetherian, and are even called $c$-Noetherians; the subject is more deeply studied in Section 3.5 of Chapter 3. When $R$ is Noetherian, an analogue subject is to understand how many elements are needed to generate an ideal $J$ such that $I^c = J^c$ (where $I$ is a previously fixed ideal).

Chapter 2 discusses the main properties of star operations, whose theory is a well-known part of multiplicative ideal theory since the works of Krull and Gilmer.

The defining property of star operations can be seen as a form of "invariance through multiplication": for every $x \in R$ (which we suppose is an integral domain) and every ideal $I \trianglelefteq R$, we have that $x \cdot I^c = (xI)^c$. It naturally leads to the idea of multiplying not only by elements of $R$, but also by elements of the quotient field $K$; however, in this case, the ideal $xI$ is usually no more contained in $R$. This motivates the introduction of the concept of fractional ideals, which are $R$-submodules of $K$ that can be multiplied into $R$ (i.e., $J \subseteq K$ for which there is a $y \in R$ such that $yJ \subseteq R$). Every star operation $c$ can be canonically extended as a closure $\star$ on the set of fractional ideals by $I^\star := \frac{1}{x}(xI)^c$, where $x$ is an element such that $xI \subseteq R$; if we insist that $\star$ verifies $x \cdot I^\star = (xI)^\star$ even for $x \in K$ and fractional ideals $I$, then this is the unique way to extend $c$. Moreover, the set of fractional ideals is the biggest set where this extension is unique.

Maybe the most important star operation is the $v$-operation, also called *divisorial closure*: it can be defined either as the intersection of all principal fractional ideals containing $I$, or as the double dual $(R :_K (R :_K I))$ of $I$. Its importance relies mainly on the fact that $v$ is the biggest star operation, thus giving an explicit bound for the other star operations. Section 2.3 proves the equivalence between the two definitions and gives a condition for an ideal $I$ to have $I^v = R$:

**Proposition 2.17.** *Suppose that there is an element $x \in I$ such that $I/(x) \nsubseteq \mathcal{Z}(R/(x))$ (where $\mathcal{Z}(A)$ is the set of zerodivisors of a ring $A$). Then $I^v = R$.*

An explicit special case (Proposition 2.19) is when $R$ is a Noetherian domain and $P$ is a prime ideal of height $\geq 2$ such that $R_P$ is integrally closed: in this case, $P^v = R$.

The next two sections focuses on the concepts of $\star$-invertibility and of $\star$-class group, especially in the case when $\star$ is of finite type. For the former, the results that can be obtained are very closely analogous to those valid for the notion of invertibility, leading to the definition of the class of *Prüfer $\star$-multiplication domains* (P$\star$MDs) as a generalization of the class of Prüfer domains: P$\star$MDs are domains such that every finitely generated ideal of $R$ is $\star$-invertible, or equivalently such that $R_M$ is a valuation domain for every $\star$-maximal ideal $M$; when $\star$ is the identity, this characterization "collapses" on that of Prüfer domains.

For the class group, the picture is much less clear, especially if $\star$ is not taken to be equal to the identity or to $t$ (the finite type closure associated to $v$). Even in this last case, the analogy with the identity is not perfect: for example, an homomorphism $\phi : R \longrightarrow S$ (or even an inclusion) does not always induce a map between the corresponding $t$-class groups. However, the $t$-class group is relevant when considering conditions equivalent to certain properties of factorization: for example, unique factorization domains are those Krull domains whose $t$-class group vanishes.

In the last section, we analyse $v$-invertibility. The criterion assumes a different form with respect to other $\star$-invertibilities: $I$ is $v$-invertible if and only if $(I :_K I) = R$. This leads to the notions of *completely integrally closed rings*, as the rings where each ideal is $v$-invertible, and of *complete integral closure* of a ring, as the union $\bigcup\{(I :_K I) \mid I \text{ is an ideal of } R\}$, which can be seen as an extension of the usual notion of integral closure (where the union ranges only among finitely generated ideals); moreover, just like integral closure can be defined through equations of linear dependence, the complete integral of $R$ can be seen as the set of elements such that $cx^n \in R$ for all $n \in \mathbb{N}$ and for an element $c \in R$ ($c \neq 0$). However, complete integral closure is much less well-behaved than integral closure: for example, there are rings for which the complete integral closure is not completely integrally closed, i.e., complete integral closure is not always idempotent.

Chapter 3 is mainly about closure operation induced by a family of rings, that is, closures $c$ that can be written as $I^c = \bigcap IS \cap R$, where $S$ ranges among a (given) family of rings containing $R$. Although not every star operation can be constructed this way, closures of this type provides a wide set of examples that are usually simpler and more "regular" than an arbitrary closure operation: for example, for these closure it is always true that a $c$-ideal is contained in a prime $c$-ideal, even if $c$ is not of finite type. (However, $c$-maximal ideals need not to exist.)

