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Closure operations and star operations in commutative rings

Candidato
Dario Spirito

Relatore
Prof. Marco Fontana

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CONTENTS

INTRODUCTION

A *closure operation* is a map f between the elements of a partial ordered set that verifies three axioms: extension ($x \leq f(x)$), order-preservation ($x \leq y \implies f(x) \leq f(y)$) and idempotence ($f(f(x)) = f(x)$ for every x). The most known closure operation is perhaps the operation of closure between sets of a topological space; another common operation is the one that assigns to a subset of an algebraic structure (a group, a ring, a vector space...) the smallest substructure that contains the subset. Moreover, almost every field of mathematics has some construction that can be seen as a closure operation between structures: algebra has integral closure of rings and ring completion (of Noetherian local rings), topology has compactifications and completion of metric spaces, analysis has completion of measure spaces.

This thesis is about closure operation in the partial ordered set of ideals of a commutative unitary ring R.

Due to their generality, these closure operations does not satisfy many ring-theoretic properties, and thus they have rarely been the subject of a general theory: however, single closures (like integral closure or tight closure) and smaller sets of closures (such as star operations) have been studied in detail. The case of star operations is somewhat emblematic: many of their basic properties are in fact valid for wider class of closures, that of semi-prime closure operations (neither requiring different proofs), but the concept was instead generalized to semistar operations.

Recently, some authors have found useful to consider closure operations as an autonomous subject; new definitions has been given, pursuing generalizations of properties of previously known closures, or trying to understand the structure of some subsets of the set of closure operations. However, the various fields of study are still very far, partly because of different assumptions and problems, partly because of very different techniques.

The thesis is divided into four chapters, each one narrowing down the subject: the first is dedicated to arbitrary closure operation, trying to identify some general properties; the second to a special class of closures, star operations, and the third to closures arising from overrings of the original ring, focusing on those induces by localizations. The last one deals with one specific closure, integral closure, and with its links to two other operations, which can be seen as variants: complete integral closure and tight closure.

All rings will be assumed to be commutative and with unity; in Chapters 2 and 3, and in the most of Chapter 4, we will consider only integral domains. We will not assume (if not for specific results) that the rings are Noetherian, as many definitions and theorems become trivial in the Noetherian context; in fact, Sections 1.6 and 3.5 study to what extent some properties of Noetherian rings can be transferred to some kind of non-Noetherian ones.

Chapter 1 is dedicated to general closure operation, to their set and to three properties that they can have: finite type, semi-primality, and c-finiteness of ideals.

The set $C(R)$ of closure operations is a very big set, due to the very low requirement for a map to be a closure; it can be naturally identified as a subset of the power set of $\mathcal{I}(R)$ (the set of ideals of R) by the application that send a closure c to the set of c-closed ideals (I is said to be c-closed, or a c-ideal, if $I = I^c$); the image of this map is constituted by the sets of ideals closed by arbitrary intersections. This correspondence gives a natural partial ordering on $C(R)$; however, since having few closed ideals means that the closure of an ideal I is usually big (with respect to I), we reverse the ordering, and say that a closure c is smaller than d if the set of c-ideals contains the set of d-ideals or, in another way, if $I^c \subseteq I^d$ for every ideal I. On the contrary, $C(R)$ fails to be a monoid, because the composition of two closure operation is not always idempotent, even on very simple rings.

Finite type closure operations are closures whose behaviour is determined by the finitely generated ideals; closures of finite type are usually much more similar to the identity then the others, in the sense that all the information about the closure of an ideal I can be recovered by the closures of finitely generated ideals: there is no "jump" between finitely and non-finitely generated ideals. Every closure operation can be modified into a new closure, which is of finite type and agree with the original one on finitely generated ideals.

Semi-primality is probably the most general property that uses effectively the ring structure of R: a closure c is *semi-prime* if $x \cdot I^c \subseteq (xI)^c$ for every $x \in R$ and every ideal I. This property, for example, is connected to the structure of the c-spectrum of R, that is, the set of prime ideals that are also c-ideals: if c is also of finite type, many classical results (existence of maximal ideals, primality of maximal ideals, representation of an ideal as intersection of its extensions in localizations) can be expanded to c-ideals, giving existence of c-maximal ideals (i.e., maximal elements of the set of c-ideals), primality of c-maximal ideals, representation of an ideal as intersection of its extensions in the localizations at c-maximal ideals.

Semi-primality is a very natural concept: many constructions (analysed in more detail in Chapter 3, Section 3.1) yield naturally semi-prime operations. On the other hand, semi-prime operation are a natural generalization of star operations (which are defined by the equality $x \cdot I^c = (xI)^c$), and in fact the properties cited above are usually proved in the star operation setting, although the proofs need almost no change to adapt them to the semi-prime case. Moreover, some closures (for example integral closure or tight closure) happens to be star operations only in certain rings (typically integrally closed ones), but they are always semi-prime, and thus this more general setting permits to use some techniques also when the closure fails to be a star operation.

c-finiteness of an ideal is a more strict form of the finite type property, although it is local (on a single ideal) rather than global (on all ideals). An ideal I is c-finite if its closure I^c is also the closure of a finitely generated ideal; when this happens, for the study of c we have that I can be (almost) considered finitely generated. The condition that all the ideals are c-finite is a much more specific condition than c being of finite type: for example, the identity is always of finite type, but an ideal is c-finite only if it is finitely generated. In fact, those rings where all ideals are c-finite (or rather strictly c-finite) are somewhat "close" to being Noetherian; the subject is more deeply studied in Chapter 3, Section 3.5. When R is Noetherian, an analogue subject is to understand how much elements are needed to generate an ideal J such that $I^c = J^c$ (where I is a previously fixed ideal).

Chapter 2 discusses the main properties of star operations, whose theory is a well-known part of multiplicative ideal theory since the works of Krull and Gilmer.

The defining property of star operations can be seen as a form of "translation by multiplication": for every $x \in R$ and every ideal $I \trianglelefteq R$, we have that $x \cdot I^c = (xI)^c$. When restricted to integral domains, it naturally leads to the idea of multiplying not only by elements of R, but also by elements of the quotient field K; since in this way it is possible to make the ideals no more contained in R, the concept of fractional ideal is introduced as R-submodules of K that can be multiplied into R (i.e., $J \subseteq K$ for which there is a $y \in R$ such that $yJ \subseteq R$). Every star operation can be uniquely extended as a closure on the set of fractional ideals (if we insist that the closure verify $x \cdot I^c = (xI)^c$ even for $x \in K$ and fractional ideals I), and moreover it is the biggest set such that said extension is unique.

Maybe the most important star operation is the v-operation, also called *divisorial closure*: it can be defined either as the intersection of all principal fractional ideals containing I, or as the double dual $(R :_K (R :_K I))$ of I. Its importance relies mainly on the fact that v is the biggest star operation, thus giving an explicit bound for the other star operations. Section 2.3 proves the equivalence between the two definitions and gives a condition (Proposition 2.17, with a more explicit special case as Proposition 2.19) for an ideal I to not be contained in any other divisorial ideal contained in R.

The next two sections focuses on the concepts of $\star$-invertibility and of $\star$-class group, especially in the case when $\star$ is of finite type. For the former, the results that can be obtained are very closely analogous to those valid for the notion of invertibility, leading to the definition of the class of *Prüfer $\star$-multiplication domains* as a generalization of the class of Prüfer domain; for the class group, the picture is much less clear, especially if $\star$ is not taken to be equal to the identity or to t (the finite type closure associated to v). Even in this last case, the analogy with the identity is not perfect: for example, an homomorphisms $\phi : R \longrightarrow S$ (or even an inclusion) does not always induce a map between the corresponding t-class groups. However, the t-class group is relevant when considering conditions equivalent to certain properties of factorization: for example, unique factorization domains are those Krull domains whose t-class group vanishes.

In the end, we analyse v-invertibility. The criterion assumes a different form with respect to other $\star$-invertibilities: I is v-invertible if and only if $(I :_K I) = R$. This leads to the notion of *completely integrally closed rings*, as the rings where each ideal is v-invertible, and to the notion of *complete integral closure* of a ring, as the union $\bigcup\{(I :_K I) \mid I \text{ is an ideal of } R\}$, which can be seen as an extension of the usual notion of integral closure (where the union ranges only among finitely generated ideals); moreover, just like integral closure can be defined through equations of linear dependence, the complete integral of R can be seen as the set of elements such that $cx^n \in R$ for all $n \in \mathbb{N}$ and for an element $c \in R$. However, complete integral closure is much less well-behaved than integral closure: for example, there are rings for which the complete integral closure is not completely integrally closed, i.e., complete integral closure is not always idempotent.

Chapter 3 is mainly about closure operation induced by a family of rings, that is, closures c that can be written as $I^c = \bigcap IS \cap R$, where S ranges among a (given) family of rings containing R. Although not every star operation can be constructed this way, closures of this type provides a wide set of examples that are usually simpler and more "regular" than an arbitrary closure operation: for example, for these closure it is always true that a c-ideal is contained in a prime c-ideal, even if c is not of finite type. (In this last case, however, it could be that c-maximal ideals do not exist.)

We begin by two even more general constructions: the first uses homomorphisms from R (dropping the condition that R is contained in the rings) and closure operations also on the image, while the second uses modules. Usually, these construction yield semi-prime operations, but more rarely star operations: for example, to have that the closure $I^c = \bigcap IS \cap R$ is a star operation we must suppose that $\bigcap S = R$.

We proceed by giving some properties of closures induced by a family of rings, successively shifting to the case when each of these rings is a localization of R: these are called *spectral operations*. Spectral operations have been more thoroughly studied than closures induced by a general family of rings, mostly because of their characterization as closures that distributes over finite intersection (along with another condition: see Proposition 3.10), and because of the possibility to assign to every star operation of finite type $\star$ a spectral star operation $\star_w$, which is in many case simpler but close enough to the original $\star$. The construction of $\star_w$ is detailed in Section 3.4.

We also prove (Proposition 3.13) a characterization of finite type spectral operations (among all the spectral operations) in terms of the compactness of the $\star$-spectrum in the Zariski topology inherited from $\mathrm{Spec}(R)$.

In the last two sections of the chapter, we continue the investigation of two subjects: operations c that satisfies the ascending chain condition on c-ideals (called c-Noetherian) and construction of star operation.

c-Noetherian rings are called this way because some theorems, classically proved for Noetherian rings, can be carried over to them, although we usually have to restrict to the set of c-ideals. However, the properties of the closure c are important to determine how many properties we can transfer: the stronger results are obtained when c is a spectral star operation, due to the fact that, in this case, each localization at a c-prime ideal is Noetherian. This imply that, as a rule of thumb, if a theorem about Noetherian rings depends only on the local structure of the ring, then it can be transferred to c-ideals of c-Noetherian domains: this is the case, for example, of the Principal Ideal Theorem and of Krull Intersection Theorem.

The last section shows how to build new star operations from an old one and from prime ideals not fixed by it: this gives, for example, a bound on the number of non-divisorial prime ideals, in the case that R has only a finite number of star operations. Moreover, in this way it is sometimes possible to count the number of spectral operations on a ring.

Chapter 4 deals with integral closure of ideals and with two variations, complete integral closure and tight closure.

Integral closure of ideals is an old concept, first considered by Krull, which extends the notion of integral closure of rings. Just like the integral closure of R is the set of elements of its quotient ring that verify a monic polynomial equation $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with coefficients in R, the integral closure of an ideal $I \trianglelefteq R$ is the set of elements of R that verify an analogous equation, but with each $a_i \in I^i$.

Integral closure is linked to many topics in commutative algebra: this leads to a great number of different views on the subject, often with a new (but equivalent) definition, and often mirroring what happens for integral closure of rings. We prove the equivalence of the above definition with two other different approaches: Definition 4.1 (via Proposition 4.10) and Proposition 4.34.

The latter shows that, like the integral closure of R is equal to $\bigcup(J :_K J)$, where the union ranges among all the finitely generated ideals $J \trianglelefteq R$, the integral closure of an ideal I is equal to $\bigcup(IJ :_R J)$, with J varying in the same set. This equivalence leads to a more general class of closures, called Δ-closures.

The former is the analogue of the fact that the integral closure of a domain R is equal to the intersection of all valuation rings contained between R and its field of fractions K: the integral closure of I is equal to the intersection $\bigcap IV \cap R$ (where the V are the valuation overrings of K). However, the set of valuation overrings is not the unique that can be used to obtain integral closure: any suitable set is said to be a *b-set*. We study what are sufficient conditions for R to have a b-set composed by discrete valuation rings: in addition to the "classical" theorem that this is true if R is Noetherian, we prove this for domains that have an integral extension which is locally Noetherian (Propositions 4.14 and 4.18).

Next we introduce the concept of complete integral closure of an ideal, similarly to complete integral closure of rings: an element x is said to be in the complete integral closure I^{cic} of I if there is an element $c \in R$, $c \neq 0$ such that $cx^n \in I^n$. This definition, although a natural generalization of both complete integral closure of rings and integral closure of ideals, has received little attention: perhaps the only result is the old theorem stating that, if R is a Noetherian domain, I^{cic} coincides with the integral closure of I, just like it happens for the complete integral closure and the integral closure of R (as a ring). We extend this result to every ring such that integral closure of ideals can be obtained only by Noetherian valuation rings. However, it is not known if complete integral closure is idempotent and, therefore, a closure operation; since the complete integral closure of a ring is not necessarily completely integrally closed, it can be expected that idempotence fails also in this case.

Tight closure is a more recent closure, developed in the context of Noetherian rings. Unlike the other closures, it is only defined when the characteristic of the ring is a prime number $p > 0$: an element x is in the tight closure of I if there is an element $c \in R$, $c \neq 0$, such that $cx^{p^e} \in I^{[p^e]}$ for every $e \geq 1$

(where $I^{[n]}$ is the ideal generated by the nth powers of the elements of I). For Noetherian rings, there is a very deep theory of tight closure, which links it to regular rings, regular sequences (and thus Cohen-Macaulay rings) and homological results; if R is non-Noetherian, little is known, just like for complete integral closure, and it is entirely possible that tight closure fails, in general, to be idempotent.

1. GENERAL PROPERTIES OF CLOSURE OPERATIONS

1.1 The definition

All rings will be commutative and unitary; $\mathcal{I}$ (or $\mathcal{I}(R)$ if needed for clarity) will denote the set of ideals of R.

Definition 1.1. *A map $c : \mathcal{I} \to \mathcal{I}$ ($I \mapsto I^c$) is a* closure operation *if, for every $I, J \in \mathcal{I}$,*

1. $I \subseteq I^c$ (extension)*;*
2. $I \subseteq J \Longrightarrow I^c \subseteq J^c$ (order-preservation)*;*
3. $(I^c)^c = I^c$ (idempotence)*.*

*If $I = I^c$, we say that I is a c-*closed ideal*, or, briefly, a c-*ideal*; the set of c-ideals is denoted by $\mathcal{I}^c$.*

Proposition 1.2. *Let R be a ring, c a closure operation, $\{I_\alpha \mid \alpha \in A\}$ a nonempty set of ideals of R.*

1. $$I^c = \bigcap \{J \in \mathcal{I}^c \mid I \subseteq J\}. \tag{1.1}$$
2. $\left(\bigcap_{\alpha \in A} I_\alpha^c\right)^c = \bigcap_{\alpha \in A} I_\alpha^c$*. In particular, $\mathcal{I}^c$ is closed under intersections; that is, if all the I_α are c-closed, then $\bigcap_{\alpha \in A} I_\alpha$ is c-closed as well.*
3. $\left(\sum_{\alpha \in A} I_\alpha^c\right)^c = \left(\sum_{\alpha \in A} I_\alpha\right)^c$.

Proof. 1. Let $\mathcal{A}$ be the set of c-ideals that contain I. By order-preservation, $I \subseteq J$ implies that $I^c \subseteq J^c = J$ for every $J \in \mathcal{A}$; it follows that I^c is in the intersection.

For the reverse containment, we observe that I^c is in $\mathcal{A}$, because $I \subseteq I^c$ by the extension property. Than we have $\bigcap_{J \in \mathcal{A}} J \subseteq I^c$.

2. ($\supseteq$) is the extension property; for ($\subseteq$), we note that $\bigcap I_\alpha^c \subseteq I_\beta^c$ for every $\beta \in A$; by order preservation and idempotence, $(\bigcap I_\alpha^c)^c \subseteq \left(I_\beta^c\right)^c = I_\beta^c$; intersection on all the βs,
$$\left(\bigcap_{\alpha\in A} I_\alpha^c\right)^c \subseteq \bigcap_{\beta\in A} I_\beta^c = \bigcap_{\alpha\in A} I_\alpha^c. \tag{1.2}$$
Suppose that $I_\alpha = I_\alpha^c$ for every α; then, by the above part of the proposition,
$$\left(\bigcap_{\alpha\in A} I_\alpha\right)^c = \left(\bigcap_{\alpha\in A} I_\alpha^c\right)^c = \bigcap_{\alpha\in A} I_\alpha^c = \bigcap_{\alpha\in A} I_\alpha \tag{1.3}$$
is c-closed.

3. The ($\supseteq$) inclusion is a consequence of the extension property: $I_\alpha \subseteq I_\alpha^c$ for every α, and the containment passes to the sum and to the closure. For ($\subseteq$), $I_\beta \subseteq \sum I_\alpha \Longrightarrow I_\beta^c \subseteq (\sum I_\alpha)^c$; summing on all the βs,
$$\sum_{\beta\in A} I_\beta^c \subseteq \left(\sum_{\alpha\in A} I_\alpha\right)^c \Longrightarrow \left(\sum_{\beta\in A} I_\beta^c\right)^c \subseteq \left(\left(\sum_{\alpha\in A} I_\alpha\right)^c\right)^c = \left(\sum_{\alpha\in A} I_\alpha\right)^c \tag{1.4}$$
which is the thesis.

□

We note that the proof of the above proposition uses nothing except the fact that $\sum I_\alpha$ and $\bigcap I_\beta$ are, respectively, the smallest ideal containing all the I_α and the biggest contained in every I_β; the same properties would be fulfilled if $\mathcal{I}$ is replaced by any *complete lattice* (a partial ordered set in which every subset has an infimum and a supremum), and a closure operation is defined just as in our definition but replacing the set containment with the order relation.

This explains also why it is not possible to obtain any non-trivial property linking a generic closure operation to other ideal-theoretic notions such as multiplication of ideals: for this, we have to restrict to a smaller class of closure operations, namely those called *semi-prime*, which will be studied in Section 1.5.

1.1.1 The set of c-ideals

The first point of the above proposition implies that the set $\mathcal{I}^c$ is uniquely determined by c: if d is another closure operation such that $\mathcal{I}^c = \mathcal{I}^d$, then $\bigcap\{J \in \mathcal{I}^c \mid I \subseteq J\} = \bigcap\{J \in \mathcal{I}^d = \mathcal{I}^c \mid I \subseteq J\}$, and so $I^c = I^d$ for every I, that is, $c = d$.

Moreover, we can construct a closure operation c from any subset $\mathcal{A} \subseteq \mathcal{I}$ containing R just defining $I^c = \bigcap\{J \in \mathcal{A} \mid I \subseteq J\}$, where the closed ideals become the intersections of any subset of $\mathcal{A}$; we can call this the closure operation *generated* by $\mathcal{A}$. If $\mathcal{B} \subseteq \mathcal{I}$ is closed under intersections (and contains R), then it is the set of closed ideals of a closure operation: this is in particular true if $\mathcal{B}$ is totally ordered and contains its minimum $\bigcap_{J\in\mathcal{B}} J$.

This shows that we have an abundance of closure operations, even on very simple rings: if x is a non-zerodivisor, then every subset X of $\mathbb{N}^* = \{1, 2, \dots, n, \dots\}$ generate a different closure operation, where the closed ideals are the (x^m) for $m \in X$ (plus, if needed, the intersection $\bigcap_{n\geq 1}(x^n)$).

This point of view can also be used to transfer closure operation from R to the quotient rings R/I and back; we will see a generalization of this, along with other method to construct closure operations, in Section 3.1.

1.2 Examples

1.2.1 Basic examples

There are two closures which are both trivial and extremal: the first is the *identity*, that fixes every ideal, while the other is the *indiscrete closure*, that sends every ideal to the unit ideal R.

The simplest non-trivial closure operation is probably the *radical*, indicated with $\mathrm{rad}(I)$ or $\sqrt{I}$. It could be defined either as $\mathrm{rad}(I) = \{\alpha \in R \mid \alpha^n \in I \text{ for some } n \in \mathbb{N}\}$ or

$$\mathrm{rad}(I) = \bigcap\{P \in \mathrm{Spec}(R) \mid I \subseteq P\} \tag{1.5}$$

making it an example of the previous construction, with $\mathcal{A} = \mathrm{Spec}(R)$.

1.2.2 Integral closure

Another very used closure operation is the *integral closure*: for every ideal I, I^- is the set of elements r for which there exist a $n \in \mathbb{N}$ and elements $a_i \in I^i$

(for $1 \leq i \leq n$) such that

$$r^n + a_1 r^{n-1} + \cdots + a_n = 0. \tag{1.6}$$

There are many different equivalent characterizations of integral closure; for Noetherian rings, $r \in I^-$ if and only if there is an element c, not contained in any minimal prime of R, such that for every $n \in \mathbb{N}^*$ we have $cr^n \in I^n$. In non-Noetherian context, the elements that satisfies this definition forms an ideal, called the *complete integral closure* of I, but this may or may not be a closure operation.

Integral closure is studied in more detail in Chapter 4, along with some condition for the equality between it and complete integral closure.

1.2.3 Characteristic p closures

Some closure operations works on rings of characteristic $p > 0$, p being a prime number: the main ones are *Frobenius closure* and *tight closure*. For every ideal I and every natural number n, the *bracket power* $I^{[n]}$ of I is the ideal generated by the nth powers of the elements of I; it is generally much smaller then the ordinary power I^n.

The *Frobenius closure* I^F of I is the set of elements such that $x^{p^e} \in I^{[p^e]}$ for some $e \in \mathbb{N}$.

If R is a domain, an element r is in the *tight closure* $I^\star$ of I if there exist an element $c \neq 0$ such that $cx^{p^e} \in I^{[p^e]}$ for every $e \geq 1$. If R is not a domain, c is required not to be in any minimal prime of R.

Frobenius closure is always a closure operation, while tight closure, in non-Noetherian rings, suffers the same problems of complete integral closure. It is, however, studied exclusively in the Noetherian context.

