FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI

Graduation Thesis in Mathematics
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Mather's Theory: Fathi Siconolfi Theorem

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ACADEMIC YEAR 2007 - 2008

Abstract

In this thesis, we shall consider Hamiltonian system on tori; in other words, we shall consider the following differential equation on $\mathbb{T}^n \times \mathbb{R}^n$:

$$\begin{cases} \dot{q} = & \frac{\partial H}{\partial p}(q,p) \\ \dot{p} = & -\frac{\partial H}{\partial q}(q,p) \end{cases}$$

where $q \in \mathbb{T}^n$, $p \in \mathbb{R}^n$ are angle-action variable and $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function of class C^r, with $r \geq 2$.

In many interesting examples, the function H has the form

$$H(q,p,\epsilon) = H_0(p) + \epsilon H_1(q,p),$$

where $\epsilon \in \mathbb{R}$ is a small parameter.

When $\epsilon = 0$, the system is called integrable.
Considering Hamiltonian system on the torus may look rather restrictive; however, many interesting examples of Hamiltonian mechanics live on this space, for instance, the motions of two bodies in a plane under Newtonian attraction is an integrable system on $\mathbb{T} \times \mathbb{R}$. The solar system, which can be seen as a small perturbation of the integrable Sun-Jupiter problem, also lives on $\mathbb{T}^n \times \mathbb{R}^n$ for a suitable n depending on the number of the planets.
Integrable problems have the very simple trajectories $p(t) \equiv p_0$, $q(t) = H_0'(p)t + q_0$; in particular, they are stable: $p(t)$ remains constant throughout the evolution of the system. We can ask whether of this stability is lost when ϵ is not zero, but small. A particular case of this question is :

Are the orbits of the planets in the solar system stable? Is it possible that, over very long times, the orbits of some of the planets change greatly?
There are two important stability theorem: The Kolmogorov - Arnold - Moser theorem proven in 1963 and the Nekhoroshev theorem proven in 1979. Let us be more precise.

We define the complex domain D as follows: let B_R be the ball of radius R around the origin, then :

$$D = \left\{ (q,p) \in \mathbb{C}^{2n},\ dist(p, B_R) \leq \rho;\ |Im(q)| \leq \sigma \right\}$$

with $|Im(q)| = \sup_{1 \leq i \leq n} |Im(q_i)|$. Note that real part of D is nothing but $B_{R+\rho} \times \mathbb{R}^n$.

Theorem 0.0.1 (KAM Theorem). *Let $H(q,p,\epsilon) = H_0(p) + \epsilon H_1(q,p)$ be an Hamiltonian such that:*

1. *H_0, H_1 are real analytic on D;*

2. *$p \mapsto H_0(p)$ is strictly convex, i.e. for each $p \in \mathbb{R}^n$ the second partial vertical derivate $\frac{\partial^2 H_0}{\partial p^2}(p)$ is defined strictly positive, as a quadratic form;*

3. *$H_1(q+k,p) = H_1(q,p)$ for each $k \in \mathbb{Z}^n$.*

Then the following happens: there are $\epsilon_0 > 0$ and $\tilde{c} > 0$ such that for $\epsilon \in (0, \epsilon_0)$ we can find a set $I \subset B_{R+\rho}$ such that, if $p(0) \in I$, then

$$\|p(t) - p(0)\| \leq \widetilde{c}\sqrt{\epsilon}, \qquad \forall t \in \mathbb{R}.$$

Moreover, $\frac{|I|}{|B_{R+\rho}|} \geq 1 - \widetilde{c}\sqrt{\epsilon}$.

The KAM theorem gives stability for all times ($\|p(t) - p(0)\| \leq \widetilde{c}\sqrt{\epsilon}$, $\forall t$), but only for a set of initial conditions of large measure; the Theorem of Nekhoroshev, on the other side, gives stability for very a long time for all orbits.

Theorem 0.0.2 (Nekhoroshev Theorem)**.** *Let H be as above. Then there is $\tilde{c} > 0$ such that for any initial condition $(p(0), q(0))$, with $p(0) \in B_R$, one has*

$$\|p(t) - p(0)\| \leq \widetilde{c}\,\epsilon^{\frac{1}{2n}}, \qquad for\ |t| \leq \exp(\frac{\widetilde{c}}{\epsilon^{2n}})$$

provided ϵ is small enough.