We begin by two even more general constructions. The first uses homomorphisms from $R$ (dropping the condition that $R$ is contained in the rings) and closure operations also on the image; it can be written, in its most general form, as

$$I^c := \bigcap_{\alpha \in A} \phi_\alpha^{-1}((\phi_\alpha(I)S_\alpha)^{d_\alpha}), \tag{3.5}$$

where $\phi_\alpha : R \longrightarrow S_\alpha$ are homomorphisms and each $d_\alpha$ a closure on $S_\alpha$. The second uses modules: if $U$ is an $R$-module, the map $I \mapsto I^c := (IU :_R U)$ is a closure operation. Usually, these constructions yield semi-prime operations, but more rarely star operations: for example, to have that the closure $I^c = \bigcap IS \cap R$ is a star operation we must suppose that $\bigcap S = R$.

We proceed by giving some properties of closures induced by a family of rings, successively shifting to the case when each of these rings is a localization of $R$: these are called *spectral operations*. Spectral operations have been more thoroughly studied than closures induced by a general family of rings, mostly because the following characterization:

**Proposition 3.10.** *A a star operation $\star$ on $R$ is spectral if and only if $(I \cap J)^\star = I^\star \cap J^\star$ for every pair ideals $I, J$ of $R$ and every $\star$-ideal is contained in a $\star$-prime ideal.*

Moreover, to every star operation of finite type $\star$ is possible to assign a spectral star operation $\star_w$, which is in many case simpler but close enough to the original $\star$. The construction of $\star_w$ is detailed in Section 3.4.

We also prove a characterization of finite type spectral operations among all the spectral operations:

**Proposition 3.13.** *Let $\Delta$ be a set of prime ideals of $R$ such that $\bigcap_{P \in \Delta} R_P = R$,* *and let $\star$ be the spectral operation $I^\star := \bigcap_{P\in\Delta} IR_P$. Then $\star$ is of finite type if and only if $\Delta$ is compact in the Zariski topology inherited from* Spec$(R)$.

In the last two sections of the chapter, we continue the investigation of two subjects: operations $c$ that satisfies the ascending chain condition on $c$-ideals (called $c$-Noetherian) and construction of star operation.

$c$-Noetherian rings are called this way because some theorems, classically proved for Noetherian rings, can be carried over to them, although we usually have to restrict to the set of $c$-ideals. However, the properties of the closure $c$ are important to determine how many results we can transfer: the strongest results are obtained when $c$ is a spectral star operation, due to the fact that, in this case, each localization at a $c$-prime ideal is Noetherian. This imply that, as a rule of thumb, if a theorem about Noetherian rings depends only on the local structure of the ring, then it can be transferred to $c$-ideals of $c$-Noetherian domains. Two examples are:

**Propositions 3.26, 3.27 and 3.29.** *Let $R$ be a $\star$-Noetherian domain, where $\star$ is a spectral star operation.*

*Generalized Principal Ideal Theorem.* *If $P$ is a prime ideal minimal over $(a_1, \ldots, a_n)^\star$, then the height of $P$ is at most $n$; in particular, a minimal prime of a principal ideal has height 1.*

*Krull Intersection Theorem.* *If $I^\star \neq R$, then $\bigcap_{n\geq 1}(I^n)^\star = (0)$.*

*Powers of radical.* *Every $\star$-ideal contains a power of its radical.*

In addition, every $\star$-ideal has a primary decomposition by $\star$-primary ideals.

The last section of the chapter shows how to build new star operations from an old one and from prime ideals not fixed by it: this gives, for example, a bound on the number of non-divisorial prime ideals in the case that $R$ has only a finite number of star operations. Moreover, this allows to count the number of spectral star operations on a Noetherian integrally closed ring: they are finite if and only if $R$ is one-dimensional (in which case there is only one) or two-dimensional and semi-local (in which case there are $2^{|\operatorname{Max}(R)|}$ spectral star operations); more compactly, there are $2^{|X^{(2)}|}$ spectral star operations, where $X^{(2)}$ is the set of prime ideals of height 2.

Chapter 4 deals with integral closure of ideals and with two variations, complete integral closure and tight closure.

Integral closure of ideals is an old concept, first considered by Krull, which extends the notion of integral closure of rings. Just like the integral closure of $R$ in an extension ring $S$ is the set of elements of $S$ that verify a monic polynomial equation $x^n + a_1x^{n-1} + \cdots + a_n = 0$ with coefficients in $R$, the integral closure of an ideal $I \trianglelefteq R$ is the set of elements of $R$ that verify an analogous equation, but with each $a_i \in I^i$.

Integral closure is linked to many topics in commutative algebra: this leads to a great number of different views on the subject, often with a new (but equivalent) definition, and often mirroring what happens for integral closure of rings. We prove the equivalence of the above definition with two other different approaches.

The first uses valuation overrings of $R$: just like the integral closure of a domain $R$ in its quotient field $K$ is equal to the intersection of all valuation rings contained between $R$ and $K$, the integral closure of an ideal $I$ is equal to the intersection $\bigcap IV \cap R$ (where the $V$ are the valuation overrings of $K$): this is Proposition 4.10. However, the set of valuation overrings is not the unique that can be used to obtain integral closure: any suitable set is said to be a *$b$-set*. For example, if $L$ is a field containing $K$, then the set of valuation rings containing $R$ and having $L$ as quotient field is itself a $b$-set: this allows the study of the behaviour of integral closure in relation to an integral extension of rings.