Tight closure has been studied mainly in connection with regular sequences and system of parameters; it is linked to regularity and to the Cohen-Macaulay property, and also to integral closure by the alternative definition of the latter in Noetherian rings. The "good" properties and the many uses of tight closure in characteristic p have generated attempts to extend it in characteristic 0, and to find new closure operations analogous to it.

Some of the properties of tight closure are described in Section 4.4.

1.2.4 The v-operation

In the context of multiplicative ideal theory, the most important closure operation is the v-operation: if R is a domain, K its quotient field, then I^v is defined to be the intersection of all principal R-submodules of K. From the v-operation we can define, among others, the t and the w-operations.

Any invertible ideal is a v-ideal, and so the v-operation coincides with the identity on Dedekind domains and (but only for finitely-generated ideals) on Prüfer domains. The v-operation, and operations related to it (like t and w), can so be used to study, along various generalizations of Prüfer domains, Krull domains and properties of factorization.

1.3 The set of closure operations

For any ring R, we define $C(R)$ to be the set of the closure operations on R. It has a natural structure as a partial ordered set:

Definition 1.3. *Let $c, d \in C(R)$. We say that $c \leq d$ if $I^c \subseteq I^d$ for every ideal $I \trianglelefteq R$; equivalently, if $\mathcal{I}^d \subseteq \mathcal{I}^c$.*

The equivalence can be proved as follows: if $I^c \subseteq I^d$ for every ideal, and I is d-closed, then $I \subseteq I^c \subseteq I^d = I$, and so I is c-closed; conversely, if every d-ideal is a c-ideal, then

$$I^c = \bigcap \{J \in \mathcal{I}^c \mid I \subseteq J\} \subseteq \bigcap \{J \in \mathcal{I}^d \mid I \subseteq J\} = I^d. \tag{1.7}$$

$C(R)$ has an infimum (the identity closure) and a supremum (the indiscrete closure). Also, every subset of $C(R)$ has an infimum and a supremum: given a collection $\{c_\lambda \mid \lambda \in \Lambda\}$, we can define $\inf\{c_\lambda\}$ as the operation generated by the union $\bigcup_{\lambda\in\Lambda} \mathcal{I}^{c_\lambda}$, while $\sup\{c_\lambda\}$ will be the operation generated by the intersection $\bigcap_{\lambda\in\Lambda} \mathcal{I}^{c_\lambda}$. In the latter case, $\bigcap_{\lambda\in\Lambda} \mathcal{I}^{c_\lambda}$ is precisely the collection of closed ideals, since as every $\mathcal{I}^{c_\lambda}$ is closed under intersections, so is their intersection.

The infimum has also a simple interpretation in term of ideals: given a set $\{c_\lambda \mid \lambda \in \Lambda\}$ of closure operations, if we denote by c its infimum, we have

$$I^c = \bigcap_{\lambda\in\Lambda} I^{c_\lambda}. \tag{1.8}$$

This follows because, rewriting the intersection using (1.1), we have

$$\bigcap_{\lambda\in\Lambda} I^{c_\lambda} = \bigcap_{\lambda\in\Lambda} \bigcap_{\substack{J\in\mathcal{I}^{c_\lambda} \\ J\supseteq I}} J = \bigcap_{J\supseteq I} \bigcap_{\substack{\lambda\in\Lambda \\ J\in\mathcal{I}^{c_\lambda}}} J = \bigcap_{\substack{J\supseteq I \\ J\in\mathcal{C}}} J, \tag{1.9}$$

where $\mathcal{C} = \bigcup_{\lambda\in\Lambda} \mathcal{I}^{c_\lambda}$; that is, the closure operation generated by the set $\bigcup_{\lambda\in\Lambda} \mathcal{I}^{c_\lambda}$, namely c.

For the supremum, however, there is no such simple description. Indeed, if we put

$$I^c = \sum_{\lambda\in\Lambda} I^{c_\lambda}, \tag{1.10}$$

there is no guarantee that we obtain a closure operation; if, for example, R is a discrete valuation ring with maximal ideal $M = (p)$, put

$$(p^n)^{c_1} = \begin{cases} (p^2) & \text{if } n \geq 2 \\ R & \text{if } n = 1 \end{cases} \qquad (p^n)^{c_2} = (p) \text{ if } n \geq 1$$
$$(0)^{c_1} = (0)^{c_2} = (0). \tag{1.11}$$

Then, in this case,

$$\begin{gathered}(p^2)^c = (p^2)^{c_1} + (p^2)^{c_2} = (p^2) + (p) = (p) \quad \text{while} \\ (p)^c = (p)^{c_1} + (p)^{c_2} = R + (p) = R\end{gathered} \tag{1.12}$$

and c is not idempotent.

There is one case in which we can, however, describe the supremum:

Proposition 1.4. *Let R be a Noetherian ring and $\{c_\lambda\}$ is a* directed *set of closure operation (that is, for any $\lambda_1, \lambda_2 \in \Lambda$ exists $\mu \in \Lambda$ such that $c_{\lambda_1} \leq \mu$ and $c_{\lambda_2} \leq c_\mu$). Then*

$$I^c = \sum_{\lambda \in \Lambda} I^{c_\lambda} \tag{1.13}$$

is a closure operation, the supremum of the set $\{c_\lambda\}$.

Proof. It is clear that $I \subseteq I^c$ (because $I \subseteq I^{c_\lambda}$ for every λ) and that c is order-preserving (because $I \subseteq J$ implies $I^{c_\lambda} \subseteq J^{c_\lambda}$).

For idempotence, let J be an ideal and $J^c = (f_1, \ldots, f_n)$. For every f_i, there exists a λ_i such that $f_i \in J^{c_{\lambda_i}}$; by directedness, we can find $\mu \in \Lambda$ such that $c_{\lambda_i} \leq c_\mu$ for every i; then

$$J^c = (f_1, \ldots, f_n) \subseteq J^{c_{\lambda_1}} + J^{c_{\lambda_2}} + \cdots + J^{c_{\lambda_n}} \subseteq J^{c_\mu} \subseteq J^c \tag{1.14}$$

so that for every J, $J^c = J^{c_\mu}$ for some $\mu \in \Lambda$.

Pick an ideal I: there are $\lambda_1, \lambda_2 \in \Lambda$ such that $I^c = I^{c_{\lambda_1}}$, $(I^c)^c = (I^c)^{c_{\lambda_2}}$. For directness, we found a $\lambda \in \Lambda$ bigger then λ_1 and λ_2; with this choice,

$$(I^c)^c = (I^c)^{c_{\lambda_2}} = (I^{c_{\lambda_1}})^{c_{\lambda_2}} \subseteq (I^{c_\lambda})^{c_\lambda} = I^{c_\lambda} \subseteq I^c \tag{1.15}$$

and thus (by the extensive property) $(I^c)^c = I^c$.

To show that it is the supremum, it is sufficient to check that $\mathcal{I}^c = \bigcap_{\lambda \in \Lambda} \mathcal{I}^{c_\lambda}$. If $I \in \mathcal{I}^c$, then

$$I = I^c = \sum_{\lambda \in \Lambda} I^{c_\lambda} \supseteq \sum_{\lambda \in \Lambda} I = I \tag{1.16}$$

and $I = I^{c_\lambda}$ for every λ. Conversely, if $I \in \mathcal{I}^{c_\lambda}$ for every λ, then

$$I^c = \sum_{\lambda\in\Lambda} I^{c_\lambda} = \sum_{\lambda\in\Lambda} I = I \tag{1.17}$$

and I is c-closed. $\square$

The above proposition does not hold when R is not Noetherian, even if we suppose that the set $\{c_\lambda\}$ is totally ordered. For example, let R be a valuation ring with value group $\mathbb{Q}$ and valuation v, and let $q_1, \dots, q_n, \dots$ be a strictly decreasing sequence of rational numbers with limit 0; let $I_n := \{x \in R \mid v(x) \geq q_n\}$, $I := \{x \in R \mid v(x) > 0\}$ the maximal ideal of R, and define c_n to be the closure operation

$$J^{c_n} := \begin{cases} I_n & \text{if } J \subseteq I_n \\ J & \text{if } I_n \subsetneq J \subsetneq I \\ R & \text{if } I \subseteq J \end{cases} \tag{1.18}$$

Clearly $c_n < c_{n+1}$ for every n, because $\{q_n\}$ is decreasing; let d be the map $J \mapsto J^d := \sum_{n\in\mathbb{N}} I^{c_n}$. For every ideal $J \subsetneq I$, $J \subseteq I_n$ for some n, because, for any $x \in I \subseteq J$, $v(x) > 0$ and thus $v(x) < q_n$ for some n; moreover, $\bigcup_{n\geq 1} I_n = I$ and thus $J^d = (I_n)^d = I$. But $I^{c_n} = R$ for every c_n, because $I_n \subsetneq I$, and thus $I^d = R$; hence d is not idempotent.

1.3.1 The composition of two closure operations

Unlike for partial ordering, the set $C(R)$ has not a nice multiplicative structure: in general, the composition of two closure operations, although extensive and order-preserving, needs not to be idempotent, and so it fails to be a closure operation. Again the example (1.3) works: if $c = c_2 \circ c_1$,

$$\begin{gathered} (p^2)^c = ((p^2)^{c_1})^{c_2} = (p^2)^{c_2} = (p) \quad \text{while} \\ (p)^c = ((p)^{c_1})^{c_2} = R^{c_2} = R \end{gathered} \tag{1.19}$$

so that c is not idempotent.

Define cd to be the composition $d \circ c : I \mapsto I^{cd} = (I^c)^d$. If $c \leq d$ or $d \leq c$, then cd equals the bigger of the two: in the first case,

$$I^{cd} = (I^c)^d \subseteq (I^d)^d = I^d \tag{1.20}$$

because $I^c \subseteq I^d$ and d is idempotent; moreover,

$$I^d \subseteq (I^c)^d = I^{cd} \tag{1.21}$$

by the extension property of c, and so $I^{cd} = I^d$. If $d \leq c$, we have

$$I^{cd} = (I^c)^d \subseteq (I^c)^c = I^c \subseteq (I^c)^d = I^{cd}, \tag{1.22}$$

and hence $I^{cd} = I^c$.

This is not a necessary condition: for example, let K_1, K_2 be fields and $R = K_1 \times K_2$. Then R has four ideals: (0), the two maximals $M_1 = K_1 \times \{0\}$, $M_2 = \{0\} \times K_2$ and the whole R. Defining

$$I^c = \begin{cases} M_1 & \text{if } I \subseteq M_1 \\ R & \text{otherwise} \end{cases} \qquad I^d = \begin{cases} M_2 & \text{if } I \subseteq M_2 \\ R & \text{otherwise} \end{cases} \tag{1.23}$$

neither $c \leq d$ nor $d \leq c$, but

$$\begin{aligned} (0)^{cd} &= M_1^d = R \\ M_1^{cd} &= M_1^d = R \\ M_2^{cd} &= R^d = R \end{aligned} \tag{1.24}$$

and cd is just the indiscrete closure. However, here is a criterion:

Proposition 1.5. *Let c, d be closure operations on R. cd is a closure operation if and only if d takes the set $\mathcal{I}^c$ of c-ideals in itself.*

Proof. If cd is a closure operation, then, for every c-closed ideal I, we have $I^{cd} = I^d$ and $(I^{cd})^{cd} = I^{cd}$; but

$$(I^d)^c = I^{dc} \subseteq (I^c)^{dc} \subseteq ((I^c)^{dc})^d = I^{cdcd} = I^{cd} = I^d \tag{1.25}$$

and I^d is c-closed.

Suppose now that $d : \mathcal{I} \longrightarrow \mathcal{I}$ restricts to a map $d : \mathcal{I}^c \longrightarrow \mathcal{I}^c$. Then I^{cd} is a c-ideal and a d-ideal; it follows that $(I^{cd})^{cd} = ((I^{cd})^c)^d = (I^{cd})^d = I^{cd}$ and cd is idempotent, hence a closure operation. □

The compositions allows to express, in the Noetherian case, the supremum of two closure operations in terms of ideals. In the next proposition, we define $(cd)^n$ to be the composition of cd with itself n times.

Proposition 1.6. *Let R be a Noetherian ring and c, d two closure operations on R. Then the supremum e of $\{c, d\}$ is given by*

$$I^e = \bigcup_{n \geq 1} I^{(cd)^n}. \tag{1.26}$$

Proof. The map e is clearly extensive and order-preserving; we have to show that it is idempotent.

We observe that for every ideal J, there is an integer $n = n(J)$ such that $J^e = J^{(cd)^n}$: this follows at once because $J^{(cd)^k} \subseteq J^{(cd)^l}$ for $k \leq l$ and the ring is Noetherian, so the chain stabilizes.

Let I be an ideal and n, m such that $I^e = I^{(cd)^n}$ and $(I^e)^e = (I^e)^{(cd)^m}$. Then

$$(I^e)^e = (I^e)^{(cd)^m} = (I^{(cd)^n})^{(cd)^m} = I^{(cd)^{n+m}} = I^{(cd)^n} = I^e. \tag{1.27}$$

It follows from the definition that I^c and I^d are contained in I^e, so $c, d \leq e$ and $\mathcal{I}^e \subseteq \mathcal{I}^c \cap \mathcal{I}^d$. But now if I is both a c-ideal and a d-ideal then $I^{cd} = I$ and $I^{(cd)^n} = I$, so that I is e-closed and $\mathcal{I}^c \cap \mathcal{I}^d \subseteq \mathcal{I}^e$. This shows that $e = \sup\{c, d\}$. □

1.4 Closure operations of finite type

Definition 1.7. *A closure operation c is of* finite type *if, for every ideal I,*

$$I^c = \bigcup \{J^c \mid J \subseteq I \text{ and } J \text{ is finitely generated}\}. \tag{1.28}$$

The radical and integral closure are of finite type, just like the identity and the indiscrete closure; on the contrary, the v-operation is not, in general, of finite type. On a Noetherian ring, every operation is of finite type.

If c is a closure operations, we define

$$I^{c_f} = \bigcup \{J^c \mid J \subseteq I \text{ and } J \text{ is finitely generated}\}. \tag{1.29}$$

Proposition 1.8. *Let c be a closure operation.*

1. *c_f is a closure operation.*
2. *$c_f \leq c$ and $I^{c_f} = I^c$ for every finitely generated ideal I.*
3. *If $c \leq d$, then $c_f \leq d_f$.*
4. *c_f is of finite type; moreover, c_f is the largest closure operation d of finite type such that $d \leq c$.*

Proof. 1. Extension: if $x \in I$, $(x) \subseteq I$ is finitely generated, hence $x \in (x)^c \subseteq I^{c_f}$.

Order-preservation: if $I \subseteq J$, every finitely generated ideal contained in I is contained in J, and $I^{c_f} \subseteq J^{c_f}$.

Idempotence: if $x \in (I^{c_f})^{c_f}$, there is a $J \subseteq I^{c_f}$ finitely generated such that $x \in J^c$; suppose $J = (a_1, \ldots, a_n)$. Then, for every i, $a_i \in I^{c_f}$, and so $a_i \in K_i^c$ for finitely generated ideals $K_i \subseteq I$; it follows that $J \subseteq K_1^c + \cdots + K_n^c$, and

$$J^c \subseteq (K_1^c + \cdots + K_n^c)^c = (K_1 + \cdots + K_n)^c \tag{1.30}$$

and $x \in (K_1 + \cdots + K_n)^c$, where $K_1 + \cdots + K_n$ is contained in I and finitely generated, which implies $x \in I^{c_f}$.

2. $J^c \subseteq I^c$ for every ideal I; so every ideal in the union is contained in I^c, and $I^{c_f} \subseteq I^c$.

   If I is finitely generated, I^c is in the union, and $I^c \subseteq I^{c_f}$.

3. If $J \subseteq I$ and J is finitely generated, $J^c \subseteq J^d$; taking the union among all such J we have $I^{c_f} \subseteq I^{d_f}$.

4. Let $c' = (c_f)_f$: by the definition we have

$$I^{c'} = \bigcup \{J^{c_f} \mid J \subseteq I \text{ and } J \text{ is finitely generated}\} \tag{1.31}$$

   but $J^{c_f} = J^c$ for finitely generated ideals J, and $I^{c'} = I^{c_f}$.

   Suppose $d \leq c$ and d is of finite type. Then

$$\begin{aligned} I^d = \bigcup \{J^d \mid J \subseteq I \text{ and } J \text{ is finitely generated}\} \subseteq \\ \subseteq \bigcup \{J^c \mid I \subseteq J \text{ and } J \text{ is finitely generated}\} = I^{c_f} \end{aligned} \tag{1.32}$$

   because $J^d \subseteq J^c$; hence $d \leq c_f$.

□

Proposition 1.9. *Let $c_1, \ldots, c_n$ be a finite number of closure operations of finite type. Then the infimum $c := \inf\{c_1, \ldots, c_n\}$ is of finite type.*

Proof. Let I be any ideal and $x \in I^c$. By (1.8), $I^c = \bigcap_{i=1}^{n} I^{c_i}$; thus $x \in I^{c_i}$ for every i, and since the c_i are of finite type there are finitely generated ideals $H_1, \ldots, H_n \subseteq I$ such that $x \in H_i^{c_i}$. Set $H := H_1 + \ldots + H_n$: then H is finitely generated and contained in I, and, for every i, $x \in H^{c_i}$ because $H_i^{c_i} \subseteq H^{c_i}$. Then $x \in \bigcap_{i=1}^{n} H^{c_i} = H^c$, and c is of finite type. □

On the contrary, the infimum of an infinite number of finite type closure operations need not to be of finite type; an example is given at the end of Section 3.3.

1.5 Semi-prime operations

Definition 1.10. *A closure operation c is* semi-prime *if $xI^c \subseteq (xI)^c$ for every ideal I and for every $x \in R$. The set of semi-prime closure operations on R is denoted by $S(R)$.*

Proposition 1.11. *Let c be a closure operation. The following are equivalent:*

1. *c is semi-prime.*
2. *$IJ^c \subseteq (IJ)^c$ for every pair I, J of ideals.*
3. *$(IJ)^c = (I^cJ^c)^c$ for every I, J.*

Proof. ($1 \iff 2$) If (2) is true, then c is seen to be semi-prime by taking $I = (x)$. Conversely, let I be an ideal; then

$$IJ^c = \sum_{i \in I} iJ^c \subseteq \sum_{i \in I} (iJ)^c \subseteq \left(\sum_{i \in I} (iJ)^c \right)^c. \tag{1.33}$$

By Proposition 1.2, part 3, we have

$$\left(\sum_{i \in I} (iJ)^c \right)^c = \left(\sum_{i \in I} (iJ) \right)^c = (IJ)^c \tag{1.34}$$

and $IJ^c \subseteq (IJ)^c$.

($2 \iff 3$) If c verifies (2), then (applying it with I^c and J, and subsequently with J and I)

$$(I^cJ^c)^c \subseteq ((I^cJ)^c)^c = (I^cJ)^c \subseteq (IJ)^c \tag{1.35}$$

while $(IJ)^c \subseteq (I^cJ^c)^c$ by order-preservation; so $(IJ)^c = (I^cJ^c)^c$.

Suppose $(IJ)^c = (I^cJ^c)^c$. Then, by extension,

$$IJ^c \subseteq I^cJ^c \subseteq (I^cJ^c)^c = (IJ)^c, \tag{1.36}$$

and c is semi-prime. $\square$

The last characterization allows to define, for every semi-prime operation c, an operation $\times^c$ which makes $\mathcal{I}^c$ a monoid, by putting $I \times^c J = (IJ)^c$. In particular, the associative property follows because

$$\begin{aligned}(I \times^c J) \times^c K = ((IJ)^cK)^c = ((IJ)K)^c = (IJK)^c = \\ = (I(JK))^c = (I(JK)^c)^c = I \times^c (J \times^c K)\end{aligned} \tag{1.37}$$

In the context of star operations, this construction permits to define a class group associated to a given operation, in analogy to the "classical" class group, which could be seen as arising from the identity operation. (See Section 2.4 for more details.)

Semi-primality is the first property that is ideal-theoretic in nature; it is not in general possible to derive IJ just by the partial order on $\mathcal{I}$. For the following proposition, we introduce the ideal

$$(I :_R J) := \{x \in R \mid xJ \subseteq I\}. \tag{1.38}$$

Being semi-prime implies some nice properties:

Proposition 1.12. *Let c be a semi-prime closure operation on R, I, J ideals of R, W a multiplicatively closed subset of R.*

1. *$(I :_R J)^c \subseteq (I^c :_R J)$; if I is c-closed, then so is $(I :_R J)$, and $(I^c :_R J)$ is c-closed for every I.*

2. *$(I^c :_R J^c) = (I^c :_R J)$.*

Suppose moreover that R is Noetherian and I is c-closed.

3. *$W^{-1}I \cap R$ is c-closed.*

4. *The minimal primary components of I are c-closed.*

5. *If I has no embedded components, then it has a primary composition by c-closed ideals.*

Proof. 1. For every $x \in (I :_R J)^c$, we have $xJ \subseteq (I :_R J)^c J \subseteq (J(I :_R J))^c \subseteq I^c$, and $x \in (I^c :_R J)$.

If $I = I^c$, then $(I :_R J)^c \subseteq (I^c :_R J) = (I :_R J)$ and $(I :_R J)$ is c-closed; in particular, for every ideal I, this is true for $(I^c :_R J)$.

2. $J \subseteq J^c$, so $(I^c :_R J^c) \subseteq (I^c :_R J)$ for an arbitrary closure operation c. If, moreover, c is semi-prime and $x \in (I^c :_R J)$, then $xJ^c \subseteq (xJ)^c \subseteq (I^c)^c = I^c$ and $x \in (I^c :_R J^c)$, so that $(I^c :_R J^c) = (I^c :_R J)$.

3. Let $J = W^{-1}I \cap R = (a_1, \dots, a_n)$ (R is Noetherian, so J is finitely generated); for every a_i there is a $w_i \in W$ such that $w_i a_i \in I$; let $w := w_1 \cdots w_n$.

Then $wJ \subseteq I$ and $J \subseteq (I :_R w)$, which is c-closed by part (1). But if $x \in (I :_R w)$, then $xw \in I \subseteq W^{-1}I \cap R = J$, and $(I :_R w) \subseteq J$. Hence J is c-closed.