Both theorems do not exclude the existence of an orbit $(q(t), p(t))$ such that

$$\|p(t) - p(0)\| \geq 1, \qquad for\ |t| \gg 1.$$

V. I. Arnold has found an example (see [9]) of a perturbation of an integrable Hamiltonian System in which there are orbits with the property above. But in a general Hamiltonian System, what is the behavior of an orbit whose initial condition is not in the set I? What regions of phase space can it visit? Aubry - Mather theory tries to give an answer. We are not going to make a survey of this theory, but we shall concentrate on some relations between the Lax-Oleinik semigroup, the sub-solutions of the Hamiltonian-Jacobi equation and the Aubry set. We shall define these objects for a general Lagrangian on $\mathbb{T}^n \times \mathbb{R}^n$; our aim is to prove that there is a $C^{1,1}$ function $u : \mathbb{T}^n \to \mathbb{R}$ such that the Lagrangian $L(q, \dot{q}) - \langle d_q u, \dot{q} \rangle$ reaches its minimum exactly on the Aubry set.

We will divide our work in five chapters:

Chapter 1: Preliminaries and Notation

In the first chapter, we shall introduce the hypotheses for the Lagrangian L defined on $\mathbb{T}^n \times \mathbb{R}^n$:

- L is C^r, with $r \geq 2$;
- For each $x \in \mathbb{T}^n$ the function $v \mapsto L(x, v)$ is strictly convex, i.e. $\forall\ (x, v) \in \mathbb{T}^n \times \mathbb{R}^n$ the second partial vertical derivate $\frac{\partial^2 L}{\partial v^2}(x, v)$ is defined strictly positive, as a quadratic form;

- $L(x,v)$ is superlinear in v uniformly in x, i.e.

$$\forall x \in T^n, \qquad \lim_{||v|| \to \infty} \frac{L(x,v)}{||v||} = +\infty.$$

We say that a Lagrangian $L : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Tonelli Lagrangian if it satisfies the conditions above.

Then we shall recall some properties of Legendre's transform and the relation between the Lagrangian flow and the Hamiltonian one. We shall also introduce the important notion of minimizer curve for the Lagrangian action:

Definition 0.0.3 (Minimizer Curve)**.** *Let $A.C.([t_0,t_1],\mathbb{T}^n)$ denote the class of absolutely continuous functions from $[t_0,t_1]$ to $\mathbb{T}^n$. Let $\widetilde{\gamma} : [t_0,t_1] \to \mathbb{T}^n$ an absolutely continuous curve, we say that $\tilde{\gamma}$ is a minimizer for the Lagrangian action if*

$$\int_{t_0}^{t_1} L(\widetilde{\gamma}(s), \dot{\widetilde{\gamma}}(s))ds =$$
$$= \min \left\{ \int_{t_0}^{t_1} L(\gamma(s),\dot{\gamma}(s))ds : \gamma \in A.C.([t_0,t_1],T^n), \gamma(t_0) = x_0, \gamma(t_1) = x_1 \right\}.$$

A natural question: given two times $t_0 < t_1 \in \mathbb{R}$ and two generic points $x_0, x_1 \in \mathbb{T}^n$, can we find a minimizer curve $\widetilde{\gamma} : [t_0,t_1] \to \mathbb{T}^n$ that connects $x_0 = \tilde{\gamma}(t_0)$ with $x_1 = \tilde{\gamma}(t_1)$? In the next chapter, we shall see that the answer to the question is positive.

Chapter 2: Tonelli's Theorem and Regularity of the minimizer curves

In the first section of this chapter, we shall prove Tonelli's theorem:

Theorem 0.0.4 (Tonelli's Theorem)**.** *Let $L : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Tonelli Lagrangian. Then, given $x_0, x_1 \in \mathbb{T}^n$ and $t_0 < t_1 \in \mathbb{R}$, there exists an absolutely continuous curve $\widetilde{\gamma} : [t_0,t_1] \to \mathbb{T}^n$ which minimizes the Lagrangian action among the absolutely continuous curves connecting $\widetilde{\gamma}(t_0) = x_0$ with $\widetilde{\gamma}(t_1) = x_1$.*