Another interesting problem is studying what are sufficient conditions for $R$ to have a $b$-set composed by discrete valuation rings (called a *discrete $b$-set*): it is a "classical" theorem that this is true if $R$ is Noetherian. We show that this property is local (i.e., $R$ admits a discrete $b$-set if and only if so does every localization $R_M$ at maximal ideals – Proposition 4.14) and that it descends along integral extensions (i.e., if $R \subseteq S$ is integral and $S$ admits a discrete $b$-set, so does $R$ – Proposition 4.18); hence every domain that has an integral extension which is locally Noetherian admits a discrete $b$-set.

The other approach generalizes the fact that the integral closure of $R$ is equal to $\bigcup(J :_K J)$, where the union ranges among all the finitely generated ideals $J \trianglelefteq R$: Proposition 4.34 shows that the integral closure of an ideal $I$ is equal to $\bigcup(IJ :_R J)$, with $J$ varying in the same set. This equivalence leads to a more general class of closures, called $\Delta$-closures, which are given by

$$I^{d_\Delta} := \bigcup_{J \in \Delta} (IJ :_R J) \tag{4.28}$$

where $\Delta$ is a multiplicatively closed set of finitely generated ideals of $R$.

In Section 4.1.2 we show how to obtain integral closure of ideals of an arbitrary ring (not necessarily a domain) from the integral closure of ideals in domains.

Section 4.2 proves some properties of integral closure; it is shown that it is always a semi-prime closure operation of finite type, and that is a star operation if and only if $R$ is integrally closed in its quotient field. It is then proved that integral closure commutes with localizations, and that all ideal are integrally closed if and only if $R$ is a Prüfer domain.

Next we introduce the concept of complete integral closure of an ideal, similarly to complete integral closure of rings: an element $x$ is said to be in the complete integral closure $I^{cic}$ of $I$ if there is an element $c \in R$, $c \neq 0$ such that $cx^n \in I^n$ for every $n \in \mathbb{N}$. This definition, although a natural generalization of both complete integral closure of rings and integral closure of ideals, has received little attention: perhaps the only result is the old theorem stating that, if $R$ is a Noetherian domain, $I^{cic}$ coincides with the integral closure of $I$, just like it happens for the complete integral closure and the integral closure of $R$ (as a ring). We extend this result to every ring that admits a discrete $b$-set (Proposition 4.40). However, it is not known if complete integral closure is idempotent and, therefore, a closure operation; since the complete integral closure of a ring is not necessarily completely integrally closed, it can be expected that idempotence fails also in this case.

Tight closure is a more recent closure, developed in the context of Noetherian rings. Unlike the other closures, it is only defined when the characteristic of the ring is a prime number $p > 0$: an element $x$ is in the tight closure of $I$ if there is an element $c \in R$, $c \neq 0$, such that $cx^{p^e} \in I^{[p^e]}$ for every $e \geq 1$ (where $I^{[n]}$ is the ideal generated by the $n$th powers of the elements of $I$). For Noetherian rings, the theory of tight closure is linked to regular rings (where each ideal is tightly closed), regular sequences (and thus Cohen-Macaulay rings) and homological results; moreover, some aspects mirrors what happens for integral closure. For example, tight closure on an arbitrary ring is determined by tight closure on domains (Proposition 4.47, an analogue of Proposition 4.23), and tight closure and integral closure agree for principal ideals. Briançon-Skoda theorem (Theorem 4.49) shows how to compare integral closure and tight closure of powers of an ideal. If $R$ is non-Noetherian, little is known, just like for complete integral closure, and it is entirely possible that tight closure fails, in general, to be idempotent.

# ESSENTIAL BIBLIOGRAPHY

D. D. Anderson, *Star-operations induced by overrings*, Comm. Algebra **16** (1988), no. 12, 2535–2553.

D. D. Anderson and S. J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), no. 5, 2461–2475.

David F. Anderson, *A general theory of class groups*, Comm. Algebra **16** (1988), no. 4, 805–847.

Neil M. Epstein, *A guide to closure operations in commutative algebra*, preprint (2011), arXiv:1106.1119v3.

Robert Gilmer, *Multiplicative ideal theory*, Marcel Dekker Inc., New York, 1972, Pure and Applied Mathematics, No. 12.

Melvin Hochster and Craig Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116.

Craig Huneke and Irena Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.

Irving Kaplansky, *Commutative rings*, revised ed., The University of Chicago Press, Chicago, Ill.-London, 1974.

D. Kirby, *Closure operations on ideals and submodules*, J. London Math. Soc. **44** (1969), 283–291.

Wolfgang Krull, *Idealtheorie*, Springer-Verlag, Berlin, 1935.

Eliakim H. Moore, *Introduction to a form of general analysis*, The New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910, pp. 1–150.

D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc. **50** (1954), 145–158.