4. Let $I = Q_1 \cap \cdots \cap Q_n$ be a primary decomposition of I, where $Q_1, \ldots, Q_m$ ($m \leq n$) are the isolated components (which are unique). Now $Q_i R_{P_i} \cap R = Q_i$, because each Q_i is P_i-primary; moreover, $IR_{P_i} = Q_i R_{P_i}$ because Q_i is the only primary component contained in P_i (by its minimality). Hence Q_i is c-closed.

5. If I has no embedded components, $m = n$, every Q_i is c-closed, and so I has a primary decomposition by c-ideals.

□

Semi-primality verifies some stability properties:

Proposition 1.13. *Let $\{c_\lambda\}_{\lambda\in\Lambda}$ a set of semi-prime closure operations.*

1. $\inf\{c_\lambda\}$ *is semi-prime.*
2. *If $I^d := \sum_{\lambda\in\Lambda} I^{c_\lambda}$ is a closure operation, then it is semi-prime.*
3. *If c_1, c_2 are semi-prime and the composition $c_1 c_2$ is a closure operation, then it is semi-prime.*

Proof. 1. Let $c := \inf\{c_\lambda\}$, and pick any $x \in R$ and any ideal I. We have

$$xI^c = x\bigcap_{\lambda\in\Lambda} J^{c_\lambda} = \bigcap_{\lambda\in\Lambda} xJ^{c_\lambda} \subseteq \bigcap_{\lambda\in\Lambda} (xJ)^{c_\lambda} = (xJ)^c \tag{1.39}$$

and thus c is semi-prime.

2. In the same way,

$$xJ^c = x\sum_{\lambda\in\Lambda} J^{c_\lambda} = \sum_{\lambda\in\Lambda} xJ^{c_\lambda} \subseteq \sum_{\lambda\in\Lambda} (xJ)^{c_\lambda} = (xJ)^c. \tag{1.40}$$

3. We have

$$xJ^{c_1c_2} = x(J^{c_1})^{c_2} \subseteq (xJ^{c_1})^{c_2} \subseteq ((xJ)^{c_1})^{c_2} = (xJ)^{c_1c_2} \tag{1.41}$$

and c_1c_2 is semi-prime.

□

Although the composition of two semi-prime operations, as the composition of two general closures, need not to be a closure operation, in some simple cases it is possible to "control" the behaviour of the composition:

Proposition 1.14. *[46, Proposition 3.6] Let R be a Dedekind domain. The set $S(R)$ of semi-prime closure operation on R is the union of the two sub-monoids*

$$M_0 := \{c \in S(R) \mid (0)^c = (0)\} \tag{1.42}$$

and

$$M_f := \{c \in S(R) \mid (0)^c \neq (0)\}. \tag{1.43}$$

Moreover, if $c \in M_0$ and $d \in M_f$, then $c \circ d \in M_f$.

For the statement of the next proposition we need two definitions: (c is a closure operation).

Definition 1.15. $c-\mathrm{Spec}(R) := \{P \in \mathrm{Spec}(R) \mid P^c = P\}$*; its elements are called* c-primes.

Definition 1.16. $c-\mathrm{Max}(R)$ *is the set of maximal proper c-ideals (that is, c-ideals M such that $I^c = R$ for every $I \supsetneq M$); its elements are called* c-maximals.

In general, $c-\mathrm{Spec}(R)$ and $c-\mathrm{Max}(R)$ could be empty, and c-maximal ideals need not to be prime. For semi-prime operations, however, some of these properties hold.

Proposition 1.17. *Let c be a semi-prime operation.*

1. *c-maximal ideals are prime.*
2. *If c is of finite type, every proper c-ideal is contained in a c-maximal ideal.*
3. *If I is a c-ideal and P a minimal prime of I, P is a c_f-ideal.*
4. *Suppose R is a domain and c is of finite type. Then, for every c-ideal I,*

$$I = \bigcap_{M \in c-\mathrm{Max}(R)} IR_M. \tag{1.44}$$

Proof. 1. Suppose $xy \in I$ and I is maximal among proper c-ideals. If both x and y are not in I, we have $(I,x)^c = R = (I,y)^c$, because (I,x) and (I,y) are bigger than I; but $(I,x)(I,y) \subseteq (I,xy) = I$ and $((I,x)(I,y))^c \subseteq I$, while

$$((I,x)(I,y))^c = ((I,x)^c(I,y)^c)^c = R^c = R \tag{1.45}$$

which is impossible. Hence I is prime.

2. By Zorn lemma, is sufficient to proof that, if $\{I_\alpha\}$ is an ascending chain of c-ideals, then $I := \bigcup I_\alpha$ is a c-ideal.

   Let $J = (a_1, \ldots, a_n) \subseteq I$; there are $\alpha_1, \ldots, \alpha_n$ such that $a_i \in I_{\alpha_i}$, and an α such that $a_i \in I_\alpha$ for every i; hence $J^c \subseteq I_\alpha^c = I_\alpha \subseteq I$, and

   $$I^c = I^{c_f} = \bigcup \{J^c \mid J \subseteq I \text{ and } J \text{ is finitely generated}\} \subseteq I, \quad (1.46)$$

   so that $I^c = I$.

3. Let $J = (a_1, \ldots, a_m)$ be a finitely generated ideal contained in P. Passing to R_P, we see that PR_P is minimal over IR_P, and $\mathrm{rad}(IR_P) = PR_P \supseteq \mathrm{rad}(JR_P)$; hence there are n_i such that $a_i^{n_i} \in IR_P$, and if $n := n_1 + \cdots + n_m$ we have $J^n R_P = (JR_P)^n \subseteq IR_P$. It follows that there is a $s \in R \setminus P$ such that $sJ^n \subseteq I$, and $(sJ^n)^c \subseteq I^c = I \subseteq P$. But, by semi-primality,

   $$(sJ^n)^c = ((s)J \cdots J)^c = ((s)J^c \cdots J^c)^c = (s(J^c)^n)^c \quad (1.47)$$

   and $s(J^c)^n \subseteq (s(J^c)^n)^c = s(J^n)^c \subseteq P$. Because $s \notin P$ and P is prime, $(J^c)^n \subseteq P$; hence, taking radicals, $J^c \subseteq P$ and

   $$P^{c_f} = \bigcup \{J^c \mid J \subseteq P \text{ and } J \text{ is finitely generated}\} \subseteq P \quad (1.48)$$

   so that P is a c_f-ideal.

4. It is clear that $I \subseteq \bigcap IR_M$; let $x \in \bigcap IR_M$. For every $M \in c-\mathrm{Max}(R)$, we can write $x = \frac{y_M}{z_M}$, where $y_M \in I$ and $z_M \in R \setminus M$; hence, for every M, $z_M \in (I :_R x)$, and thus $(I :_R x) \notin M$ for every $M \in c - \mathrm{Max}(R)$. But this implies $(I :_R x)^c = R$, while $(I : x)^c = (I :_R x)$ by Proposition 1.12; hence $1 \in (I :_R x)$, that is, $x \in I$.

□

Parts 2 and 3 of the above proposition can't be expanded to cover closure operations not of finite type: for example, if R is a non-Noetherian valuation ring of Krull dimension 1, the v-operation is a semi-prime closure operation, but the maximal ideal is not v-closed, and hence there are no v-maximal ideals.

1.6 c-finiteness

When investigating the properties of a closure operation, it is often possible to replace I^c by I, or to prove properties of I^c using I; for example, if c is semi-prime, the product $(I^cJ^c)^c$ is equal to $(IJ)^c$. Hence, we can replace I with any ideal K such that $I^c = K^c$; if K is somewhat "better" than I, it is possible to obtain sharper results.

In the non-Noetherian context, finitely generated ideals are often easier to deal with; so we give the following definition (c is a closure operation):

Definition 1.18. *An ideal I is* c-finite *if there is a finitely generated ideal J such that $J^c = I^c$; it is* strictly c-finite *if there is a $K \subseteq I$ such that $K^c = I^c$.*

The two concept are really different: for example, let R be a local ring with non-finitely generated maximal ideal M, and define I^c to be I unless $I = M$, in which case $M^c = R$. Then M is not strictly c-finite (because $I^c = I \neq R$ for every $I \subsetneq M$) but it is c-finite (because $M^c = R = R^c$). However, the two definitions coincide when c is of finite type:

Proposition 1.19. *Let I be an ideal, c a closure operation. The following are equivalent:*

1. *I is strictly c-finite;*
2. *I is c_f-finite;*
3. *I is strictly c_f-finite.*

Proof. ($1 \iff 3$) We have the following picture:

$$\begin{array}{ccccc} I \subseteq & I^{c_f} \subseteq & I^c \\ \cup\!\! \mid & \cup\!\! \mid & \cup\!\! \mid \\ J \subseteq & J^{c_f} = & J^c \end{array} \tag{1.49}$$

If $I^c = J^c$, then $I^{c_f} \subseteq I^c = J^{c_f} \subseteq I^{c_f}$ and thus $I^{c_f} = J^{c_f}$.

If $I^{c_f} = J^{c_f}$, then $I^c = (I^{c_f})^c = (J^{c_f})^c = J^c$.

($3 \Longrightarrow 2$) is obvious.

($2 \Longrightarrow 3$) Suppose $I^{c_f} = J^{c_f}$, and let $J = (a_1, \ldots, a_n)$. Since $a_i \in I^{c_f}$, there is (for every $i \in \{1, \ldots, n\}$) a finitely generated ideal $H_i \subseteq I$ such that $a_i \in H_i^c = H_i^{c_f}$; thus, if $H := H_1 + \cdots + H_n$, then $H \subseteq I$ and so

$$I^{c_f} \supseteq H^{c_f} = (H_1 + \cdots + H_n)^{c_f} \supseteq ((a_1) + \cdots + (a_n))^{c_f} = (a_1, \ldots, a_n)^{c_f} = J^{c_f} = I^{c_f} \tag{1.50}$$

that is, $I^{c_f} = H^{c_f}$, and I is strictly c_f-finite. $\square$

There is an analogue of the Noetherian property:

Proposition 1.20. *Let R be a ring and c a closure operation. The following are equivalent:*

1. *Every ideal of R is strictly c-finite.*

2. *The set $\mathcal{I}^c$ satisfies the ascending chain condition.*

3. *Every subset of $\mathcal{I}^c$ has a maximal element.*

Moreover, in this case, $c = c_f$.

Proof. ($2 \iff 3$) is a standard property of posets; see e.g. [9, Proposition 6.1].

($1 \implies 2$) Let $\{I_\alpha\}_{\alpha\in A}$ be an ascending chain of c-ideals, and let $I := \bigcup_{\alpha\in A} I_\alpha$. Then I^c is strictly c-finite, and so there are $x_1, \dots, x_n \in I$ such that $(x_1, \dots, x_n)^c = I^c$. But every x_i is contained in a I_{α_i}, and hence there is a $I_{\overline{\alpha}}$ which contains all them. Thus $I^c = (x_1, \dots, x_n)^c \subseteq (I_{\overline{\alpha}})^c \subseteq I^c$, and $(I_{\overline{\alpha}})^c = I^c$.

($2 \implies 1$) If I is not strictly c-finite, then there is sequence $\{x_i\}_{i\in\mathbb{N}} \subseteq I$ such that $x_{i+1} \notin (x_1, \dots, x_i)^c$; but then $\{(x_1, \dots, x_i)^c\}_{i\in\mathbb{N}}$ is an ascending chain of c-ideals which does not stabilizes, against the hypothesis.

If condition (1) is verified, then $I^c = J^c$ for a finitely generated $J \subseteq I$, and thus $I^{c_f} = \bigcup\{H^c \mid H \subseteq I \text{ and } H \text{ is finitely generated}\} \supseteq J^c = I^c$; therefore $I^c = I^{c_f}$ for every I and $c = c_f$. □

We will study in more detail closures that satisfy this property in Section 3.5.

For Noetherian rings, these conditions are obviously uninteresting. In this context, an analogue problem is: given I, what is the minimal n such that $I^c = J^c$, where J is generated by n elements?

Define a *c-minimal reduction* of I as a $J \subseteq I$ such that $J^c = I^c$ and $K^c \neq I^c$ for every $K \subsetneq J$. Minimal reductions need not to exist: for example, if R is a Noetherian domain and c is the radical, then no nonzero ideal has a minimal reduction, since if $\mathrm{rad}(J) = \mathrm{rad}(I)$, then $\mathrm{rad}(J^2) = \mathrm{rad}(J) = \mathrm{rad}(I)$ but $J^2 \subsetneq J$.

Suppose that (R, M) is a Noetherian local ring. A sufficient condition for the existence of minimal reductions for every c-ideal is that, if I and J are ideals such that $J \subseteq I \subseteq (J + MI)^c$, it follows that $I^c = J^c$; closures that satisfy this condition are called *Nakayama closures*, because the fact that the identity verifies it is the statement of Nakayama lemma. Nakayama closures include integral closure, tight closure and Frobenius closure [17, 18]. With this hypothesis, moreover, every minimal generating set of a minimal reduction extends to a minimal generating set of the ideal.

If minimal reductions of I do exist, it is natural to ask if the are generated by the same number of elements; if this happens, and c is Nakayama, we say that I has *c-spread*, indicated with $\ell^c(I)$. In this case, we have also that if K is generated by less than $\ell^c(I)$ elements then $K^c \neq I^c$ [17, Theorem 2.4].

When does every ideal have spread? This is true when c is the integral closure and the residue field R/M is infinite (in this case, the spread coincides with the *analytic spread* of I) [40][34, Proposition 8.3.7], and when c has a so-called *special c-decomposition* (for example when c is Frobenius closure and R/M is perfect, as well as tight closure with certain hypothesis).

1.7 Historical and bibliographical note

Closure operations were first considered, as property of partial ordered sets, by E.H. Moore in 1910 [39], and became part of lattice theory (see e.g. [12]); in this context was born also the idea of the set of closure operations as a lattice [48]. General properties in Sections 1.1 and 1.3 are present there, sometimes as exercises, as well as in [22] (although stated only for star operations – see next chapter). Kirby, in 1969 [36], is probably the first to have used the general concept in ring theory, although other definitions, more ring-theoretic (like Krull's \-operations [37]), were already used.

Alike, semi-prime operations were introduced no later then 1964 [41], but they were used rarely; notably, the term was used in [43] and in [26]. Results and proof of Section 1.5 are taken from [19].

Sections 1.4 and the first half of 1.6 have long been studied in the context of star operations: results of the former and Proposition 1.19 dates at least as back as [22], while Proposition 1.20 appears in [3] and [49]; the proof is almost unchanged.

Nakayama closures were introduced in [17], along with the notion of c-reductions and c-spread; the definition of analytic spread (for integral closure) and the proof of its Nakayama-like property predate it by fifty years [40].

2. STAR OPERATIONS

From now on, if not specified, we will consider only integral domains R with quotient field K.

2.1 Fractional ideals

Definition 2.1. *An R-submodule I of K is a* fractional ideal *of R if there is a $x \in R$, $x \neq 0$ such that $xI \subseteq R$.*

Fractional ideals contained in R are precisely the ideals of R; to emphasize this fact, they are called, among the fractional ideals, *integral ideals*, while fractional ideals are simply called *ideals*; the term "fractional" is sometimes added for emphasis.

Proposition 2.2. *If $E \subseteq I$ are R-submodules of K and I is a fractional ideal, then E is a fractional ideal.*

Proof. There is a $x \in R$ such that $xI \subseteq R$; but $xE \subseteq xI \implies E$ is a fractional ideal. □

The set of fractional ideals is closed under the main ideal-theoretic operations:

Proposition 2.3. *Let $I, J, \{I_\alpha\}_{\alpha \in A}$ be fractional ideals of R. Then $\bigcap_{\alpha \in A} I_\alpha$, $I + J$ and IJ are fractional ideals.*

Proof. For every $\beta \in A$, $\bigcap_{\alpha \in A} I_\alpha \subseteq I_\beta$, so $\bigcap I_\alpha$ is a fractional ideal by the previous proposition.

If $xI \subseteq R$ and $yJ \subseteq R$, then $xy(I + J) = yxI + xyJ \subseteq yR + xR \subseteq R$.

If $xI \subseteq R$, then $IJ = (xI)(x^{-1}J) \subseteq R(x^{-1}J) = x^{-1}J$ that is a fractional ideal. □

As any cyclic R-submodule of K is a fractional ideal, it follows that every finitely generated R-submodule of K is a fractional ideal. Since every fractional ideal is isomorphic to an integral ideal (as an R-module), infinitely generated fractional ideals exist if and only if R is not Noetherian.

We recall that, for I and J ideals of R, we defined

$$(I :_R J) := \{r \in R \mid rJ \subseteq I\}. \tag{2.1}$$

The same operation works if I and J are general R-submodules of K; moreover, we introduce

$$(I : J) := (I :_K J) := \{r \in K \mid rJ \subseteq I\}. \tag{2.2}$$

From the definitions it is immediate to see that $(I :_R J) = (I :_K J) \cap R$.

It is clear that an R-submodule M of K is a fractional ideal if and only if $(R :_R M) \neq (0)$; since, for every submodule $N \neq (0)$, we have always $N \cap R \neq (0)$, this is also equivalent to $(R : M) \neq (0)$. The properties of these operations are summarized by the next proposition.

Proposition 2.4. *Let $I, J, L, \{H_\alpha\}_{\alpha \in A}$ be R-submodules of K, $x \in K$, S a multiplicatively closed subset of R.*

1. $(xI : J) = x(I : J) = (I : x^{-1}J)$.
2. *If $I \subseteq L$ then $(I : J) \subseteq (L : J)$ and $(J : I) \supseteq (J : L)$.*
3. *If I is a fractional ideal and $J \neq (0)$, then $(I : J)$ and $(I :_R J)$ are fractional ideals.*
4. $(\bigcap H_\alpha : J) = \bigcap (H_\alpha : J)$.
5. $(I : \sum H_\alpha) = \bigcap (I : H_\alpha)$.
6. $((I : J) : L) = (I : JL)$.
7. *$S^{-1}(I : J) \subseteq (S^{-1}I : S^{-1}J)$; if J is finitely generated, $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$.*

Proof.

1. $y \in (xI : J) \iff yJ \subseteq xI \iff x^{-1}yJ \subseteq I \iff y \in (I : x^{-1}J)$; moreover, $y \in x(I : J) \iff y = xz$ and $zJ \subseteq I$, which happens if and only if $yJ = xzJ \subseteq xI \iff y \in (xI : J)$, and the three modules are equal.
2. If $xJ \subseteq I$, then $xJ \subseteq L$ and $x \in (L : J)$; the same for the other containment.

3. If $xI \subseteq R$, then $(I : J) = x^{-1}(xI : J) \subseteq (R : J)$; for every $j \in J \setminus \{0\}$, $j(R : J) \subseteq R$, so $(R : J)$ is a fractional ideal and so is $(I : J)$. In addition, $(I :_R J) = (I : J) \cap R \subseteq (I : J)$ is a fractional ideal.

4. $(\subseteq)$: if $x \in (\bigcap H_\alpha : J)$, then $xJ \subseteq \bigcap I_\alpha \subseteq I_\beta$ for every $\beta \in A$, and hence $x \in \bigcap(H_\alpha : J)$.

   $(\supseteq)$: if $x \in \bigcap(H_\alpha : J)$, $xJ \subseteq I_\alpha$ for every $\alpha \in A$ and hence $xJ \subseteq \bigcap I_\alpha$, that is, $x \in (\bigcap H_\alpha : J)$.

5. $(\subseteq)$: if $x \in (I : \sum H_\alpha)$, $x \sum H_\alpha \subseteq I$ and hence $xH_\beta \subseteq I$ for every $\beta \in A$; thus $x \in \bigcap(I : H_\alpha)$.

   $(\supseteq)$: if $x \in \bigcap(I : H_\alpha)$, $xH_\alpha \subseteq I$ for every α, so $x \sum H_\alpha \subseteq I$ and $x \in (I : \sum H_\alpha)$.

6. $(\subseteq)$ If $x \in ((I : J) : L)$, then $xK \subseteq (I : J)$ and thus $xjL \subseteq I$ for every $j \in J$; that is, $xJL \subseteq I$ and $x \in (I : JL)$.

   $(\supseteq)$ If $x \in (I : JL)$, $xJL \subseteq I \Longrightarrow xL \subseteq (I : J) \Longrightarrow x \in ((I : J) : L)$.

7. If $x \in S^{-1}(I : J)$, $x = \frac{y}{s}$ where $y \in (I : J)$ and $s \in S$; for every $j = \frac{k}{t} \in S^{-1}J$,
$$xj = \frac{y}{s}\frac{k}{t} = \frac{yk}{st} \in S^{-1}I \tag{2.3}$$
   because, as $y \in (I : J)$ and $k \in J$, $yk \in I$.

   If $J = (j_1, \ldots, j_n)$ is finitely generated, since localization commutes with finite sums and intersections, we have that
$$(S^{-1}I : S^{-1}J) = (S^{-1}I : (S^{-1}j_1R + \cdots + S^{-1}j_nR)) = \bigcap_{i=1}^{n}(S^{-1}I : j_iS^{-1}R) =$$
$$= \bigcap_{i=1}^{n} j_i^{-1}(S^{-1}I : S^{-1}R) = \bigcap_{i=1}^{n} j_i^{-1}S^{-1}I = \bigcap_{i=1}^{n} S^{-1}(j_i^{-1}I) = \bigcap_{i=1}^{n} S^{-1}(I : j_iR) =$$
$$= S^{-1}\bigcap_{i=1}^{n}(I : j_iR) = S^{-1}(I : J). \tag{2.4}$$

$\square$

2.2 Star operations

Definition 2.5. *Let R be an integral domain. A closure operation $\star$ on R is a* star operation *if, for every $x \in R$ and every integral ideal I, $xI^\star = (xI)^\star$.*

Proposition 2.6. *Let $\star$ be a star operation on R.*

1. *Principal ideals are $\star$-closed.*
2. *If I is a $\star$-ideal, so is xI for every $x \in R$.*
3. *Every star operation is semi-prime.*
4. *Every height 1 prime ideal is $\star_f$-closed.*
5. *If $x \in K$ and $xI \subseteq R$, then $xI^\star = (xI)^\star$.*

Proof.