We are going to give a new proof of theorem above, due to P. Bernard. The main idea of the proof is the following: to an absolutely continuous curve $\gamma : [t_0,t_1] \to \mathbb{T}^n$ we associate a measure μ_γ, which is the push-forward of Lebesgue by γ. In other words, if f is a continuous function on $[t_0,t_1] \times \mathbb{T}^n \times \mathbb{R}^n$, we have

$$\int_{[t_0,t_1]\times\mathbb{T}^n\times\mathbb{R}^n} f \, d\mu_\gamma = \int_{t_0}^{t_1} f(s,\gamma(s),\dot{\gamma}(s))ds.$$

Let γ_k be a minimizing sequence for the Lagrangian action, i.e. let $\gamma_k : [t_0,t_1] \to \mathbb{T}^n$ be a sequence of absolutely continuous curves such that $\gamma_k(t_0) = x_0$, $\gamma_k(t_1) = x_1$ and

$$\int_{t_0}^{t_1} L(\gamma_k(s),\dot{\gamma}_k(s))ds \xrightarrow{k} \inf \left\{ \int_{t_0}^{t_1} L(\gamma(s),\dot{\gamma}(s))ds : \gamma \in A.C., \gamma(t_0) = x_0, \gamma(t_1) = x_1 \right\}.$$

To γ_k we associated the measure μ_k as above.
We shall introduce a suitable topology on the space of measures, and see that

1. μ_k converges, up to subsequences, to a measure μ;

2. $I(\mu) := \int L d\mu \leq \int L d\mu_k$, i.e. I is lower semicontinuous;

3. μ is a measure corresponding to a minimizer curve $\tilde{\gamma}$.

In the second section, we shall show that any minimizer curve is a C^r solution of the Euler-Lagrangian equation; we remark that this fact is false for general time dependent Lagrangian, as shown in [4].

Chapter 3: The Lax-Oleinik Semigroup

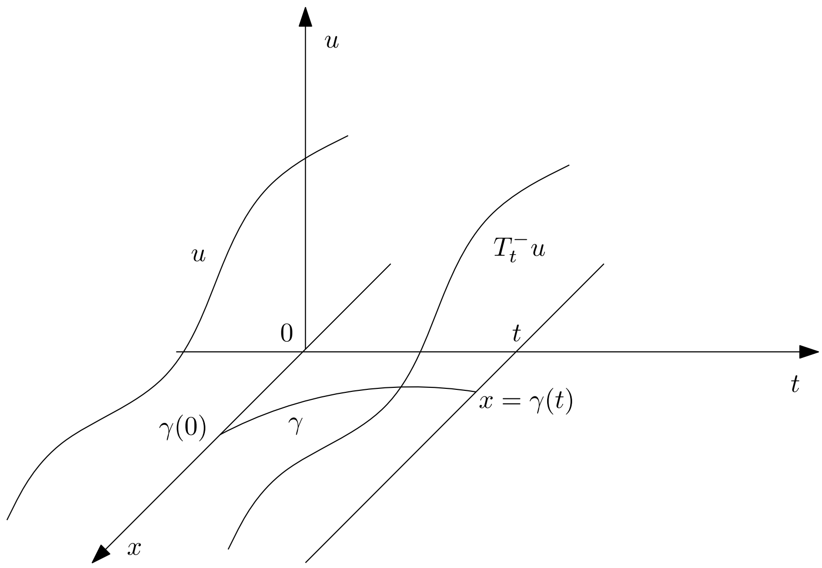
In the third chapter, we shall initially introduce a semigroup of non-linear operators $(T_t^-)_{t\geq 0}$ from
$C(\mathbb{T}^n, \mathbb{R})$ into itself. This semigroup is well known in PDE and in the Calculus of Variations, and it is called the Lax-Oleinik semigroup. To define it let us fix $u \in C(\mathbb{T}^n, \mathbb{R})$ and $t > 0$. For $x \in \mathbb{T}^n$, we set

$$T_t^- u(x) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds \right\},$$

where the infimum is taken over all the absolutely continuous curves $\gamma : [0, t] \to \mathbb{T}^n$ such that $\gamma(t) = x$. Moreover we define the Lax-Oleinik semigroup under time reversal by

$$T_t^+ u(x) = \sup_{\gamma} \left\{ u(\gamma(t)) - \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds \right\},$$

where the supremum is taken over all the absolutely continuous curves $\gamma : [0, t] \to \mathbb{T}^n$ such that $\gamma(0) = x$.
The figure shows how a function u at time 0 is brought into a functions $T_t^- u$ at time t.