1. Just take $I = R$ in the definition.
2. $(xI)^\star = xI^\star = xI$, hence xI is $\star$-closed.
3. It is a direct consequence of the definition of semi-prime operation.
4. Let P be an height 1 prime ideal. For every $x \in P$, $x \neq 0$, P is a minimal prime of (x), which is a $\star$-ideal. Hence, since $\star$ is semi-prime, P is $\star_f$-closed by Proposition 1.17.
5. Let $x = y/z$, with $y, z \in R$; then $xI = J \Longrightarrow yI = zJ \Longrightarrow (yI)^\star = (zJ)^\star \Longrightarrow yI^\star = zJ^\star \Longrightarrow J^\star = (y/z)I^\star = xI^\star$.

□

The last part of the proposition motivate the next construction, which is analogous of that in Section 1.1.1.

Let E be an R-module containing R (that is, there is an injective R-module map $R \longrightarrow E$), and let $\mathcal{M}$ be a set of R-submodules of E such that $\mathcal{I} \subseteq \mathcal{M}$. Suppose that $\mathcal{A}$ is a subset of $\mathcal{M}$ and $R \in \mathcal{A}$; then the map

$$I \mapsto I^c = \bigcap \{N \in \mathcal{A} \mid I \subseteq N\} \tag{2.5}$$

(for ideals I) is a closure operation on R: the proof is the same as in Proposition 1.2 and Section 1.1.1, with the only addition that the intersection is contained in R (and hence is an ideal) because $R \in \mathcal{A}$.

If we suppose also that $\mathcal{M}$ is closed under intersections, then the same definition works to define a closure operation *on* $\mathcal{M}$ (that is, an extensive, order-preserving, idempotent map from $\mathcal{M}$ to $\mathcal{M}$), which is an extension of the closure operation on R.

Suppose now that $E = K$ is the quotient field of R. The set of fractional ideals arises naturally as the largest set of submodules where the star operation property extends uniquely:

Proposition 2.7. *Let c be a star operation on R, $\mathcal{F}$ the set of fractional ideals of R, $\mathcal{M}$ the set of R-submodules of K.*

1. *There is a unique closure operation $\star$ which extends c on $\mathcal{F}$ and such that, for every $I \in \mathcal{F}$ and $x \in K$, $xI^\star = (xI)^\star$.*

2. *Suppose that, for every two extensions $\star_1, \star_2$ of c on $\mathcal{M}$ such that $xI^{\star_i} = (xI)^{\star_i}$, $I^{\star_1} = I^{\star_2}$. Then $I \in \mathcal{F}$.*

Proof. 1. For every fractional ideal I, let x be an element of R such that $xI \subseteq R$, and define $I^\star := \frac{1}{x}(xI)^c$. This is the unique way to extend c respecting the star operation property, so we must only show that $\star$ has the desired properties.

$\star$ is well-defined, for if xI and yI are both contained in R, then $(xyI)^c = x(yI^c) = y(xI^c)$ and

$$\frac{1}{x}(xI)^c = \frac{1}{x}\frac{1}{y}(xyI)^c = \frac{1}{y}(yI)^c. \tag{2.6}$$

It is a direct consequence of the properties of c that $\star$ is a closure operation on $\mathcal{F}$; for the other property, suppose $I \in \mathcal{F}$ and $x \in K$, and pick an $r \in R$ such that $rI, rxI \subseteq R$. Then

$$(xI)^\star = \frac{1}{r}(rxI)^c = \frac{1}{r}x(rI)^c = xI^\star. \tag{2.7}$$

2. Let $\star$ be the (unique) extension of c to $\mathcal{F}$. We define

$$I^{\star_1} := \begin{cases} I^\star & \text{if } I \in \mathcal{F} \\ I & \text{otherwise} \end{cases} \qquad I^{\star_2} := \begin{cases} I^\star & \text{if } I \in \mathcal{F} \\ K & \text{otherwise} \end{cases} \tag{2.8}$$

They both extends c, and it is clear that they are both extensive and idempotent (since $I^\star \in \mathcal{F}$ for every $I \in \mathcal{F}$); moreover, they are order-preserving because an R-submodule of K contained in a fractional ideals is itself a fractional ideal.

Take $x \in K$. If $I \in \mathcal{F}$, $xI^{\star_i} = xI^\star = (xI)^\star = (xI)^{\star_i}$ by the first part; if $I \notin \mathcal{F}$, then $xI \notin \mathcal{F}$ and

$$\begin{gathered} xI^{\star_1} = xI = (xI)^{\star_1} \\ xI^{\star_2} = xK = K = (xI)^{\star_2} \end{gathered} \tag{2.9}$$

so that $\star_1$ and $\star_2$ are different extensions of c with the desired property. ∎

It is natural to give the next definition:

Definition 2.8. *A map* $\star : \mathcal{F} \longrightarrow \mathcal{F}$ *($I \mapsto I^\star$) is a* star operation *if*

1. $I \subseteq I^\star$;
2. $I \subseteq J \Longrightarrow I^\star \subseteq J^\star$;
3. $(I^\star)^\star = I^\star$;
4. $R^\star = R$;
5. *for every* $x \in K$, $(xI)^\star = xI^\star$.

Part 1 of Proposition 2.7 can thus be rephrased as follows: every star operation on R admits a unique extension to a star operation on $\mathcal{F}$. Moreover, it is clear that a star operation on $\mathcal{F}$, when restricted to the set of integral ideals, becomes a star operation on R; hence there is a one-to-one correspondence between star operations on R and star operations on $\mathcal{F}$, and we can drop the distinction between the two.

We define the $\star$-ideals to be the fractional ideals I such that $I^\star = I$, and we denote their set by $\mathcal{F}^\star$.

Many of the results and definitions of the previous chapter carries over without differences replacing integral ideals with fractional ideals, and $\mathcal{F}^\star$ with $\mathcal{I}^\star$.

For the former case, the standard technique is to find a x such that xI is an integral ideal, apply the result to xI and then multiplying back by x^{-1}. Sometimes additional care is needed: for example, for part 3 of Proposition 1.2 we must add a new hypothesis, that $\sum I_\alpha$ is fractional ideal. Also, we can expand Proposition 1.12 to cover a new case:

Proposition 2.9. *Let* I, J *be fractional ideals of* R*. Then*

1. $(I :_K J)^\star \subseteq (I^\star :_K J)$.
2. *If* I *is a* $\star$*-ideal, so is* $(I :_K J)$*, and* $(I^\star :_K J)$ *is* $\star$*-closed for every* I.
3. $(I^\star : J) = (I^\star : J^\star)$.

The proof is a verbatim copy of part 1 and 2 of Proposition 1.12.

Likewise, the order relation between star operation can be restated with the set of $\star$-fractional ideals: $\star_1 \leq \star_2$ if and only if $\mathcal{F}^{\star_1} \supseteq \mathcal{F}^{\star_2}$; the construction for the infimum and the supremum requires no modification to adapt them to fractional ideals, as do the definition of star operations of finite type (we will show in Proposition 2.12 that these constructions actually yield star operations).

The next proposition shows a necessary condition for a set to be the set of closed ideals of a star operation.

Proposition 2.10. *Let $\star$ be a closure operation on $\mathcal{F}$. Then $\star$ is a star operation if and only if, for every $I \in \mathcal{F}^\star$ and $x \in K$, $xI \in \mathcal{F}^\star$.*

Proof. The "only if" part is obvious. Suppose $xI \in \mathcal{F}^\star$ for every I and x, and let J be a fractional ideal. We have $J^\star = \bigcap\{I \in \mathcal{F}^\star \mid J \subseteq I\}$, and thus

$$\begin{aligned}(xJ)^\star = \bigcap\{I \in \mathcal{F}^\star \mid xJ \subseteq I\} = \bigcap\{I \in \mathcal{F}^\star \mid J \subseteq x^{-1}I\} = \\ = \bigcap\{xI \in \mathcal{F}^\star \mid J \subseteq I\} = x\bigcap\{I \in \mathcal{F}^\star \mid J \subseteq I\} = xJ^\star \quad (2.10)\end{aligned}$$

because $x\bigcap I_\alpha = \bigcap(xI_\alpha)$. □

There is also a version of this proposition for integral ideals, although more clumsy: $\star$ is a star operation if and only if, for every $I \in \mathcal{I}^\star$ and $x \in K$ such that $xI \subseteq R$, $xI \in \mathcal{I}^\star$.

A form of converse of this statement also holds; the proof is the same of the above proposition.

Proposition 2.11. *Suppose $\mathcal{M}$ is a set of fractional ideals of R such that $R \in \mathcal{M}$ and, for every $J \in \mathcal{M}$ and $x \in K$, $xJ \in \mathcal{M}$. Then the map $I \mapsto \bigcap\{J \in \mathcal{F}^\star \mid I \subseteq J\}$ is a star operation.*

Special $\mathcal{M}$ are (a) the set $\mathcal{P}$ of principal ideals, (b) the set of finitely generated ideals, (c) the set of ideals generated by k or less elements, (d) the set of invertible ideals.

The next proposition is an analogue of Proposition 1.13.

Proposition 2.12. *Let $\{\star_\lambda\}_{\lambda\in\Lambda}$ be a set of star operation on an integral domain R. Then $\inf\{\star_\lambda\}$, $\sup\{\star_\lambda\}$ and $(\star_\lambda)_f$ (for any λ) are star operations.*

Proof. For $\star := \sup\{c_\lambda\}$, the set of $\star$-ideals is $\bigcap \mathcal{F}^{\star_\lambda}$; by Proposition 2.10, $xI \in \mathcal{F}^{\star_\lambda}$ for every $I \in \mathcal{F}^{\star_\lambda}$ and $x \in K$. If $I \in \mathcal{F}^\star$, then $I \in \mathcal{F}^{\star_\lambda} \Longrightarrow xI \in \mathcal{F}^{\star_\lambda}$ for every λ and thus $xI \in \mathcal{F}^\star$.

For $\star := \inf\{c_\lambda\}$, $\mathcal{F}^\star$ is generated by the union of the $\mathcal{F}^{\star_\lambda}$; if $I \in \mathcal{F}^\star$, then $I = \bigcap I_\lambda$ where $I_\lambda \in \mathcal{F}^{\star_\lambda}$; for every $x \in K$,

$$xI = x\bigcap I_\lambda = \bigcap xI_\lambda \quad (2.11)$$

and every $xI_\lambda \in \mathcal{F}^{\star_\lambda}$ because $\star_\lambda$ is a star operation. Hence $xI \in \mathcal{F}^\star$ and $\star$ is a star operation.

Let $\star := \star_\lambda$.

$$\begin{aligned} xI^{\star_f} &= x\bigcup\{J^\star \mid J \subseteq I \text{ and } J \text{ is finitely generated}\} = \\ &= \bigcup\{xJ^\star \mid J \subseteq I \text{ and } J \text{ is finitely generated}\} = \\ &= \bigcup\{(xJ)^\star \mid J \subseteq I \text{ and } J \text{ is finitely generated}\} = \\ &= \bigcup\{(xJ)^\star \mid xJ \subseteq xI \text{ and } J \text{ is finitely generated}\} = \\ &= \bigcup\{K^\star \mid K \subseteq xI \text{ and } K \text{ is finitely generated}\} = (xI)^{\star_f}. \end{aligned} \tag{2.12}$$

Hence $\star_f$ is a star operation. □

2.3 The v-operation

Definition 2.13. *For every fractional ideal I,*

$$I^v := \bigcap\{xR \mid I \subseteq xR\} \tag{2.13}$$

that is, v is the star operation generated by the set of principal ideals. If $I = I^v$, I is also called divisorial. *The t-operation is the star operation of finite type associated to v.*

Since every principal ideal is a $\star$-ideal for any star operation $\star$, $I^\star \subseteq I^v$; that is, v is the largest star operation. To work with it, we need another characterization:

Proposition 2.14. *For any ideal $I \neq (0)$, $I^v = (R : (R : I))$.*

For now, we define $I^c := (R : (R : I))$. To prove that $c = v$, we need two lemmas.

Lemma 2.15. *c is extensive.*

Proof. If $x \in I$, by definition $xi \in R$ for each $i \in (R : I)$; then $x(R : I) \subseteq R$ and $x \in I^c$. □

Lemma 2.16. $(R : I^c) = (R : I)^c = (R : I)$

Proof. $(R : I^c) = (R : I)^c$ because they are both equal to $(R : (R : (R : I)))$. By the last lemma, $(R : I) \subseteq (R : I)^c$; but $(R : I) \supseteq (R : I^c)$ because the map $J \mapsto (R : J)$ reverses the inclusions. Hence $(R : I^c) \subseteq (R : I) \subseteq (R : I)^c = (R : I^c)$ and the three ideals are equal. □

Proof of Proposition 2.14. We will show that v and c are both the largest star operation, and thus are equal.

c is a star operation: extension is one of the lemmas, order preservation follows from the fact that $J \mapsto (R : J)$ reverse inclusions, and we apply it twice. For idempotence,

$$(I^c)^c = (R : (R : I^c)) = (R : (R : I)) = I^c \tag{2.14}$$

by the previous lemma. It is clear that $R^c = R$; if $x \in K$,

$$(xI^c) = (R : (R : xI)) = (R : x^{-1}(R : I)) = x(R : (R : I)) = xI^c \tag{2.15}$$

and hence c is a star operation.

Pick any star operation $\star$. For any ideal I, $(R : I^\star)$ is $\star$-closed and equal to $(R : I)$ by Proposition 2.9 (R is $\star$-closed), and hence

$$(I^\star)^c = (R : (R : I^\star)) = (R : (R : I)) = I^c \tag{2.16}$$

and so $\mathcal{F}^\star \supseteq \mathcal{F}^c$, that is, $\star \leq c$. We have previously noted that v is the largest star operation; thus $v = c$. □

In the next proposition, we give some sufficient conditions to have $I^v = R$, for a given ideal I. We denote by $\mathcal{Z}(A)$ the set of zerodivisors of a ring A.

Proposition 2.17. *Let R be a domain and I an ideal; suppose that there is a $x \in I$ such that $I/(x) \nsubseteq \mathcal{Z}(R/(x))$. Then $I^v = R$.*

Proof. It is sufficient to show that $(R : I) = R$; pick x like in the hypothesis, and suppose there is an $\alpha \in (R : I) \setminus R$. Since $\alpha x \in R$, then $\alpha = \frac{y}{x}$ for a $y \in R$; y is not contained in (x) because, otherwise, $\alpha \in R$. Let z be any element of I that is not a zerodivisor in $R/(x)$; then $zw \notin (x)$ for every $w \in R \setminus (x)$, and thus in particular $zy \notin (x)$. But then $z\alpha = \frac{zy}{x} \notin R$, while it should be $z\alpha \in R$ since $\alpha \in (R : I)$; hence $(R : I) = R$ and $I^v = R$. □

Corollary 2.18. *Suppose that I is an ideal properly containing a principal prime. Then $I^v = R$.*

Proof. Let $(p) \subset I$, $(p) \in \mathrm{Spec}(R)$. Then $R/(p)$ is a domain, hence $\mathcal{Z}(R/(p)) = (0)$ and $I/(p)$ cannot be contained in it. □

When R is Noetherian, the hypothesis of Proposition 2.17 can be rewritten as "depth $I > 1$"; a particular case is the following:

Proposition 2.19. *Let R be a Noetherian domain and P a prime ideal of height > 1 such that R_P is integrally closed. Then $P^v = R$.*

To prove this, we need two facts:

Theorem 2.20. *Let A be a Noetherian ring and E be a finitely generated A-module; let $\mathcal{Z}_A(E) := \{x \in R \mid xm = 0$ for some $m \in E\}$, and I an ideal contained in $\mathcal{Z}_A(E)$. Then there is an $\overline{m} \in E$, $\overline{m} \neq 0$, such that $\overline{m}I = (0)$.*

Proof. This is a standard theorem of the theory of Noetherian rings; see e.g. [35, Theorem 82]. □

Lemma 2.21. *Let (R, P) be an integrally closed Noetherian local domain and suppose that for every $x \in P$ we have $P/(x) \subseteq \mathcal{Z}(R/(x))$. Then P is principal and R has dimension 1.*

Proof. $P(R : P)$, lying between P and R, is equal to one of these two.

If $P(R : P) = P$, then for every $x \in (R : P)$ we have $xP \subseteq P$; since R is Noetherian, P is also a finitely generated faithful $R[x]$-module, and thus x is integral over R and $x \in R$ because R is integrally closed; hence $(R : P) \subseteq R$, and $(R : P) = R$. because $1 \in (R : P)$.

However, by hypothesis, we have $\mathcal{Z}_R(R/(x)) = P$; thus, by Theorem 2.20 there is a $\overline{y} \in P/(x)$, $\overline{y} \neq 0$, such that $y\frac{P}{(x)} = 0$, that is, a $y \in P \setminus (x)$ such that $yP \subseteq (x)$. Thus $\frac{y}{x}P \subseteq R$ and $\alpha := \frac{y}{x} \in (R : P)$. But $y \notin (x)$, and hence $\alpha \notin R$, that is, $R \subsetneq (R : P)$. This is a contradiction, and $P(R : P) = R$.

If $P(R : P) = R$, then P is an invertible ideal, and thus is principal (since R is local). Suppose $\dim R > 1$: then there is a nonzero prime ideal Q properly contained in $P = (p)$. But then, for every $x \in Q$, $x = px_1$ for a $x_1 \in Q$ (since Q is prime and $p \notin Q$), and by induction there is a $x_n \in Q$ such that $x = p^n x_n$. Thus $Q \subseteq \bigcap_{n\geq 1}(p^n) = (0)$, against the hypothesis that Q is nonzero. Hence $\dim R = 1$. □

Proof of Proposition 2.19. Suppose firstly that P is the maximal ideal of R; then, by the previous lemma, $P/(x) \nsubseteq \mathcal{Z}(R/(x))$, since otherwise $\dim R = 1$, that is, P has height 1, in contrast with the hypothesis. By Proposition 2.17, $P^v = R$.

Let now R be any ring, and pick any $x \in P$. R_P is as above, so $\frac{PR_P}{xR_P} \nsubseteq \mathcal{Z}\left(\frac{R_P}{xR_P}\right)$; thus there is a $y \in PR_P \setminus xR_P$ such that $yz \notin xR_P$ for every $z \in R_P \setminus xR_P$. Let $y = \frac{y'}{s}$, with $y' \in P$; then, for every $z' \in P \setminus xR$, we have that $sz' \notin xR_P$, because otherwise $z' \in xR_P$ since $s \notin P$; thus $y'z' = ysz' = y(sz') \notin xR$, and y is not a zerodivisor in R/xR. Therefore, by Proposition 2.17, $P^v = R$. □

In particular, if R is Noetherian and integrally closed, then every prime ideal of height ≥ 2 is not divisorial; being v of finite type in this context, the

v-maximal ideals are the height 1 primes. If R is not Noetherian, both Lemma 2.21 and Proposition 2.19 need not to hold: if (R, M) is a non-Noetherian valuation domain with principal maximal ideal (e.g., if its value group is $\mathbb{Z}^2$) then $M^v = M$ (since M is principal) and every prime ideal is divisorial.

The v-operation can be defined also if R is not a domain, replacing K by the total fraction ring Q and putting I^v to be the intersection of all principal R-submodules of Q containing I (for any integral ideal I). However, in this case, v is not necessarily semi-prime: for example, if R is an Artinian local ring with non-principal maximal ideal M, then R is equal to its total ring of fractions, and the unique principal ideal containing M is R, and thus $M^v = R$, while $(0)^v = (0)$. But if v were semi-prime, then $(M^n)^v = ((M^v)^n)^v$; taking n such as $M^n = (0)$, it would follow that $(0) = (0)^v = (M^n)^v = R^v = R$, which is impossible. Hence v is not semi-prime.

2.4 Invertibility

As we have noted in Section 1.5 after Proposition 1.11, for every semi-prime closure c the operation $I \times^c J = (IJ)^c$ on the set $\mathcal{I}^c$ is associative, with identity R. With this operation, the structure of $\mathcal{I}^c$ is not very rich: for $IJ \subseteq I$ for every J, and thus $I \times^c J = (IJ)^c \subseteq I$. From this follows that there are no invertible elements, except for R itself.

When $c = \star$ is a star operation, however, we can define this operation on $\mathcal{F}^\star$, and even on the whole $\mathcal{F}$; from this we get the following definition.

Definition 2.22. *Let $\star$ be a star operation. A fractional ideal I is* $\star$-invertible *if there is a fractional ideal J such that $(IJ)^\star = R$. The set of $\star$-ideals $\star$-invertible is denoted by* $\mathrm{Inv}^\star$ *(or* $\mathrm{Inv}^\star(R)$ *if there is more than one ring involved).*

Note that a $\star$-invertible ideal is *not*, in general, a $\star$-ideal; but, as $(IJ)^\star = (I^\star J)^\star$, I is $\star$-invertible if an only if $I^\star$ is $\star$-invertible.

It is also clear that $\mathrm{Inv}^\star$ is the set of invertible elements of the monoid $(\mathcal{F}^\star, \times^\star)$, and hence is a group.

Since $(R : I)$ is the biggest ideal such that $I(R : I) \subseteq R$, the definition could be rephrased as follows: I is $\star$-invertible if $(I(R : I))^\star = R$. If, moreover, $\star$ is of finite type, so that every $\star$-ideal is contained in a $\star$-maximal ideal, this is also equivalent to the following: I is $\star$-invertible if $I(R : I)$ is not contained in any $\star$-maximal ideal. From this, we get that if $\star_1 - \mathrm{Max} = \star_2 - \mathrm{Max}$ then $\star_1$- and $\star_2$-invertible ideals are the same.

As the set of $\star$-ideals shrinks when we consider bigger star operations, we expect that the set of invertible ideals becomes larger. More precisely, suppose $\star_1 \leq \star_2$ are star operation and I is $\star_1$-invertible. Hence $(I(R:I))^{\star_2} \supseteq (I(R:I))^{\star_1} = R$; but $(I(R:I))^{\star_2} \subseteq R$ because $I(R:I) \subseteq R$, and thus $(I(R:I))^{\star_2} = R$, that is, I is $\star_2$-invertible. We can say more:

Proposition 2.23. *If $I \in \mathrm{Inv}^\star$ then $I^v = I$.*

Proof. $I(R:I) \subseteq I^v(R:I)$; but $I^v(R:I) = (R:(R:I))(R:I) \subseteq R$, and hence

$$R = (I(R:I))^\star \subseteq (I^v(R:I))^\star \subseteq R^\star = R \tag{2.17}$$

and I^v is $\star$-invertible. But I^v is a $\star$-ideal, and thus I and I^v are both inverses of $(R:I)$ in $\mathrm{Inv}^\star$, which is a group: hence $I = I^v$ is a v-ideal. □

Corollary 2.24. *If $\star_1 \leq \star_2$ then $\mathrm{Inv}^{\star_1} \subseteq \mathrm{Inv}^{\star_2}$.*

Proof. Every $I \in \mathrm{Inv}^{\star_1}$ is $\star_2$-invertible (from the remark above) and is a v-ideal (by the previous proposition), hence a $\star_2$-ideals, that is, $I \in \mathrm{Inv}^{\star_2}$. □

If $\star = d$ is the identity operation, the concept of d-invertible ideals coincides with the usual definition of invertible ideal, and so there is not conflict of notation. The above proposition shows that every invertible ideal is a v-ideal, and a $\star$-ideal for every star operation; thus, if R is a Dedekind domain (that is, if every ideal is invertible) then the only star operation on R is the identity. The same happens (for finitely generated ideals) when R is a Prüfer domain: since star operations of finite type are determined by their action of the set of finitely generated ideals, the unique star operation of finite type on a Prüfer domain is the identity. In both cases, the reverse implication is not true, because there are domains where the v-operation coincides with the identity, but are not Dedekind nor Prüfer: for example, in $L[[t^2,t^3]]$ (where L is a field), every ideal is divisorial, but $L[[t^2,t^3]]$ is not integrally closed, and thus is not Prüfer [46].

More generally, in a Noetherian domain R every ideal is divisorial if and only if R is a *Gorenstein ring* of dimension 1, i.e., if K/R is an injective R-module [35, Theorem 222][11, Theorem 6.3]; in an integrally closed domain R, every ideal is divisorial if and only if R is Prüfer, every nonzero ideal is contained in finitely many maximal ideals, every prime ideal is contained in only one maximal ideal and every maximal ideal is invertible [25].

It is well known that an ideal is invertible if and only if it is finitely generated and locally principal, that is, IR_M is principal for every $M \in \mathrm{Max}(R)$. There is an analogous criterion for $\star$-invertibility:

Proposition 2.25. *Let $\star$ be a star operation of finite type. Then a fractional ideal I is $\star$-invertible if and only if it is $\star$-finite and $I^\star R_M$ is principal for every $M \in \star - \mathrm{Max}(R)$.*

Proof. Without loss of generality, we can suppose that I is an integral ideal and that $I = I^\star$.

($\Longrightarrow$) Suppose I is $\star$-invertible. Then

$$1 \in R = (I(R:I))^\star = \bigcup\{H^\star \mid H \subseteq I(R:I) \text{ and } H \text{ is finitely generated}\} \tag{2.18}$$

and thus there is a finitely generated ideal $H = (x_1, \ldots, x_n)$ such that $1 \in H^\star$. As $H \subseteq I(R:I)$, for every x_i there are elements $a_{ij} \in I$ and $b_{ij} \in (R:I)$ such that $x_i = \sum_{j=1}^m a_{ij}b_{ij}$; define $F := \sum_{i,j} a_{ij}R \subseteq I$ and $G = \sum_{i,j} b_{ij}R \subseteq (R:I)$. Then

$$R = H^\star = (FG)^\star \subseteq (F(R:I))^\star \subseteq (I(R:I))^\star = R \tag{2.19}$$

so that $(F(R:I))^\star = R$. Thus $F^\star$ and $I^\star = I$ are both $\star$-inverses of $(R:I)^\star = (R:I)$; then $F^\star = I^\star$ and I is $\star$-finite.

As I is $\star$-invertible, $I(R:I)$ is not contained in any $\star$-maximal ideal, and thus $I(R:I)R_M = R_M$ for every $M \in \star-\mathrm{Max}(R)$. Hence IR_M is an invertible ideal; being R_M a local ring, this imply that $IR_M = I^\star R_M$ ($I = I^\star$ by hypothesis) is principal.

($\Longleftarrow$) As $I = I^\star$, $\star = \star_f$ and I is $\star$-finite, I is strictly $\star$-finite (Proposition 1.19), and thus there is a finitely generated ideal $J \subseteq I$ such that $J^\star = I^\star$. We have $J(R:J) \subseteq R$, so $(J^\star(R:J))^\star = (J(R:J))^\star \subseteq R$; by extension, $J^\star(R:J) \subseteq R$. Let $M \in \star-\mathrm{Max}(R)$: then

$$R_M \supseteq J^\star(R:J)R_M = (J^\star R_M)(R_M : JR_M) \tag{2.20}$$

(because J is finitely generated); we have also $J \subseteq J^\star$, so $JR_M \subseteq J^\star R_M$ and $(R_M : JR_M) \supseteq (R_M : J^\star R_M)$. Thus

$$(J^\star R_M)(R_M : JR_M) \supseteq (J^\star R_M)(R_M : J^\star R_M) = R_M \tag{2.21}$$

because $J^\star R_M = IR_M$ is principal by hypothesis. Then $J^\star(R:J)$ is not contained in any $\star$-maximal ideal, that is, $(J^\star(R:J))^\star = R$. But $(J^\star(R:J))^\star = (I(R:J))^\star$ and so I is $\star$-invertible. $\square$

Along the many ways to characterize Prüfer domains, two of the most useful are: every finitely generated fractional ideal is invertible, and R_M is a valuation domain for every $M \in \mathrm{Max}(R)$. This fact can be generalized as follows:

Proposition 2.26. *Let $\star$ be a star operation of finite type, R a domain. Then every finitely generated ideal of R is $\star$-invertible if and only if R_M is a valuation domain for every $M \in \star-\mathrm{Max}(R)$.*

Domains that verify one condition (hence both) of the proposition are called *Prüfer $\star$-multiplication domains*, or in short *P$\star$MD*; if $\star$ is not of finite type, P$\star_f$MDs are also called simply P$\star$MDs. The most important case is that of Prüfer v-multiplication domains (PvMD), when $\star = v$ (that is, $\star_f = t$). It is clear that if $\star$ is the identity P$\star$MDs are just Prüfer domains, and that if $\star_1 \leq \star_2$ then every P$\star_1$MD is a P$\star_2$MD.

Proof. ($\Longrightarrow$) Let $M \in \star-\mathrm{Max}(R)$ and J be a finitely generated ideal of R_M; there is a finitely generated ideal I of R such that $J = IR_M$. By hypothesis, I is $\star$-invertible, that is, $I(R:I)$ is not contained in any $\star$-maximal ideal, and in particular $I(R:I) \nsubseteq M$. This imply $I(R:I)R_M = R_M$; but $I(R:I)R_M = IR_M(R_M:IR_M) = J(R_M:J)$, and so J is an invertible ideal of R_M, and hence principal. Since every finitely generated ideal is principal, R_M is a Bézout domain; being local, it is a valuation ring.

($\Longleftarrow$) Let I be a finitely generated ideal. Then

$$\begin{aligned}(I(R:I))^\star = & \bigcap_{M\in\star-\mathrm{Max}(R)} ((I(R:I))^\star)R_M \supseteq \\ & \supseteq \bigcap_{M\in\star-\mathrm{Max}(R)} (I(R:I))R_M = \bigcap_{M\in\star-\mathrm{Max}(R)} (IR_M)(R_M:IR_M) \quad (2.22)\end{aligned}$$

But IR_M is principal because it is a finitely generated ideal of a valuation ring; hence it is invertible and $(IR_M)(R_M:IR_M) = R_M$. Thus $(I(R:I))^\star \supseteq \bigcap_{M\in\star-\mathrm{Max}(R)} R_M = R$; but $(I(R:I))^\star \subseteq R$, and so $I(R:I) = R$, and I is $\star$-invertible. $\square$

Since a $\star$-invertible ideal is $\star$-finite, the proposition shows also that R is a P$\star$MD if and only if every $\star$-finite ideal is $\star$-invertible, or, expressed differently, if and only if the set of $\star$-finite ideals is a group under $\star$-multiplication.

If R is Noetherian, then it is a PvMD if and only if it is integrally closed: every P$\star$MD is integrally closed (because it is the intersection of valuation domains, which are integrally closed) and the v-maximal ideals ($v = t$) of an integrally closed are the height 1 prime ideals (by Proposition 2.19) and so the R_M are discrete valuation rings.

Many more characterizations of P$\star$MD are known: fourteen equivalence for PvMDs are listed in [2], while [32] lists some for general P$\star$MDs.

2.5 $\mathrm{Inv}^\star$ *and the class group*

As we have seen, $\mathrm{Inv}^\star(R)$ is a commutative group; it is also a partially order group under reverse containment, and every pair of elements has an upper bound: if $I, J \in \mathrm{Inv}^\star$, then $I \cap R$ and $J \cap R$ are integral ideals of R bigger than (0) and thus (since R is an integral domain) have nonzero intersection; but if $x \in I \cap J$, then (x) is an invertible $\star$-ideal contained both in I and in J, so $I \leq (x)$ and $J \leq (x)$. The positive cone of $\mathrm{Inv}^\star$ is the set $\mathrm{Inv}^\star \cap \mathcal{I}$ of integral $\star$-invertible $\star$-ideal.

$\mathrm{Inv}^\star$ is, however, too big to be a good subject of study; hence we give a new definition:

Definition 2.27. *Let $\star$ be a star operation on R, and let $\mathcal{P}(R)$ be the set of principal ideals of R. The* $\star$-class group *of R is* $\mathrm{Cl}^\star(R) := \dfrac{\mathrm{Inv}^\star(R)}{\mathcal{P}(R)}$.

Corollary 2.24 is immediately translated as: if $\star_1 \leq \star_2$, then $\mathrm{Cl}^{\star_1} \leq \mathrm{Cl}^{\star_2}$.

The class group generalizes two precedent notions: the *Picard group* of R, defined as $\mathrm{Pic}(R) = \frac{\mathrm{Inv}(R)}{\mathcal{P}(R)}$, and also called "class group" in the context of Dedekind domains, and the *divisor class group* of Krull domains, which is defined as the analogue of $\frac{\mathrm{Inv}^v(R)}{\mathcal{P}(R)}$. As a matter of fact, the Picard group of a Dedekind domain and the divisor class group of a Krull domain both coincide with the t-class group.

Since the identity is the smallest star operation, we always have an inclusion $\mathrm{Pic}(R) \subseteq \mathrm{Cl}^\star(R)$; the quotient $G^\star(R) := \frac{\mathrm{Cl}^\star(R)}{\mathrm{Pic}(R)}$ is called the *local $\star$-class group* of R.

Among the class groups, the most prominent and the most useful is undoubtedly the t-class group $\mathrm{Cl}^t(R)$, which is often denoted simply as $\mathrm{Cl}(R)$. It is closely linked to the arithmetical and factorization properties of domains; in particular, its vanishing "signals" some important properties:

Theorem 2.28. *Let R be a domain,* Cl *the t-class group. Then*

1. *[44] R is a UFD $\iff$ R is a Krull domain and* $\mathrm{Cl}(R) = 0$.
2. *[13] R is a GCD domain (i.e., every pair of elements has a greatest common divisor) $\iff$ R is a PvMD and* $\mathrm{Cl}(R) = 0$.
3. *R is a Bézout domain $\iff$ R is a Prüfer domain and* $\mathrm{Cl}(R) = 0$.

The third part of the theorem is just a corollary of the second: if R is Bézout, then it is Prüfer (since every finitely generated ideal is principal, and hence invertible) and $\mathrm{Cl}(R) = 0$ because it is a GCD domain. Conversely, if R is Prüfer, then it is a PvMD, and since $\mathrm{Cl}(R) = 0$ every t-invertible ideal is principal: but t-invertible ideals are invertible ideals, because $d = t$, and hence every finitely generated idea, being invertible, is principal, and R is Bézout.

The t-class group has also some functorial properties that allows to link $\mathrm{Cl}(R)$ to the class group of the localizations or of polynomial rings over R or of pullbacks; however, an extension $R \subseteq T$ does not always give rise to the "canonical" map

$$\begin{aligned} \mathrm{Cl}(R) &\longrightarrow \quad \mathrm{Cl}(T) \\ [I] &\mapsto \quad [(IT)^t] \end{aligned} \tag{2.23}$$

because $(IT)^t$ is not, in general, t-invertible if I does. For example, let K be a field, $R = K[[x^2, xy, y^2]]$, $T = K[[x^2, x^3, xy, y]]$ and $I = (x^2, xy)R$; $t = v$ because all the rings are Noetherian. Then R is the quotient of $K[[X, Y, Z]]$ by $(XZ - Y^2)$, and thus I, being the image of (X, Y), is a prime ideal of height 1, and hence a t-ideal; moreover, the maximal ideal M of R is not a t-ideal (this can be seen either by direct calculation, or by noting that R is Cohen-Macaulay because it is a quotient of a Cohen-Macaulay ring by an ideal generated by a regular element, implying that M has depth 2 and $M^v = R$ by Proposition 2.17), IR_I is principal (it is generated by $\frac{y}{x}$) and thus I is t-invertible. On the contrary, the maximal ideal N of T is a t-ideal because $x \in (T : N) \setminus T$, and thus t-invertible ideals coincide with invertible ideals, i.e., are the principal ideals. But $IT = (x^2, xy)T$ is not contained in any integral principal ideal different from T, and thus $(IT)^t$ is t-invertible if and only if $(IT)^t = T$. But $IT \subseteq N$ and thus $(IT)^t \subseteq N$, that is, $(IT)^t$ is not t-invertible.

A sufficient condition for the existence of the canonical map is that $((IJ)^t T)^t = (IJT)^t$ for every ideals I, J of R (here t represent both the t-operation on R and the t-operation on T), and in particular if $(I^t T)^t = (IT)^t$ for every finitely generated $I \trianglelefteq R$; in this last case, the extension $R \subseteq T$ is said to be *t-compatible*. Flat extensions are t-compatible [50]: thus, $R \subseteq R_S$ (for S a multiplicatively closed subset of R) and $R \subseteq R[X]$ are t-compatible.

In the former case, $\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(R_S)$ needs not, in general, to be neither injective nor surjective; a sufficient condition to be injective is that S is generated by prime elements [3, Theorem 2.3], while to be surjective is that R is a PvMD [6, Proposition 6.5]; if P is t-prime ideal, then $\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(R_P)$ is the zero map [8, Proposition 2.3]. In the latter case, $\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(R[\mathbf{X}])$ (for any set $\{\mathbf{X}\}$ of indeterminates) is an isomorphism if and only if R is integrally closed [21]. More results on the map $\mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(T)$ can be found in [7] and in [50].

2.6 v-invertibility and complete integral closure

While the general criterion 2.25 works for star operations of finite type, there is also one for the v-invertibility:

Proposition 2.29. *I is v-invertible if and only if $(I^v : I^v) = R$.*

Proof. Since $(I^v : I^v) = (I^v : I)$, and using Proposition 2.4, part 6, we have

$$(I^v : I^v) = ((R : (R : I)) : I) = (R : I(R : I)). \tag{2.24}$$

If $(I^v : I^v) = R$, then $R = (R : I(R : I))$ and $(I(R : I))^v = R$, that is, I is v-invertible; conversely, if I is v-invertible, then $(R : I(R : I)) = (R : (I(R : I))^v) = (R : R) = R$. $\square$

$(I : I)$ is at the same time an overring and a fractional ideal of R; it is also easy to see that an R-submodule S is an overring of R if and only if $(S : S) = S$. This imply that an overring S of R is a fractional ideal if and only if there is an (integral) ideal I such that $(I : I) = S$ (since $(xS : xS) = (S : S)$).

The union of all these overrings is denoted by $\tilde{R}$ and is called the *complete integral closure* of R, and its elements are said to be *almost integral* over R:

$$\tilde{R} := \bigcup\{(I : I) \mid I \in \mathcal{I}(R)\}. \tag{2.25}$$

By definition, $R = \tilde{R}$ if and only if every ideal of R is v-invertible; in this case, R is said to be *completely integrally closed.*

We also have that

$$\tilde{R} = \bigcup\{(I^v : I^v) \mid I \in \mathcal{I}(R)\} = \bigcup\{(I^\star : I^\star) \mid I \in \mathcal{I}(R)\} \tag{2.26}$$

for every star operation: $\bigcup\{(I^v : I^v)\} \subseteq \bigcup\{(I^\star : I^\star)\} \subseteq \bigcup\{(I : I)\}$ because each union has more terms than the previous one, while, for every I, $(I : I) \subseteq (I^v : I) = (I^v : I^v)$, and thus $\bigcup\{(I : I)\} \subseteq \bigcup\{(I^v : I^v)\}$.

$\tilde{R}$ is also a ring: for every K and L, we have $(K : K) \subseteq (KL : KL)$, and so if $i, j \in \tilde{R}$ there are I, J such that $i \in (I : I)$ and $j \in (J : J)$; both $i + j$ and ij are in $(IJ : IJ)$.

It is useful to compare complete integral closure with integral closure: an element x is integral over R if and only if there is a finitely generated ideal I such that $xI \subseteq I$, that is, $x \in (I : I)$; thus

$$\overline{R} := \bigcup\{(I : I) \mid I \text{ is a finitely generated ideal of } R\} \tag{2.27}$$

is contained in $\tilde{R}$, and $\tilde{R} = \overline{R}$ if R is Noetherian, since every fractional ideal of a Noetherian ring is finitely generated. Since $v = t$ if R is Noetherian, this characterization permits a different proof (which does not uses Proposition 2.19) that a Noetherian ring is a PvMD if and only if it is integrally closed.

Also the "equational" definition of integral element has an analogue:

Proposition 2.30. *$x \in \tilde{R}$ if and only if there is a $c \in R$ such that $cx^n \in R$ for every $n > 0$.*

Proof. If $x \in \tilde{R}$, then $x \in (I : I)$ for an integral ideal I; thus $x^n \in (I : I)$ for every $n > 0$ (since $(I : I)$ is an overring) and for every $i \in I$ we have $ix^n \in I \subseteq R$.

If $cx^n \in R$, then $cR[x] \subseteq R$, and thus $R[x]$ is a fractional ideal of R; in particular $x \in (cR[x] : cR[x]) \subseteq \tilde{R}$. $\square$

$\tilde{R}$ itself need not to be a fractional ideal of R; if it does, then its fractional ideals coincide with those of R (since if $xI \subseteq \tilde{R}$ and $c\tilde{R} \subseteq R$, then $xcI \subseteq R$) and so the complete integral closure of $\tilde{R}$ is equal to the complete integral closure of R, that is, $\tilde{\tilde{R}} = \tilde{R}$. If $\tilde{R}$ is not a fractional ideal, however, it could be that its complete integral closure is strictly larger than itself: thus complete integral closure is not a "real" closure, since it is not, in general, idempotent; an example is $L[\{X^{2n+1}Y^{n(2n+1)} \mid n \in \mathbb{N}\}]$, where L is a field [23]. In fact, there are examples of domains where complete integral closure never stabilizes, that is, if $R_1 := \tilde{R}$ and R_n is the complete integral closure of R_{n-1}, then the chain $R \subsetneq R_1 \subseteq R_2 \subsetneq \cdots \subsetneq R_n \subsetneq \cdots$ is strictly ascending [28].

We end by a small proposition concerning completely integrally closed local domains and the divisoriality of their maximal ideal.

Proposition 2.31. *Let (R, M) be a completely integrally closed local domain. M is divisorial if and only if R is a discrete valuation ring.*

Proof. If R is a DVR then M is principal and hence divisorial. Conversely, suppose $M^v = M$, and let I be any ideal of R; then I is v-invertible because R is completely integrally closed. Thus $(I(R : I))^v = R$ and $I(R : I) = R$ (otherwise $(I(R : I))^v \subseteq M^v = M$), that is, I is invertible. Since R is local, I is principal; hence every ideal of R is principal, and R is a local PID, i.e., a DVR. $\square$

2.7 Historical and bibliographical note

Star operations were introduced by Krull [37] under the name "‵-operations", and received their name by Gilmer. They have also been called *prime operations*, by analogy with semi-prime operations [41, 36].

Results in Sections 2.1, 2.2 and the first half of 2.3 are standard, and can be found, for example, in [22], although I elected to define star operations as closure operations on R and then extend them to fractional ideals rather then start from these (as is commonly done).

The second half of Section 2.3 is largely built upon [35] (where the star operation language is not used).

Proposition 2.26 was first proved in [24] for PvMD (where they were called "v-multiplication rings"); the term "Prüfer" was added no later than Gilmer's book [22].

Complete integral closure was also introduced by Krull [37]; results in Section 2.6 are in [22, Chapters 13 and 34].

3. STAR OPERATIONS AND OVERRINGS

3.1 Homomorphisms and modules

In this section, we do not suppose that the rings considered are integral domains.

A very productive way to construct closure operations on a ring R is by using homomorphisms from R to other rings.

Proposition 3.1. *Let $\phi : R \longrightarrow S$ be a homomorphism and d a closure operation on S. Then the map*

$$I \mapsto I^c := \phi^{-1}((\phi(I)S)^d) \tag{3.1}$$

is a closure operation on R. Moreover, if d is semi-prime (respectively, of finite type) then so is c.

When ϕ is an inclusion, c can be written simply as $I^c = (IS)^d \cap R$.

Proof. Extension and order-preservation are clear:

$$I \subseteq \phi^{-1}(\phi(I)S) \subseteq \phi^{-1}((\phi(I)S)^d) = I^c, \tag{3.2}$$

while $I \subseteq J$ implies that $\phi(I)S \subseteq \phi(J)S$, and the inclusion is preserved under d and under contraction.

Suppose that $x \in (I^c)^c$. Then $\phi(x) \in \phi(I^c) = \phi(\phi^{-1}((\phi(I)S)^d))$, i.e.

$$x \in (\phi^{-1} \circ \phi \circ \phi^{-1})((\phi(I)S)^d). \tag{3.3}$$

But, for every ideal $J \trianglelefteq S$,

$$(\phi^{-1} \circ \phi \circ \phi^{-1})(J) = \phi^{-1}(J); \tag{3.4}$$

thus $x \in \phi^{-1}((\phi(I)S)^d)$, that is, $x \in I^c$.

Suppose that d is semi-prime, and let $y \in xI^c$: then $y = xz$ with $z \in I^c$, and thus $\phi(y) = \phi(x)\phi(z) \in \phi(x)(\phi(I)S)^d$. Since d is semi-prime, the last ideal is contained in $(\phi(x)(\phi(I)S))^d = (\phi(xI)S)^d$, and $y \in (xI)^c$. Thus $xI^c \subseteq (xI)^c$, and c is semi-prime.