By Tonelli's theorem we have that the infimum (resp. supremum) above is a minimum (resp. maximum); moreover the Lax-Oleinik semigroup satisfies the following properties:

1. Each $T_t^{\pm}$ maps $C(\mathbb{T}^n, \mathbb{R})$ into itself.

2. **(Semigroup Property)** We have $T_{t+t'}^{\pm} = T_t^{\pm} \circ T_{t'}^{\pm}$, for each $t, t' > 0$.

3. **(Monotonicity)** For each $u, v \in C(\mathbb{T}^n, \mathbb{R})$ and $t > 0$, we have

$$u \leq v \;\Rightarrow\; T_t^{\pm} u \leq T_t^{\pm} v.$$

4. If c is a constant and $u \in C(\mathbb{T}^n, \mathbb{R})$ we have $T_t^{\pm}(c + u) = c + T_t^{\pm} u$.

5. **(Non-expansiveness)** The maps $T_t^{\pm}$ are non-expansive

$$\forall\, u, v \in C(\mathbb{T}^n, \mathbb{R}),\ \forall t \geq 0,\ \left\| T_t^{\pm} u - T_t^{\pm} v \right\|_{\infty} \leq \| u - v \|_{\infty}$$

6. For each $u \in C(\mathbb{T}^n, \mathbb{R})$, we have $\lim_{t \to 0} T_t^{\pm} u = u$.

7. For each $u \in C(\mathbb{T}^n, \mathbb{R})$, the map $t \mapsto T_t^{\pm} u$ is uniformly continuous.

8. For each $u \in C(\mathbb{T}^n, \mathbb{R})$, the function $(t, x) \mapsto T_t^{\pm} u(x)$ is continuous on $[0, +\infty) \times \mathbb{T}^n$ and locally Lipschitz on $(0, +\infty) \times \mathbb{T}^n$. In fact, for each t_0, the family of functions $(t, x) \mapsto T_t^{-} u(x)$, $u \in C(\mathbb{T}^n, \mathbb{R})$, is equi-Lipschitz on $[t_0, +\infty) \times \mathbb{T}^n$.

We say that u is a sub-solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$ (HJ), if u is Lipschitz and $H(x, d_x u) \leq c$ at every point x where $d_x u$ is defined; we shall show that

- u is a sub-solution of $H(x, d_x u) = c$ if and only if $u \leq T_t^{-} u + ct$.

We shall see that there is a number $c[0] \in \mathbb{R}$ such that the Hamilton-Jacobi equation has critical sub-solutions if $c \geq c[0]$, and it has not if $c < c[0]$. We say that a sub-solution of $H(x, d_x u) = c[0]$ is a critical sub-solution.
The value $c[0]$ has a more dynamic characterization:

$$c[0] = -\min_{\mu} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x, v)\, d\mu(x, v),$$

where μ varies among Borel probability measures on $\mathbb{T}^n \times \mathbb{R}^n$ invariant for the Euler-Lagrange flow.
In the last part of the chapter, we shall see two statements, that will be fundamental for the proof of Fathi and Siconolfi theorem in chapter five.

1. For each $t > 0$ and each $u \in C(\mathbb{T}^n, \mathbb{R})$ we have that $T_t^{-} u$ (resp. $T_t^{+} u$) is semi-concave (resp. semi-convex).

2. A function $u : \mathbb{T}^n \to \mathbb{R}$ is both semi-convex and semi-concave if and only if it is $C^{1,1}$.

Chapter 4: Conjugate Functions and the Aubry set

We sketch the definition of the Aubry set: we have seen that

$$c[0] = -\min_{\mu} \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,v) d\mu(x,v), \tag{1}$$

where μ varies among Borel probability measures on $\mathbb{T}^n \times \mathbb{R}^n$ invariant for the Euler-Lagrange flow.
Let μ be a minimal measure and let ϕ_s denote the Lagrangian flow; by the ergodic theorem we have that for μ a.e. initial condition (x,v),

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \left(L(\phi_s(x,v)) + c[0] \right) ds = 0.$$