Suppose that d is of finite type, and let I be an ideal. For every $y \in \phi(I)S$, $y = \phi(i_1)s_1 + \ldots + \phi(i_n)s_n$, and thus there is a finitely generated ideal J of R contained in I (namely $(i_1, \ldots, i_n)$) such that $y \in \phi(J)S$. Pick $x \in I^c$: then $\phi(x) \in (\phi(I)S)^d$, so that there is a finitely generated ideal $H \subseteq \phi(I)S$ such that $\phi(x) \in H^d$; let $H = (h_1, \ldots, h_m)$. Each h_i is contained in a finitely generated ideal $J_i \subseteq I$; hence $\phi(x) \in H^d \subseteq ((J_1 + \cdots + J_m)S)^d$, so that $x \in (J_1 + \cdots + J_m)^c$. Since $J_1 + \cdots + J_m$ is finitely generated, c is of finite type. □

We note that the above proposition does not work if we replace "semi-prime" with "star operation", even if ϕ is an inclusion, S is an overring of a domain R and d is the identity, because $aS \cap R$ need not to be equal to aR: for example, if $R = \mathbb{Z}$ and $S = \mathbb{Z}_{(2)} = \mathbb{Z}_{2\mathbb{Z}}$, then $6S = 2S = 2\mathbb{Z}_{(2)}$, and $(6)^c = 2\mathbb{Z}_{(2)} \cap \mathbb{Z} = 2\mathbb{Z}$. A slightly different construction that actually yields star operations is given in Section 3.6.

If S is equal to R/I, where I is an ideal of R, then the closure operation induced by the quotient map $\pi : R \longrightarrow S$ and by the closure operation d on R/I is the closure c on R whose c-ideals are the ideals containing I that project to the d-ideals of S.

If we have a whole family of rings S_α, closure operations d_α and homomorphisms $\phi_\alpha : R \longrightarrow S_\alpha$, we can take the infimum of the corresponding closures c_α, obtaining

$$I^c := \bigcap_{\alpha \in A} \phi_\alpha^{-1}((\phi_\alpha(I)S_\alpha)^{d_\alpha}) \tag{3.5}$$

which (by Proposition 1.13) is semi-prime if all the d_α are.

Closure operations that arise in this way are, for example, the radical (if the family $\{S_\alpha\}$ is the family of fields), and integral closure (if $\{S_\alpha\}$ is the family of valuation rings; see Proposition 4.24).

Further on this way, there is also a method to obtain closure operations from modules: rewriting $\phi^{-1}(\phi(I)S)$ as $\{x \in R \mid \phi(x) \in \phi(I)S\} = \{x \in R \mid \phi(x)S \subseteq \phi(I)S\}$, we can replace S by any R-module U, and the map becomes

$$I \mapsto I^c := \{x \in R \mid xU \subseteq IU\} = (IU :_R U) \tag{3.6}$$

This is really a closure operation: extension and order-preservation are clear, while for idempotence we suppose $x \in (I^c)^c$. Then $xU \subseteq I^cU$; but since $iU \subseteq IU$ for every $i \in I^c$, we have $I^cU \subseteq IU$, and hence $xU \subseteq IU$, that is, $x \in I^c$.

For every module U, c is semi-prime: if $y \in xI^c$; then $y = xz$ for a $z \in I^c$, and $yU = xzU \subseteq xIU$, so that $y \in (xI)^c$ and $xI^c \subseteq (xI)^c$.

If, moreover, U is finitely generated, then c is of finite type: let $x \in I^c$ and suppose that U is generated by $u_1, \ldots, u_m$. For every i, $xu_i \subseteq H_iU$ for a finitely generated ideal $H_i \subseteq I$; hence $xU \subseteq (H_1 + \cdots + H_m)U$ and $x \in H^c$, where $H := H_1 + \cdots + H_m$ is finitely generated; it follows that $I^c = \bigcup\{H^c \mid H \subseteq I, H \text{ is finitely generated}\}$, and c is of finite type.

If U is not finitely generated, it may happen that c is not of finite type: for example, if I is a non-finitely generated ideal such that $I^2 = I$ (e.g., the non-principal maximal ideal of a valuation ring), then $I^c = (I^2 :_R I) = (I :_R I) = R$, but, for every finitely generated ideal $H \subseteq I$, we have that $HI \subseteq H \subsetneq I$, and thus $H^c = (HI :_R I) \neq R$ because $1 \notin (HI :_R I)$.

A closure arising in this way is the Frobenius closure: we recall that, for a ring R of prime characteristic p, an element x is in the Frobenius closure I^F of I if there is an $e \in \mathbb{N}$ such that $x^{p^e} \in I^{[p^e]}$, where $I^{[p^e]}$ is the ideal generated by the p^eth powers of elements of I.

Define the R-module ${}^eR$ with the same additive structure as R (with elements denoted ${}^er$ for each $r \in R$), and R-multiplication as follows:

$$a \cdot {}^er = {}^e(a^{p^e}r). \tag{3.7}$$

Then we have closure operations c_e such that $I^{c_e} := (I \cdot {}^eR : {}^eR)$, and $c_e \leq c_{e+1}$ for every e; their supremum $I^F := \bigcup_{e\in\mathbb{N}} I^{c_e}$ is Frobenius closure, because, as sets, $I \cdot {}^eR = I^{[p^e]}$. (Proposition 1.4 actually guarantees that F is idempotent only if R is Noetherian; however, the closure works in every case.)

3.2 Extension rings

An *extension ring* of R is just a ring S with a injective homomorphism $i : R \longrightarrow S$. To avoid any problem, when we will talk about a family $\{R_\alpha\}$ of extension rings, we will tacitly assume that all the R_α are domains and that they are contained in a bigger field F; in this way, the injective homomorphisms become just inclusions, and it is meaningful to speak about IS for a fractional ideal I. The main case is when every R_α is an overring of R, i.e., when R_α is contained between R and its quotient field K.

Definition 3.2. *Let R be an integral domain and $\{R_\alpha\}_{\alpha\in A}$ a set of extension rings of R. The* closure operation induced by $\{R_\alpha\}$ *is*

$$I^c := \bigcap_{\alpha\in A} IR_\alpha \cap R. \tag{3.8}$$

By the results of the above section, c is indeed a closure operation; moreover, if $\bigcap R_\alpha = R$, then it is a star operation, because, for every $x \in R$,

$$xI^c = x\left(\bigcap_{\alpha\in A} IR_\alpha\right) = \bigcap_{\alpha\in A} xIR_\alpha = (xI)^c. \qquad (3.9)$$

It is easy to see that Definition 3.2 works also for fractional ideals.

A little more precise form of idempotence is the following:

Lemma 3.3. *$I^c = J^c$ if and only if $IR_\alpha = JR_\alpha$ for every $\alpha \in A$.*

Proof. One implication is clear; for the other, it is sufficient to prove that $IR_\alpha = I^cR_\alpha$; the $\subseteq$ containment comes from the extensive property of closure operations, while

$$I^cR_\alpha = \left(\bigcap_{\alpha\in A} IR_\alpha\right) R_\alpha \subseteq IR_\alpha R_\alpha = IR_\alpha. \qquad (3.10)$$

□

Lemma 3.4. *Let c be the closure operation induced by $\{R_\alpha\}$, and let I be an ideal of R contracted from some $J \trianglelefteq R_\beta$ (that is, $I = J \cap R$). Then I is a c-ideal.*

Proof. Since $IR_\beta = (J \cap R)R_\beta \subseteq J$, we have

$$I^c = \bigcap_{\alpha\in A} IR_\alpha \cap R = IR_\beta \cap \bigcap_{\alpha\in A} IR_\alpha \cap R \subseteq J \cap \bigcap R_\alpha \cap R = J \cap R = I \quad (3.11)$$

and thus I is a c-ideal. □

Closure operations induced by a family of extension rings need not to be of finite type, although every $IR_\alpha \cap R$ is of finite type. A sufficient condition is that every $x \in R$ is a non-unit only in finitely many R_α: in this case, if $I \trianglelefteq R$, pick any $a \in I$, and let $R_1, \ldots, R_n$ be the rings where a is not invertible; then $IR_\alpha = R_\alpha$ if $R_\alpha \neq R_i$ for every i, and thus $I^c = \bigcap_{i=1}^n IR_i$. The map

$$J \mapsto J^d := \bigcap_{i=1}^n JR_i \qquad (3.12)$$

is a closure operation of finite type (by Proposition 1.9) such that $J^d = J^c$ if $a \in J$. For every $x \in I^c$ there is a $J \subseteq I$ finitely generated such that $x \in J^d$; hence $x \in (J,a)^d = (J,a)^c$, and, since $(J,a) \subseteq I$, c is of finite type.

If every R_α is a localization of R, we can get a characterization of finite type closure operation in terms of the topology of $c-\mathrm{Spec}(R)$: see Proposition 3.13.

However, this closures share an important property with finite type closure operations:

Proposition 3.5. *Let c be the closure operation induced by $\{R_\alpha\}$. Every c-ideal is contained in a prime c-ideal.*

Proof. Let $I = I^c$, and let $R_\beta \in \{R_\alpha\}$ such that $IR_\beta \neq R_\beta$ (it exists, because otherwise we would have $I^c = \bigcap R_\alpha \cap R = R$). Let M be a prime ideal of R_β containing IR_β; then $P := M \cap R$ is a prime ideal of R, and is a c-ideal by Lemma 3.4. □

Note that the previous proposition is not true if we replace c-prime ideals with c-maximal ideals, because these need not to exist: for example, set $R = K[X_1, \ldots, X_n, \ldots]$, let Δ be the set of finitely generated prime ideals and c be the closure operation induced by the set $\{R_P \mid P \in \Delta\}$. Every $P \in \Delta$ is a c-ideal, while every $Q \notin \Delta$ is not contained in any $P \in \Delta$, and so $Q^c = R$; hence every c-maximal ideal M should be finitely generated, but, if X_m does not appear among the generators of M, then (M, X_m) is a c-prime ideal bigger then M, which is absurd.

Corollary 3.6. *Let c be the closure operation induced by $\{R_\alpha\}$. For every c-ideal I,*

$$I = \bigcap_{P \in c-\mathrm{Spec}(R)} IR_P. \tag{3.13}$$

The proof is completely analogous to that of Proposition 1.17, part 4.

It is to be noted that it is not true that $I^c = \bigcap_{P \in c-\mathrm{Spec}(R)} IR_P$: for example, if $M^c = M$ for every maximal ideal M, then $\bigcap IR_P = I$, so if c is different from the identity there is an ideal such that $I^c \neq I = \bigcap IR_P$. An explicit example is integral closure (see next chapter).

3.3 Spectral operations

Definition 3.7. *Let $\Delta \subseteq \mathrm{Spec}(R)$ be a (nonempty) set such that $\bigcap_{P\in\Delta} R_P = R$. The star operation $\star_\Delta$ induced by $\{R_P\}_{P\in\Delta}$ is called the* spectral operation induced by Δ*, and a star operation is called* spectral *if $\star = \star_\Delta$ for a set $\Delta \subseteq \mathrm{Spec}(R)$.*

A spectral operation can be associated to more than one subset. To study this case, we introduce the following terminology: a set $\Delta \subseteq \mathrm{Spec}(R)$ is *closed under generization* if every prime ideal contained in a $Q \in \Delta$ is a member of Δ, and the *generization* $\overline{\Delta}$ of a set $\Delta \subseteq \mathrm{Spec}(R)$ is the smallest set closed under generization containing Δ, or, more explicitly, $\overline{\Delta} = \{Q \in \mathrm{Spec}\, R \mid \exists\, P \in \Delta \text{ such that } Q \subseteq P\}$.

Proposition 3.8. *If $\star = \star_\Delta$ is a spectral star operation, then* $\star - \mathrm{Spec}(R) = \overline{\Delta}$.

Proof. If $P \in \overline{\Delta}$, then there is a $Q \in \Delta$ such that $P \subseteq Q$; thus $PR_Q \cap R = P$, and $P^\star = P$ by Lemma 3.4, i.e., $P \in \star - \mathrm{Spec}(R)$.

Conversely, suppose $P \notin \overline{\Delta}$. Then P is not contained in any $Q \in \Delta$, and thus $PR_Q = R_Q$ for every $Q \in \Delta$; it follows that $P^\star = \bigcap R_Q = R$ and P is not a $\star$-ideal. □

Given $\star_\Delta$ and the star operation $\star_{\overline{\Delta}}$ induced by the generization of Δ, the previous proposition shows that they have the same spectrum. Moreover, they are equal:

Proposition 3.9. *Suppose that Δ and Λ are subsets of* $\mathrm{Spec}(R)$ *such that* $\bigcap_{P \in \Delta} R_P = \bigcap_{Q \in \Lambda} R_Q = R$. *Then the corresponding star operations $\star_\Delta$ and $\star_\Lambda$ are equal if and only if $\overline{\Delta} = \overline{\Lambda}$.*

Proof. If $\star_\Delta = \star_\Lambda$ then $\star_\Delta - \mathrm{Spec} = \star_\Lambda - \mathrm{Spec}$, which is equal (respectively) to $\overline{\Delta}$ and $\overline{\Lambda}$, that thus coincide.

For the other implication it suffices to show that $\star_\Delta = \star_{\overline{\Delta}}$. Since $\Delta \subseteq \overline{\Delta}$,

$$I^{\star_{\overline{\Delta}}} = \bigcap_{M \in \overline{\Delta}} IR_M \subseteq \bigcap_{M \in \Delta} IR_M = I^{\star_\Delta}. \tag{3.14}$$

On the other hand, set $\Delta' = \overline{\Delta} \setminus \Delta$; for each $P \in \Delta'$ there is a $Q \in \Delta$ such that $P \subseteq Q$, and thus $R_P \supseteq R_Q$. Hence $\bigcap_{M \in \Delta'} IR_M \supseteq \bigcap_{M \in \Delta} IR_M$ and

$$\bigcap_{M \in \overline{\Delta}} IR_M = \bigcap_{M \in \Delta} IR_M \cap \bigcap_{M \in \Delta'} IR_M \supseteq \bigcap_{M \in \Delta} IR_M. \tag{3.15}$$

Therefore $\star_{\overline{\Delta}} = \star_\Delta$. □

The previous proposition establishes a one-to-one correspondence between spectral star operations and subset of $\mathrm{Spec}(R)$ closed under generization with the property that $\bigcap R_P = R$.

One one the main features of spectral star operations is that they distributes over finite intersections: that is, $(I \cap J)^\star = I^\star \cap J^\star$ for every pair of fractional ideals I, J; this does not happens with general operations induced by extension rings, or even by overrings. In fact, this property "almost" characterizes them:

Proposition 3.10. *Let $\star$ be a star operation on R. The following are equivalent:*

1. $\star = \star_\Delta$ *is spectral.*
2. (a) $(I \cap J)^\star = I^\star \cap J^\star$ *for all integral ideals* I, J.
   (b) *Each proper* $\star$*-integral is contained in a* $\star$*-prime ideal.*
3. (a) $(I :_R J)^\star = (I^\star :_R J^\star)$ *for all integral ideals* I, J *with* J *finitely generated.*
   (b) *Each proper* $\star$*-integral is contained in a* $\star$*-prime ideal.*
4. (a) $(I :_R x)^\star = (I^\star :_R x)$ *for all integral ideals* I.
   (b) *Each proper* $\star$*-integral is contained in a* $\star$*-prime ideal.*

Proof. ($1 \Longrightarrow 2$) (a) follows because localization commutes with finite intersections, while (b) follows from Proposition 3.5.

($2 \Longrightarrow 3$) Let $J = (j_1, \ldots, j_n)$. We have

$$(I :_R J)^\star = (R \cap (I : J))^\star = \left(R \cap \left(I : \sum_{i=1}^n j_i R\right)\right)^\star = \\ = \left(R \cap \bigcap_{i=1}^n (I : j_i R)\right)^\star = R \cap \bigcap_{i=1}^n (I : j_i R)^\star. \quad (3.16)$$

But $(I : jR)^\star = (j^{-1} I^\star) = j^{-1} I^\star = (I^\star : jR)$, and thus

$$(I :_R J)^\star = R \cap \bigcap_{i=1}^n (I^\star : j_i R) = R \cap \left(I^\star : \sum_{i=1}^n j_i R\right) = \\ = R \cap (I^\star : J) = (I^\star :_R J) = (I^\star :_R J^\star) \quad (3.17)$$

the last equality coming from Proposition 2.9.

($3 \Longrightarrow 4$) Obvious.

($4 \Longrightarrow 1$) Let $\Delta = \star - \mathrm{Spec}(R)$, and let $\star_1$ be the map (from the set of R-submodules of K to itself)

$$I^{\star_1} := \bigcap_{P \in \Delta} I R_P. \quad (3.18)$$

By Corollary 3.6, $R = R^{\star_1}$ and thus $\star_1$ is a star operation, which clearly is spectral.

Let I be any ideal. If $x \in I^\star$, then $(I :_R x)^\star = (I^\star :_R x) = R$, and thus (by property (b)) $(I :_R x)$ is not contained in any $P \in \Delta$, and $(I :_R x)R_P = R_P$ for every $P \in \Delta$. This imply $(I :_R x)^{\star_1} = \bigcap (I :_R x)R_P = \bigcap R_P = R$; but (as $\star_1$ is a spectral operation) $(I :_R x)^{\star_1} = (I^{\star_1} :_R x)$; hence the last one contains 1, and $x \in I^{\star_1}$.

Suppose $x \in I^{\star_1}$. In the same way, $(I :_R x)^{\star_1} = R$ and $(I :_R x)$ is not contained in any $P \in \Delta$; hence also $(I :_R x)^\star$ is not contained in any $P \in \Delta$, and by property (b) $(I :_R x)^\star = R$. By property (a), $(I :_R x)^\star = (I^\star :_R x)$ and so $x \in I^\star$; thus $I^{\star_1} = I^\star$ and $\star_1 = \star$. $\square$

It is to be noted that property (b) is used only in the last implication, and thus we have that (2a)$\Longrightarrow$(3a)$\Longrightarrow$(4a). Moreover, it is also true that (4a)$\Longrightarrow$(2a) [5], so that conditions (2a)-(4a) are equivalent.

It is not true that if $\star$ distributes over finite intersections then it is spectral: for example, let (R, M) be a valuation domain. Then the set of ideals of R is totally ordered; in particular, if $I \subseteq J$, then $I \cap J = I$ and $I^\star \cap J^\star = I^\star$ because $\star$ is extensive, and thus

$$(I \cap J)^\star = I^\star = I^\star \cap J^\star. \tag{3.19}$$

But if $\star = v$ and M is non-principal, then $\star$ cannot be spectral, since $M^v = R$ while there are principal ideals directly below M, so that they are not contained in any $\star$-prime ideal. However, if $\star$ is of finite type and distributes over intersections, it is necessary spectral.

In this example, v also distributes over arbitrary intersections; this is in general not true for spectral operations.

Proposition 3.11. *[1, Theorem 7] Suppose that $\star$ is a finite type star operation that distributes over arbitrary intersections. Then it is the identity.*

Proof. Since in particular it distributes over finite intersections, $\star$ is spectral; the thesis would follow if we prove that every maximal ideal is $\star$-closed.

Suppose M is maximal and $M^\star \neq M$. Then $M^\star = R$ and, since $\star$ is of finite type, there is a finitely generated ideal $I \subseteq M$ such that $I^\star = R$. Let $\mathcal{J} := \{I \subseteq M \mid I^\star = R\} \neq \varnothing$, $x \in M$ and define $J := \bigcap \{(x) + I \mid I \in \mathcal{J}\}$. Then

$$J^\star = \left(\bigcap_{I \in \mathcal{J}} (x) + I \right)^\star = \bigcap_{I \in \mathcal{J}} ((x) + I)^\star \supseteq \bigcap_{I \in \mathcal{J}} I^\star = R \tag{3.20}$$

and hence there is a $J_0 \subseteq J$ finitely generated such that $J_0^\star = R$; in particular, since $J_0 \subseteq J \subseteq M$, $J_0 \in \mathcal{J}$. Moreover, $(J_0^2)^\star = ((J_0^\star)^2)^\star = R^\star = R$, so that $J_0^2 \in \mathcal{J}$. We have that $J_0 \subseteq (x) + I$ for every $I \in \mathcal{J}$, and in particular $J_0 \subseteq (x) + J_0^2$. If P is a minimal prime of (x), then $J_0 \subseteq (x) + J_0^2 \subseteq P + J_0^2$

and thus $P+J_0=P+J_0^2$; in R/P, this means that $\overline{J_0}=\overline{J_0}^2$, and since $\overline{J_0}$ is finitely generated it is the zero ideal, i.e., $J_0\subseteq P$. But $P^\star=P$ because it is a minimal prime over the $\star$-ideal (x), and thus $J_0^\star\subseteq P$, which is a contradiction. Hence $M^\star=M$. ☐

Proposition 3.12. *A star operation generated by a family of flat extension rings is spectral.*

A more explicit way to view this proposition is: if $\star$ is induced by a family of R-flat extension rings, then there is a family of localizations that induces it.

Proof. By Proposition 3.10 and Proposition 3.5, it is sufficient to prove that, if T is a flat R-module, then $(I\cap J)T=IT\cap JT$.

There is an exact sequence

$$0\longrightarrow I\cap J\longrightarrow R\longrightarrow\frac{R}{I}\times\frac{R}{J} \tag{3.21}$$

where the last map sends r to $(r+I,r+J)$. Tensoring with T, we get an exact sequence

$$0\longrightarrow(I\cap J)\otimes_R T\longrightarrow R\otimes_R T\longrightarrow\left(\frac{R}{I}\times\frac{R}{J}\right)\otimes_R T \tag{3.22}$$

that is,

$$0\longrightarrow(I\cap J)T\longrightarrow T\longrightarrow\frac{T}{IT}\times\frac{T}{JT}. \tag{3.23}$$

But the kernel of the last map is exactly $IT\cap JT$, so that $(I\cap J)T=IT\cap JT$. ☐

There is a very neat characterization of finite type spectral operations:

Proposition 3.13. *Let $\star$ be a spectral operations associated to a set Δ. Then $\star$ is of finite type if and only if Δ is compact (in the Zariski topology inherited from* $\mathrm{Spec}(R)$*).*

Proof. Set $V_\Delta(I):=\{P\in\Delta\mid I\subseteq P\}=V(I)\cap\Delta$ and $D_\Delta(I):=\Delta\setminus V(I)=D(I)\cap\Delta$. The closed set of Δ are precisely the $V_\Delta(I)$, and the open set are the $D_\Delta(I)$.

Suppose $\star$ is of finite type; let $\mathcal{D}:=\{D_\Delta(I_\alpha)\}$ be a cover of Δ and $J:=\sum I_\alpha$. Then J is not contained in any prime of Δ: if $J\subseteq Q$, then $I_\alpha\subseteq Q$ for every I_α, and thus $Q\in V_\Delta(I_\alpha)$ and $Q\notin D_\Delta(I_\alpha)$ for every I, against the hypothesis that $\mathcal{D}$ is a cover. Since any $\star$-ideal is contained in a member of Δ, it follows that $J^\star = R$, and since $\star$ is of finite type and J is $\star$-finite, then by Proposition 1.19 J is strictly $\star$-finite, i.e., there is a finitely generated ideal $H \subseteq J$ such that $H^\star = R$; let $H = (h_1, \dots, h_n)$. Each h_i is contained in the sum of a finite number of the I_α; hence H is contained in a sum $I_1 + \cdots + I_n$. This implies that

$$\varnothing = V_\Delta(H) \supseteq V_\Delta(I_1 + \cdots + I_n) = V_\Delta(I_1) \cap \cdots \cap V_\Delta(I_n) \tag{3.24}$$

and thus $\{D(I_1), \dots, D(I_n)\}$ is a finite subcover of $\mathcal{D}$.

Conversely, suppose that Δ is compact and $x \in I^\star$. Consider the set $\mathcal{D} := \{D_\Delta((H :_R x)) \mid H \subseteq I \text{ and } H \text{ is finitely generated}\}$: we say that it is a cover of Δ. Otherwise, if $P \notin \bigcup D_\Delta((H :_R x))$, then $P \in \bigcap V_\Delta(H :_R x)$, that is, $(H :_R x) \subseteq P$ for every finitely generated ideal $H \subseteq I$. Since $\star$ is spectral, by Proposition 3.10 $(I :_R x)^\star = (I^\star :_R x) = R$ (because $x \in I^\star$), and thus $(I :_R x) \nsubseteq P$ (since P is a $\star$-ideal); if $y \in (I :_R x) \setminus P$, then $yx \in I$ and $y \in (yx :_R x)$; but $(yx) \subseteq I$ is finitely generated, and thus we should have $(yx :_R x) \subseteq P$, which is absurd. Therefore $\mathcal{D}$ is a cover of Δ.

Since Δ is compact, there is a finite subcover $\{D_\Delta((H_i :_R x))\}_{i=1}^n$; set $H := H_1 + \cdots + H_n$. Then $H \subseteq I$ is finitely generated; if $x \notin H^\star$, then $R \neq (H^\star :_R x) = (H :_R x)^\star$, and thus $(H :_R x)$ is contained in a prime ideal $P \in \Delta$; but then, for every i, $(H_i :_R x) \subseteq (H :_R x) \subseteq P$ and thus $P \in V_\Delta((H_i :_R x))$; therefore $P \notin \bigcup_i D_\Delta((H_i :_R x))$, against the hypothesis. Thus $(H :_R x)^\star = R$, and $x \in H^\star$. □

See also Proposition 3.19.

We recall that a topological space is *Noetherian* if the the family of open sets satisfies the ascending chain condition or, equivalently, the family of closed sets satisfies the descending chain condition; since there is a one-to-one correspondence between open sets and radical ideals, $\mathrm{Spec}(R)$ is Noetherian if and only if the radical ideals of R satisfy the ascending chain condition. Since a subset of a Noetherian space is again Noetherian, from the previous proposition we get at once:

Corollary 3.14. *If* $\mathrm{Spec}(R)$ *is Noetherian, then every spectral operation on* R *is of finite type.*

Corollary 3.15. *Suppose that R is a Prüfer domain with Noetherian spectrum, and let* $\Delta \subseteq \mathrm{Spec}(R)$ *be a set such that* $\bigcap_{P \in \Delta} R_P = R$. *Then* $\mathrm{Max}(R) \subseteq \Delta$.

Proof. Since $\mathrm{Spec}(R)$ is Noetherian, by the previous corollary $\star = \star_\Delta$ is of finite type and, since R is Prüfer, it equals the identity. Thus the generization $\overline{\Delta}$ coincides with $\mathrm{Spec}(R)$; but a maximal ideal is in $\overline{\Delta}$ if and only if it is in Δ, and hence $\mathrm{Max}(R) \subseteq \Delta$. □

If Spec(R) is not Noetherian, then the last corollary could not hold: for example, let R be the ring of entire functions, that is, functions that are holomorphic on the whole $\mathbb{C}$. Then R is a Bézout domain whose maximal ideals are either principal, generated by $X-\alpha$ for some $\alpha \in \mathbb{C}$, or non-finitely generated and of infinite height [27]. Let $\Delta = \{(X-\alpha) \mid \alpha \in \mathbb{C}\}$. For any $r \in K$, $r = f/g$ with $f, g \in R$; if $r \notin R$, then g has a zero β (otherwise it would be invertible), and thus $r \notin R_{(X-\beta)}$. Then $\bigcap_{P \in \Delta} R_P = R$, while Δ does not contain all the maximal ideals of R. This is also an example of a spectral operation which is not of finite type.

3.4 The $\star_w$ construction

A special case is when $\Delta = \star - \mathrm{Max}$ for a (finite type) star operation $\star$.

Definition 3.16. *Let $\star$ be a star operation of finite type; then $\star_w$ is*

$$I^{\star_w} := \bigcap_{M \in \star-\mathrm{Max}(R)} IR_M. \tag{3.25}$$

If $\star = t$, t_w is simply called the w-operation*.*

Since $\star$ is of finite type, $\bigcap_{M \in \star-\mathrm{Max}(R)} R_M = R$, so that $\star_w$ is a star operation; the same construction would work also if $\star$ were only semi-prime, yielding a star operation.

A less transparent but useful characterization is the following:

Proposition 3.17. $I^{\star_w} = \bigcup \{(I : J) \mid J^\star = R,$ *J is finitely generated*$\}$.

Proof. If x is in the right hand union, then $xJ \subseteq I$ for a finitely generated J such that $J^\star = R$; since $\star$ is of finite type, J is not contained in any $\star$-maximal ideal, and

$$\bigcap_{M \in \star-\mathrm{Max}(R)} (xJR_M) = x \bigcap_{M \in \star-\mathrm{Max}(R)} (JR_M) = x \bigcap_{M \in \star-\mathrm{Max}(R)} R_M = xR. \tag{3.26}$$

But $xJR_M \subseteq IR_M$ for every $\star$-maximal ideal, and therefore

$$\bigcap_{M \in \star-\mathrm{Max}(R)} (xJR_M) \subseteq \bigcap_{M \in \star-\mathrm{Max}(R)} IR_M = I^{\star_w} \tag{3.27}$$

and $xR \subseteq I^{\star_w}$, that is, $x \in I^{\star_w}$.

Conversely, suppose $x \in I^{\star_w}$; then, for every $\star$-maximal ideal M, $x \in IR_M$, and $x = \frac{i_M}{s_M}$ for some $i_M \in I$ and $s_M \notin M$. Let H be the ideal generated by all the s_M; then H is not contained in any M and thus $H^\star = R$, and in particular H is $\star$-finite; since $\star$ is of finite type, H is strictly $\star$-finite by Proposition 1.19, and thus there is a finitely generated ideal $J \subseteq H$ such that $J^\star = R$. Since $xs_M = i_M \in I$ for every M, $xH \subseteq I$ and thus $xJ \subseteq I$, i.e., $x \in (I : J)$ and x is the union. □

This characterization allows a simple proof that $\star_w$ is of finite type: since $x \in \bigcup (I : J)$, $x \in (I : J)$ for a J; but then xJ is a finitely generated ideal contained in I and $x \in (xJ : J)$, so that $x \in (xJ)^{\star_w}$ and $\star_w$ is of finite type.

Another feature of this characterization is that it can be generalized to arbitrary star operations $\star$, giving rise to two star operations, $\overline{\star}$ and $\star_w$ [5]:

$$I^{\overline{\star}} := \bigcup \{(I : J) \mid J^\star = R\}, \tag{3.28}$$

$$I^{\star_w} := \bigcup \{(I : J) \mid J^\star = R,\ J \text{ is finitely generated}\}. \tag{3.29}$$

There is no ambiguity of notation, because, if $\star$ is not of finite type, then the last definition of $\star_w$ coincides with the above definition of $(\star_f)_w$; it follows that $\star_w$ is spectral for any $\star$.

It can be shown that both these maps are star operations that distributes over intersections, although $\overline{\star}$ need not to be spectral; moreover, the proof of Proposition 3.17 shows that $\overline{\star} = \star_w$ if $\star$ is of finite type.

The map $\star \mapsto \star_w$ bears resemblance to the map $c \mapsto c_f$ that associates every closure operation with a closure operation of finite type; the following is an analogue of Proposition 1.8.

Proposition 3.18. *Let $\star$ be a star operation of finite type.*

1. $\star_w \leq \star$.
2. *If* $\star_1 \leq \star_2$ *then* $(\star_1)_w \leq (\star_2)_w$.
3. $\star_w - \mathrm{Spec}(R)$ *is the generization of* $\star - \mathrm{Spec}(R)$*; in particular,* $\star - \mathrm{Max}(R) = \star_w - \mathrm{Max}(R)$.
4. $(\star_w)_w = \star_w$*, and* $\star_w$ *is the largest spectral operation smaller than* $\star$.

Proof. 1. By Proposition 1.17, $I^\star := \bigcap_{M \in \star - \mathrm{Max}(R)} I^\star R_M$; thus every $\star$-ideal is a $\star_w$-ideal, and $\star_w \leq \star$.

2. As $\star_1 \leq \star_2$, $\mathcal{F}^{\star_1} \supseteq \mathcal{F}^{\star_2}$ and thus every $\star_2$-maximal ideal M is contained in a $\star_1$-maximal ideal M'. Since $IR_M \supseteq IR_{M'}$, we have

$$I^{(\star_2)_w} = \bigcap_{M \in \star_2 - \mathrm{Max}} IR_M \supseteq \bigcap_{M' \in \star_1 - \mathrm{Max}} IR_{M'} = I^{(\star_1)_w} \tag{3.30}$$

and $(\star_1)_w \leq (\star_2)_w$.

3. If $P \in \overline{\star - \mathrm{Spec}(R)}$, then $P \subseteq Q$ for some $Q \in \star - \mathrm{Spec}(R)$; since $PR_Q capR = P$, by Lemma 3.4 $P^{\star_w} = P$. Conversely, if $P^{\star_w} = P$, then $PR_M \neq R_M$ for some $M \in \star - \mathrm{Spec}(R)$; hence $P \subseteq M$ and P is in the generization.

   The second claim follows because, if $\Delta \subsetneq \mathrm{Spec}(R)$, Δ and $\overline{\Delta}$ share the same maximal elements.

4. Since $\star - \mathrm{Max}(R) = \star_w - \mathrm{Max}(R)$, the intersections $\bigcap_{M \in \star - \mathrm{Max}(R)} IR_M$ and $\bigcap_{M \in \star_w - \mathrm{Max}(R)} IR_M$ are the same, and $\star_w = (\star_w)_w$.

   If $\star_1 \leq \star$, $(\star_1)_w \leq \star_w$; but, being $\star_1$ spectral, $(\star_1)_w = \star_1$, because they are two spectral operations with the same set of maximal ideals, and $\star_1 \leq \star_w$.

□

Part 4 implies that every spectral operation of finite type is obtained as $\star_w$ of itself; since each $\star_w$ is of finite type, we have that a spectral operation is of finite type if and only if it is in the form $\star_w$.

Proposition 3.19. *If $\star$ be a star operation of finite type, then $\star - \mathrm{Spec}$ is compact in $\mathrm{Spec}(R)$.*

Proof. $\star - \mathrm{Spec}$ induces the spectral operation $\star_w$, which is of finite type; by Proposition 3.13, $\star - \mathrm{Spec}$ is compact. □

3.5 Chain conditions

In this section, we pursue the investigation of closure operations that satisfy Proposition 1.20, that is, closure operations c such that $\mathcal{I}^c$ satisfies the ascending chain condition or, equivalently, such that every ideal is strictly c-finite. To shorten the notation, in this case we say that R is *c-Noetherian.*

Proposition 3.20. *If R is c-Noetherian and $c \leq d$, then R is d-Noetherian; in particular, each closure operation bigger than c is of finite type.*

Proof. If $c \leq d$, then $\mathcal{I}^d \subseteq \mathcal{I}^c$; thus every ascending chain of d-ideals is a chain of c-ideals, and hence stabilizes. □

Proposition 3.21. *If R is c-Noetherian, then $c - \mathrm{Spec}(R)$ is a Noetherian topological space.*

Proof. Let $\Delta = c - \mathrm{Spec}(R)$. With the same notation of the proof of Proposition 3.13, $V_\Delta(I) = V_\Delta(I^c)$ for every I, because, if P is a c-ideal, then $I \subseteq P$ if and only if $I^c \subseteq P$.

Let $\{D_\Delta(I_\alpha)\}_{\alpha\in A}$ be an ascending chain of open sets; without loss of generality, we can assume that each I_α is a c-ideal. Thus $\{I_\alpha\}_{\alpha\in A}$ is an ascending chain of c-ideals, which stabilizes by hypothesis, and also $\{D_\Delta(I_\alpha)\}_{\alpha\in A}$ stabilizes; i.e., $c - \mathrm{Spec}(R)$ is Noetherian. $\square$

The condition that $c-\mathrm{Spec}(R)$ is Noetherian is far from sufficient: if c is the identity, there are non-Noetherian rings such that $c-\mathrm{Spec}(R) = \mathrm{Spec}(R)$ is Noetherian (for example, finite-dimensional valuation domains).

Corollary 3.22. *Let c be a semi-prime operation. If R is c-Noetherian, then every c-ideal has only a finite number of minimal primes.*

Proof. Let $I = I^c$ and $\Delta := c - \mathrm{Spec}(R) \cap V(I) = \{P \in \mathrm{Spec}(R) \mid I \subseteq P = P^c\}$: then Δ is a Noetherian topological space (since it is a subspace of $c - \mathrm{Spec}(R)$, which is Noetherian by the previous proposition) and thus has only a finite number of minimal elements. But the minimal elements of Δ are exactly the minimal primes of I that are c-ideals; since R is c-Noetherian, c is of finite type, and thus every minimal prime of a c-ideal is c-closed by Proposition 1.17. Hence I has only a finite number of minimal ideals. $\square$

Corollary 3.23. *Let c be a semi-prime operation, and suppose that R is c-Noetherian. If P is a c-prime ideal of height n, then there is a n-generated ideal I such that P is minimal over I^c.*

Proof. We proceed by induction: if $h(P) = 1$ the thesis is immediate. Suppose $h(P) = n$: let Q be a prime of height $n-1$ contained in P, and let I be a $(n-1)$-generated ideal such that Q is minimal over I^c. I^c has only a finite number of minimal ideals and thus there is a $x \in P$ not contained in any minimal prime of I; then (I, x) is n-generated and $(I, x)^c \subseteq P$, but (I, x) is not contained in any minimal prime of I, and thus P is minimal over $(I, x)^c$. $\square$

If $\star$ is a star operation, Proposition 3.20 shows that if R is $\star$-Noetherian then it is also v-Noetherian; these are called *Mori domains*, and have been extensively studied (see e.g. [10]).

If $\star$ is spectral, the similarity between $\star$-Noetherian and Noetherian rings becomes even more marked.

Theorem 3.24. *[15, Theorem 2.6] Let $\star$ be a finite type spectral operation on R. Then R is $\star$-Noetherian if and only if every prime $\star$-ideal is $\star$-finite.*

Proof. If R is $\star$-Noetherian, every ideal is $\star$-finite, and hence so are prime ideals.

Suppose that every prime $\star$-ideal is $\star$-finite, and let $\mathcal{J}$ be the set of $\star$-ideals that are not $\star$-finite; suppose that it is not empty. If $\{J_\alpha\}$ is a chain in $\mathcal{J}$, then $J := \bigcup J_\alpha$ is in $\mathcal{J}$: it is a $\star$-ideal (every finitely generated ideal contained in J is contained in some J_α, thus every $\star$-finite ideal is contained in some J_α and $J^\star$ is their union, because $\star$ is of finite type), and is not $\star$-finite (otherwise it would be equal to some J_α, that would be $\star$-finite). By Zorn lemma, $\mathcal{J}$ has a maximal element I.

By hypothesis, I is not prime; let $a, b \in R \setminus I$ such that $ab \in I$, and let $H := (I :_R a)$: then H is a $\star$-ideal ($H^\star = (I :_R a)^\star = (I^\star :_R a) = (I :_R a) = H$) and is $\star$-finite, because $I \subseteq H$ and $b \in H \setminus I$; set $H^\star = H_0^\star$, with H_0 finitely generated. Similarly, $I \subsetneq (I, a)$ and thus $(I, a)^\star = (x_1, \ldots, x_n, a)^\star$ for some $x_i \in I$ (if $(I, a)^\star = (y_1, \ldots, y_n)^\star$, with $y_i \in (I, a)$, then $y_i = x_i + \alpha_i a$ for some $x_i \in I$).

We claim that $I^\star = (x_1, \ldots, x_n, Ha)^\star$: the $\supseteq$ containment follows because every $x_i \in I$ and $Ha = a(I :_R a) \subseteq I$; for the other, let $y \in I$. Then $y \in (I, a)^\star = (x_1, \ldots, x_n, a)^\star$ and thus $y \in (x_1, \ldots, x_n, a)R_M$ for any $\star$-maximal ideal M; hence there is a $s \in R \setminus M$ such that $sy = x + \beta a$, where $x \in (x_1, \ldots, x_n)$ and $\beta \in R$; but then sy and x are in I, and thus $\beta \in (I :_R a) = H$, and $y \in (x_1, \ldots, x_n, Ha)R_M$. Hence $I \subseteq (x_1, \ldots, x_n, Ha)R_M$ for every M, and $I^\star \subseteq (x_1, \ldots, x_n, Ha)^\star$.

Now

$$
\begin{aligned}
(x_1, \ldots, x_n, Ha)^\star &= ((x_1, \ldots, x_n) + Ha)^\star = ((x_1, \ldots, x_n)^\star + (Ha)^\star)^\star = \\
&= ((x_1, \ldots, x_n)^\star + H^\star a)^\star = ((x_1, \ldots, x_n)^\star + H_0^\star a)^\star = (x_1, \ldots, x_n, H_0 a)^\star
\end{aligned} \tag{3.31}
$$

which is $\star$-finite because H_0 is finitely generated. Hence I is $\star$-finite, against the hypothesis, and $\mathcal{J}$ is empty. □

The hypothesis that $\star$ is of finite type is necessary: if R the ring of entire functions, and $\star$ is the spectral operation defined after Corollary 3.15, then every $\star$-prime ideal is principal, but $\star$ is not of finite type and thus R is not $\star$-Noetherian.

Proposition 3.25. *Let $\star$ be a spectral operation on R, and suppose that R is $\star$-Noetherian. Then R_P is a Noetherian ring for every $P \in \star - \mathrm{Spec}(R)$.*

Proof. If J is an ideal of R_P, there is a $I \trianglelefteq R$ such that $J = IR_P$; since $I^\star$ is $\star$-finite (by Proposition 1.20), $I^\star = H^\star$ for a $H \subseteq I$ finitely generated. Then, by Proposition 3.3, $J = IR_P = HR_P$ is finitely generated, and R_P is Noetherian. □

The previous proposition shows that every statement about Noetherian rings, depending only on the local structure of the ring, can be carried over to $\star$-Noetherian domains, although restricted to $\star$-ideals. Two almost immediate corollaries are the Principal Ideal Theorem and the Krull Intersection Theorem.

Proposition 3.26. *Let $\star$ be a spectral operation on R, and suppose that R is $\star$-Noetherian.*

1. *R satisfies the Principal Ideal Theorem, i.e., every prime minimal over a principal ideal has height 1.*
2. *If P is a prime ideal minimal over $(a_1, \ldots, a_n)^\star$, then $h(P) \leq n$.*
3. *If P is a $\star$-prime ideal, then $h(P)$ is finite.*
4. *Let $P \subsetneq Q$ be $\star$-prime ideals. The set of prime ideals properly contained between P and Q is either empty or infinite.*

Proof. Part 1 is an immediate consequence of part 2, because every principal ideal is $\star$-closed.

For part 2, since R_P is Noetherian, PR_P (which is minimal over IR_P) has height at most n, by the (generalized) Principal Ideal Theorem. Moreover, since $P = (x_1, \ldots, x_m)^\star$ for some $x_1, \ldots, x_m \in P$, the height of P is $\leq m$, and in particular is finite. (Alternatively, $h(P) = h(PR_P)$, which is finite because R_P is a Noetherian local ring.)

In the same way, the last part follows because it is valid in the Noetherian ring R_Q, and because there is a one-to-one order-preserving correspondence between prime ideals in R_Q and prime ideals of R contained in Q. $\square$

Proposition 3.27. *Let $\star$ be a spectral operation on R, and suppose that R is $\star$-Noetherian. If $I^\star \neq R$, then $\bigcap_{n \geq 1} (I^n)^\star = (0)$.*

Proof. Suppose that z is in the intersection, and let M be a $\star$-maximal ideal containing $I^\star$; then, for every integer n, $z \in (I^n)^\star \subseteq I^n R_M = (IR_M)^n$. But R_M is Noetherian, and thus $z = 0$. $\square$

Another theorem which remains valid for $\star$-Noetherian rings is primary decomposition.

Proposition 3.28. *Let $\star$ be a spectral operation on R, and suppose that R is $\star$-Noetherian. Then, for each $\star$-integral ideal, there are primary ideals $Q_1, \ldots, Q_n$, which are $\star$-ideals, such that $I = Q_1 \cap \cdots \cap Q_n$.*

Proof. The proof mirrors the corresponding proof for Noetherian rings: we show that every $\star$-ideal can be decomposed in irreducible $\star$-ideals, and that these are primary.

Say that a $\star$-ideal I is *$\star$-irreducible* if $I = J \cap H$, with $J, H \in \mathcal{I}^\star$, imply that $J = I$ or $H = I$, and let $\mathcal{R}$ be the set of $\star$-ideals which can't be written as a finite intersection of $\star$-irreducible ideals. If this set is nonempty, it has a maximal element M, which is not irreducible; hence $M = M_1 \cap M_2$, where M_1 and M_2 are decomposable as intersection of $\star$-invertible ideals. Hence also M is, and every ideal is decomposable.