The points of the projected Aubry set $\mathcal{A}_0 \subset \mathbb{T}^n$ enjoy a stronger property:
we say that $x \in \mathcal{A}_0$ if there is a sequence $t_n \to +\infty$ and a sequence of absolutely continuous curves $\gamma_n : [0, t_n] \to \mathbb{T}^n$ such that $\gamma_n(0) = \gamma_n(t_n) = x$,

$$\int_0^{t_n} \left(L(\gamma_n(s), \dot{\gamma}_n(s)) + c[0] \right) ds \xrightarrow{n} 0.$$

This set can be proven to be not empty, and it play an important role in Aubry-Mather theory.
We hall see an alternative characterization: the projected Aubry set is the set of points $x \in \mathbb{T}^n$ such that $H(x, d_x u) = c[0]$ for every critical C^1 sub-solution u of the (HJ) equation; where a critical C^1 sub-solution is simply a C^1 function u such that $H(y, d_y u) \leq c[0]$ for each $y \in \mathbb{T}^n$. By this definition, we can see easily that if $x \in \mathcal{A}_0$ and u_1, u_2 are two critical C^1 sub-solutions, we have that $d_x u_1 = d_x u_2$; thus we can define in a natural way the Aubry set $\tilde{\mathcal{A}}_0 \subset \mathbb{T}^n \times \mathbb{R}^n$ by

$$\tilde{\mathcal{A}}_0 := \{(x, d_x u) \mid x \in \mathcal{A}_0\}$$

where u is any critical C^1 sub-solution of (HJ) equation.
We shall see that $\tilde{\mathcal{A}}_0$ is compact set invariant by the Hamiltonian flow; actually, all orbits in the Aubry set are minimizer curves. Moreover, this set contains the support of all the minimal measures μ of (1).

Chapter 5: Fathi-Siconolfi Theorem

The goal of this chapter is to prove the theorem of Fathi and Siconolfi. We say that a sub-solution u is strict on the open set $U \subset \mathbb{T}^n$ if there exists a continuous non negative function $V : \mathbb{T}^n \to \mathbb{R}$ which is positive on U and such that u is also a sub-solution of the equation $H(x, d_x u) + V(x) = c$.

Theorem 0.0.5 (Fathi Siconolfi Theorem)**.** *There exists a critical $C^{1,1}$ sub-solution of the (HJ) equation which is strict outside of the projected Aubry set.*

Our result is optimal in the sense that we shall see an example where there is an unique critical sub-solution, which is $C^{1,1}$, but not C^2. We briefly sketch the proof: to begin with, we shall prove that there exists a critical sub-solution u which is strict outside of the projected Aubry set; actually, we shall see that there exists a continuous non negative function $V : \mathbb{T}^n \to \mathbb{R}$ which is positive outside of $\mathcal{A}_0$ and such that u is a sub-solution of the equation

$$H(x, d_x u) + V(x) = c[0]. \tag{2}$$

Then we fix $t > 0$ and we shall see that $T_t^+ u$ is a sub-solution of (2); moreover, we saw before that $T_t^+ u$ is semi-convex. Now we consider $T_\epsilon^- T_t^+ u$ and we shall show that it is still a sub-solution of (2); again it is semi-concave by standard properties of the Lax-Oleinik semigroup. Moreover, we shall prove that, if ϵ is small enough, $T_\epsilon^- T_t^+ u$ is semi-convex too. To end the proof, it is sufficient to recall that a function which is both semi-concave and semi-convex is $C^{1,1}$. Hence $T_\epsilon^- T_t^+ u$, that we denote with w, is a critical $C^{1,1}$ sub-solution which is strict outside of the projected Aubry set; therefore w satisfies

$$H(x, d_x w) = \begin{cases} = c[0] & \text{if } (x, d_x w) \in \tilde{\mathcal{A}}_0, \\ \\ < c[0] & \text{otherwise.} \end{cases}$$

Passing to the Lagrangian through Legendre's transform, we obtain that

$$L_{d_x w}(x, v) := L(x, v) - \langle d_x w, v \rangle = \begin{cases} = c[0] & \text{if } (x, v) \in \tilde{\mathcal{A}}_0, \\ \\ > c[0] & \text{otherwise.} \end{cases}$$

We saw before that the Aubry set is invariant and all the orbits in this set are minimizer curves for the Lagrangian L. The formula above shows that every orbit in the Aubry set is minimizing for the Lagrangian action $L_{d_x w}$ in a trivial way; but since $\langle d_x w, v \rangle$ is an exact differential 1-form, $L_{d_x w}$ and L have the same set of minimizer curves.

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