Suppose now that I is a $\star$-irreducible ideal, and define on $R' := R/I$ the closure operation c given by

$$J^c := \frac{\tilde{J}^\star}{I} \tag{3.32}$$

for every $J \trianglelefteq R'$, where $\tilde{J}$ is the (unique) ideal containing I and projecting to J under the quotient map; the c-ideals are just the quotients of the $\star$-ideals containing I, and so the set $\mathcal{I}^c(R')$ satisfies the ascending chain condition. Let $\tilde{x}, \tilde{y} \in R$, and suppose that $\tilde{x}\tilde{y} \in I$, and let x, y be the respective images in R'; the chain $\{\mathrm{Ann}(x^m)^c\}$ is ascending and hence stabilizes at $\mathrm{Ann}(x^n)^c$ (say). Let H_m be ideals of R such that $H_m/I = \mathrm{Ann}(x^m)$, and set $H := H_n$.

Suppose that $z \in (x^n) \cap (y)$; then $xz \in x(y) = (xy) = (0)$, and $z \in \mathrm{Ann}(x)$. Moreover, since $z = \alpha x^n$ (for an element $\alpha \in R'$), then $\alpha x^{n+1} = \beta x^n x = zx = 0$, and thus $\alpha \in \mathrm{Ann}(x^{n+1}) \subseteq \mathrm{Ann}(x^n)^c$. Let $\tilde{\alpha}$ be an element of R which projects to α; then $\tilde{\alpha} \in H^\star$, and since $\star$ is spectral, $\tilde{\alpha} \in H^\star = \bigcap HR_M$. We have that $H/I = \mathrm{Ann}(x^n)$; thus $x^n(H/I) = 0$, and $\tilde{x}^n H \subseteq I$. Hence

$$\tilde{x}^n H^\star = \tilde{x}^n \bigcap HR_M = \bigcap \tilde{x}^n HR_M \subseteq \bigcap IR_M = I^\star = I \tag{3.33}$$

and in particular $\tilde{\alpha}\tilde{x}^n \in I$. But then $\alpha x^n = 0$ in R', that is, $z = 0$; hence $(x^n) \cap (y) = (0)$, and (0) is primary in R'. But then I is primary in R, and every $\star$-irreducible ideal in R is primary; in particular, every $\star$-ideal has a decomposition in primary $\star$-ideals. □

If R is Noetherian, the proposition provides an extension of Proposition 1.12, where there were no control over embedded components.

Corollary 3.29. *Let $\star$ be a spectral operation on R, and suppose that R is $\star$-Noetherian. Then every $\star$-ideal contains a power of its radical.*

Proof. Let $P_1 \ldots, P_m$ the minimal primes of I (there are only a finite number by Proposition 3.22). For each i, $P_iR_{P_i}$ is minimal over IR_{P_i}, and thus is its radical; since R_{P_i} is Noetherian, there are n_i such that $(P_iR_{P_i})^{n_i} \subseteq IR_{P_i}$.

Moreover, $\mathrm{rad}(I) = P_1 \cap \cdots \cap P_m$, and thus $\mathrm{rad}(I)R_{P_i} = P_iR_{P_i}$; let $n := \max_i\{n_i\}$. Then

$$\mathrm{rad}(I)^n \subseteq (\mathrm{rad}(I)^n)^\star = \bigcap_{i=1}^m \mathrm{rad}(I)^n R_{P_i} = \bigcap_{i=1}^m (\mathrm{rad}(I)R_{P_i})^n = \\ = \bigcap_{i=1}^m (P_iR_{P_i})^n \subseteq \bigcap_{i=1}^m (P_iR_{P_i})^{n_i} \subseteq \bigcap_{i=1}^m IR_{P_i} \quad (3.34)$$

Now $IR_{P_i} \cap R = Q_iR_{P_i} \cap R = Q_i$ because Q_i is P_i-primary, and thus $\bigcap_i IR_{P_i} = \bigcap_i Q_i = I$; hence $\mathrm{rad}(I)^n \subseteq I$. □

Both the Principal Ideal Theorem and primary decomposition need not to hold if R is $\star$-Noetherian, but $\star$ is not spectral. For example, if $R = K[\{XY^n \mid n \geq 0\}]$, then R is a Mori (i.e. v-Noetherian) domain of dimension 2, $M = (\{XY^n \mid n \geq 0\})$ is maximal of height 2 and minimal over (X); v-primary decomposition holds if and only if every divisorial prime ideal has height 1, and thus a counterexample is a Noetherian ring with a height 2 divisorial prime ideal, for example $R = K[[x^2, x^3, xy, y^2]]$ (if $M = (x^2, x^3, xy, y^2)$, then $x \in (R : M) \setminus R$ and so $M^v = M$) [10, Section 3].

We note that the results of this section can be generalized to *semistar operations* [42].

3.6 Constructions of star operations

Definition 3.30. *Let $\star$ be a star operation on an integral domain R and $\star'$ a star operation on an extension ring S of R. Then the map $\delta(\star, \star')$ is defined by*

$$I \mapsto I^{\delta(\star,\star')} := (IS)^{\star'} \cap I^\star. \quad (3.35)$$

Proposition 3.1 implies that $\delta(\star, \star')$ is a closure operations; moreover

$$xI^{\delta(\star,\star')} = x((IS)^{\star'} \cap I^\star) = x(IS)^{\star'} \cap xI^\star = ((xI)S)^{\star'} \cap (xI)^\star = (xI)^{\delta(\star,\star')} \quad (3.36)$$

and thus $\delta(\star, \star')$ is a star operation. Clearly, $\delta(\star, \star') \leq \star$.

A productive way to build star operations is to use this construction with different localization of R. For a star operation $\star$ and a set Λ of prime ideals, we say that the star operation *induced by $\star$ and Λ* is

$$I \mapsto I^{\delta(\star,\Lambda)} := I^\star \cap \bigcap_{P \in \Lambda} IR_P. \quad (3.37)$$

Proposition 3.31. *Let $\star$ be a star operation on R and $\Lambda \subseteq \mathrm{Spec}(R) \setminus \star-\mathrm{Spec}(R)$ a finite set of cardinality n. There are $n+1$ sets $\Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_{n+1} = \Lambda$ such that $\delta(\star, \Lambda_i) < \delta(\star, \Lambda_j)$ for $i > j$.*

Proof. Put $\Lambda := \{P_1, \ldots, P_n\}$ and $\delta_i := \delta(\star, \Lambda_i)$.

Define $\Lambda_0 := \varnothing$ and $\Lambda_{i+1} := \Lambda_i \cup \{Q_i\}$, where $Q_i \in \Lambda$ is minimal in $\Lambda \setminus \Lambda_i$. Then $\delta_0 = \star$, and δ_i is the infimum of δ_{i-1} and $IR_{P_i} \cap R$, so that $\delta_i \leq \delta_{i-1}$.

Suppose $P \in \Lambda_i$. Since $P = PR_P \cap R$, $P^{\delta_i} = P$ by Lemma 3.4, and thus P is a δ_i-prime. On the contrary, if $P \in \Lambda \setminus \Lambda_i$, then each P is not contained in any $Q \in \Lambda_i$, and thus $PR_Q = R_Q$. It follows that

$$P^{\delta_i} = P^\star \cap \bigcap_{Q \in \Lambda_i} PR_Q = P^\star \cap \bigcap_{Q \in \Lambda_i} R_Q = P^\star \cap \bigcap_{Q \in \Lambda_i} R_Q \cap R = P^\star \cap R = P^\star \tag{3.38}$$

and hence $P^{\delta_i} \neq P$, since $P \notin \star - \mathrm{Spec}(R)$ by hypothesis. In particular, $\delta_i < \delta_{i-1}$, since P_i is a δ_i-ideal but not a δ_{i-1}-ideal. □

As a corollary, we get that if $\mathrm{Spec}(R) \setminus \star - \mathrm{Spec}(R)$ is infinite, then there are an infinite number of star operations smaller than $\star$. The result of the proposition is the best possible if Λ is totally ordered, because each choice is forced; on the other hand, if the primes of Λ are not comparable, we get more:

Proposition 3.32. *Let $\star$ be a star operation on R, and suppose that $\Lambda \subseteq \mathrm{Spec}(R) \setminus \star - \mathrm{Spec}(R)$ is a set of non-comparable prime ideals. Then every subset of Λ induces a different star operation on R, which is smaller than $\star$.*

Proof. For each set $\Sigma \subseteq \Lambda$, put $\delta_\Sigma := \delta(\star, \Sigma)$. If $P \in \Sigma$, then $P^{\delta_\Sigma} = P$ because the intersection includes R_P; if $P \notin \Sigma$, then $PR_Q = R_Q$ for each $Q \in \Sigma$ and hence

$$P^{\delta_\Sigma} = P^\star \cap \bigcap_{Q \in \Sigma} PR_Q = P^\star \cap R = P^\star. \tag{3.39}$$

Hence the action of δ_Σ on Λ is uniquely determined by Σ, and thus $\delta_\Sigma \neq \delta_\Gamma$ for any pair of subsets $\Sigma \neq \Gamma$ of Λ. □

A suitable Λ is the set of maximal ideals which are not $\star$-ideals. We give three applications.

The first is obtained by choosing $\star = v$: since v is the largest star operation, this choice gives the greatest number of star operations.

Corollary 3.33. *If Λ is the set of non-divisorial maximal ideals, then there are at least $2^{|\Lambda|}$ star operations on R.*

The second is obtained by choosing $\star = w$. Since we are intersecting I^w with other IR_P, we get still a spectral operation; the same construction can be obtained by "adding" prime ideals to $w - \mathrm{Spec}(R)$, and taking the spectral operation associated to $w - \mathrm{Spec}(R) \cup \Sigma$; the choice of the Λ_i in the proof of Proposition 3.31 can be seen as a construction that avoids that P_i is contained in the generization of $w - \mathrm{Spec}(R) \cup \Lambda_{i-1}$. Moreover, since w is the largest finite type spectral operation, all the others can be obtained by adding prime ideals to $w - \mathrm{Spec}$; if we restrict to Noetherian integrally closed domains, we obtain a precise statement:

Corollary 3.34. *Let R be a Noetherian integrally closed domain, and let $\mathcal{S}$ be the set of spectral operations on R. Then:*

- *If* $\dim R = 1$ *then* $|\mathcal{S}| = 1$.
- *If* $\dim R = 2$ *then* $|\mathcal{S}| = 2^{|\mathrm{Max}(R)|}$.
- *If* $\dim R \geq 3$*, then* $|\mathcal{S}| = \infty$.

Proof. By Proposition 2.19, the t-maximal ideals of R are the height 1 prime ideals; moreover, since every operation is of finite type, every $\star - \mathrm{Spec}(R)$ contains the height 1 prime ideals (by Proposition 2.6).

If $\dim R = 1$, then $w - \mathrm{Max} = \mathrm{Max}$ and w is the identity.

If $\dim R = 2$, then each subset of $\mathrm{Max}(R)$ yield a different star operation, and thus $|\mathcal{S}| \geq 2^{|\mathrm{Max}(R)|}$. Conversely, if $\star$ is a spectral star operation, then $\star - \mathrm{Spec}(R) = w - \mathrm{Spec}(R) \cup \Lambda$ for a $\Lambda \subseteq \mathrm{Max}(R)$ (since $\star - \mathrm{Spec}$ contains the height 1 primes, and those are the w-primes), that is, each spectral star operation is associated to a subset of $\mathrm{Max}(R)$. Hence $|\mathcal{S}| = 2^{|\mathrm{Max}(R)|}$.

If $\dim R \geq 3$, then R has an infinite number of height 2 prime ideals, because, in a Noetherian ring, the set of primes properly contained between two prime ideals is either empty or infinite (this is a consequence of the Principal Ideal Theorem: see e.g. [35, Theorem 144]; see also Proposition 3.26). By Proposition 3.31 or Proposition 3.32, $\mathcal{S}$ is infinite. □

The third application concerns the number of spectral operations on a polynomial ring.

Proposition 3.35. *Let R be a domain, $\mathbf{X} = \{X_\alpha\}$ a nonempty set of indeterminates and $\mathcal{S}$ the set of spectral operations on $R[\mathbf{X}]$. Then $\mathcal{S}$ if finite if and only if $|\mathbf{X}| = 1$ and R is a field; moreover, in this case, $R[\mathbf{X}]$ admits only one star operation.*

Proof. If $X \in \mathbf{X}$ and $\mathbf{X}' = \mathbf{X} \setminus X$, then $R[\mathbf{X}] = R[\mathbf{X}'][X]$, so we can assume that $\mathbf{X} = \{X\}$.

If R is a field, then $R[X]$ is a Principal Ideal Domain and hence has only one star operation.

Suppose that R is not a field. Since $R[X] = K[X] \cap R[[X]]$, the map $I \mapsto I^\star = IK[X] \cap IR[[X]]$ is a star operation of finite type. For any maximal ideal M of R, $MR[X]$ is prime in $R[X]$ but not maximal, since $R[X]/MR[X] \simeq (R/M)[X]$, which is not a field; moreover, for a polynomial $f \in R[X]$, the ideal $(M, f)R[X]$ is maximal if and only if the image $\overline{f}$ of f in $(R/M)[X]$ is irreducible, and $(M, f)R[X] = (M, g)R[X]$ if and only if $\overline{f} = \overline{g}$.

For any irreducible polynomial $\alpha \in (R/M)[X]$, $\alpha = \alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n \neq X$, we have that $\alpha_0 \neq 0$, and so α_0 is invertible; hence there is a $f \in R[X]$ such that $\overline{f} = \alpha$ and the term of degree 0 is equal to 1. Let $N := (M, f)R[X]$ for any such f.

Both $NK[X]$ and $NR[[X]]$ are not proper ideals, because N contains some elements that become invertible in $K[X]$ (every $r \in M$; $M \neq (0)$ because R is not a field) and some that become invertible in $R[[X]]$ (f is invertible because the term of degree 0 is invertible). Hence $N^\star = R[X]$, and thus $N^{\star_w} = R[X]$; but there are an infinite number of such N (because $(R/M)[X]$ has an infinite number of maximal ideals), and so, by Proposition 3.31 or Proposition 3.32, $\mathcal{S}$, is infinite. □

Note that, in the above proof, if R is Noetherian $\star$ is itself spectral, because both $K[X]$ (which is a localization of $R[X]$) and $R[[X]]$ (which is the (X)-completion of $R[X]$: see e.g. [9, Proposition 10.14]) are flat $R[X]$-modules.

These results are useful when searching conditions on R equivalent to have only a fixed number of star operations (or only finitely many star operations): for example, if R has exactly n star operations, then it has at most $\lfloor \log_2(n) \rfloor$ non-divisorial maximal ideals. For $n = 2$ or $n = 3$, it follows that R has at most one non-divisorial maximal ideal.

3.7 Historical and bibliographical note

The first star operation defined through overrings has been integral closure, where the family $\{R_\alpha\}$ is the family of valuation overrings of R; see next chapter. The case of arbitrary family of overrings has been treated in [1]; Section 3.2 and the backbone of Section 3.3 (Propositions 3.10, 3.11 and 3.12) comes from there and from [4].

The $\star_w$ operation was first considered, in the general case, in [5], building upon the definition of the w-operation in [47], where it was defined in the way of our Proposition 3.17; Section 3.4 follows [5].

Propositions 3.26, 3.27 and 3.28 are present in [5], where they are proved by the theory of Noether lattices.

Section 3.6 has been inspired by [31].

4. INTEGRAL CLOSURE

4.1 Equivalent definitions

Definition 4.1. *Let R be a domain. The* b-operation *on R is the closure operation associated to the set $\{V_\alpha\}$ of valuation overrings of R; that is,* $I^b := \bigcap_{\alpha \in A} IV_\alpha \cap R$.

Definition 4.2. *Let R be an arbitrary ring. An element $r \in R$ is* integral *over I if there are elements $a_i \in I^i$ and an $n \in \mathbb{N}$ such that*

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \tag{4.1}$$

The set of integral elements over I is the integral closure I^- *of I; if $I = I^-$, it is said to be* integrally closed.

We will prove that, for integral domains, the two definitions coincide and, moreover, the integral closure of an ideal in a ring R is determined by the integral closure in suitable domains arising as quotients modulo prime ideals of R, so that the latter construction, although more general, is determined by the former. This will prove, in particular, that the set of integral elements over an ideal is itself an ideal, and that integral closure is a closure operation.

It should be noted that the name "b-operation" is generally used to denote the map $I \mapsto \bigcap_{\alpha\in A} IV_\alpha$, without the intersection with R; this means that, if R is not integrally closed, I^b is not always an ideal of R but rather it is an ideal of its integral closure $\overline{R}$, and, likewise, $R \neq R^b = \overline{R}$. With this definition, b is no more a closure operation on R but a *semistar operation*: these are closure operation defined on the set of R-submodules of K satisfying all the properties of the star operations (Definition 2.8), except for $R^\star = R$. I have chosen to use this definition to continue with the assumptions and notations of the previous chapters.

We begin with three lemmas: the first will also be useful later, while the third provides an important property of integral closure.

Lemma 4.3. *Let R be any ring, I an ideal and $r \in I^k$. There is a finitely generated ideal $J \subseteq I$ such that $r \in J^k$.*

Proof. Since $r \in I^k$, $r = \alpha_1\pi_1 + \cdots + \alpha_n\pi_n$, where $\alpha_i \in R$ and every π_i is a monoid of degree k in elements of I; if $\pi_i = x_1^{(i)} \cdots x_k^{(i)}$, for $x_j^{(i)} \in I$, then $\pi_i \in (x_1^{(i)}, \dots, x_k^{(i)})^k$, and $r \in J^k$, where $J := (\{x_j^{(i)} \mid i = 1, \dots, n, j = 1, \dots, k\})$ is a finitely generated ideal. □

Lemma 4.4. *In a valuation ring V, every ideal is integrally closed.*

Proof. It is clear that every $i \in I$ is in I^-; suppose $r \in I^-$, and take an equation of integral dependence over I: $r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0$. Passing to the valuation v, we have

$$v(r^n) = v(a_1r^{n-1} + \cdots + a_{n-1}r + a_n) \geq \min\{v(a_ir^{n-i})\} = v(a_kr^{n-k}) \quad (4.2)$$

for a $k \in \{1, \dots, n\}$, $a_k \in I^k$. Thus

$$nv(r) = v(r^n) \geq v(a_kv^{n-k}) = v(a_k) + (n-k)v(r) \Longrightarrow v(r^k) \geq v(a_k) \quad (4.3)$$

By the previous lemma, there is a finitely generated ideal $J \subseteq I$ such that $a_k \in J^k$; but in a valuation ring every finitely generated ideal is principal, and so $a_k = \alpha j^k$ for some $j, \alpha \in V$. Hence $v(r^k) \geq v(\alpha j^k)$, and thus $\frac{r^k}{j^k} = \alpha \in V$. Since V is integrally closed, this imply $\frac{r}{j} \in V$, that is, $r \in jV \subseteq I$. □

Lemma 4.5. *Integral closure is* persistent*: that is, if $\phi : R \longrightarrow S$ is a ring homomorphism and I an ideal of R, then $\phi(I^-) \subseteq (\phi(I)S)^-$.*

Proof. Let $r^n + a_1r^{n-1} + \cdots + a_n = 0$ be an equation of integral dependence of r over I; then, applying ϕ, we have $\phi(r)^n + \phi(a_1)\phi(r)^{n-1} + \cdots + \phi(a_n) = 0$; but since $a_i \in I^i$, $\phi(a_i) \in \phi(I^i)S = (\phi(I)S)^i$ and $\phi(r)$ is integral over $\phi(I)S$. □

Definition 4.6. *Let Δ be a set of extension domains of R. Δ is a b-*set *of R (or for R) if each $V \in \Delta$ is a valuation ring that contains R and, for all ideals I of R,*

$$I^- = \bigcap_{V \in \Delta} IV \cap R. \quad (4.4)$$

A b-set is discrete *if each $V \in \Delta$ is a discrete valuation ring.*

The above definition is obviously useless without a criterion to determine if a Δ is a b-set:

Proposition 4.7. *Let Δ be a set of valuation rings containing R, and suppose that it satisfies the following condition:*

> *for every ideal $I \trianglelefteq R$, every $x \in R$ and every maximal ideal M of $S := R\left[\frac{I}{x}\right]$ (that is, the R-algebra generated by the elements $\frac{i}{x}$ for $i \in I$) there is a $V \in \Delta$ such that $S \subseteq V$ and $MV \neq V$.*

Then Δ is a b-set.

Proof. We have to show that $I^- = \bigcap_{V\in\Delta} IV \cap R$.

($\subseteq$) By persistence (Lemma 4.5) $I^- \subseteq (IV_\alpha)^-$; by Lemma 4.4, $(IV)^- = IV$, and thus $I^- \subseteq IV_\alpha$ for every α, and I^- is contained in the intersection.

($\supseteq$) Suppose r is in the intersection, and take the ring $S := R\left[\frac{I}{r}\right]$. For every $V \in \Delta$ such that $S \subseteq V$, we have $\left(\dfrac{I}{r}S\right) V = V$ because $r \in IV$ and so $\frac{r}{r} \in \left(\frac{I}{r}S\right) V$.

Suppose that $\left(\frac{I}{r}\right) S \neq S$. Then $\frac{I}{r}S \subseteq M$ for a maximal ideal M of S, and thus there is a valuation ring $V \in \Delta$ such that $MV \neq V$; but this implies that $\frac{I}{r}V \subseteq MV \neq V$, against what we have proved. Then $\frac{I}{r}S = S$; hence $1 \in \frac{I}{r}S$ and

$$1 = \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots + \frac{a_n}{r^n} \tag{4.5}$$

where $a_i \in I^i$. Multiplying by r^n, we get an equation of integral dependence of r over I, and $r \in I^-$. $\square$

Thus to prove that $I^b = I^-$ what we have to show is that the set of valuation overrings satisfies the condition of the previous proposition. We do this through a well-known result:

Theorem 4.8. *Let P be a prime ideal of a domain R and let F be a field containing R. There is a valuation ring V containing R and with quotient field F such that $PV \neq V$.*

Proof. Let $\mathcal{R}$ be the set of rings between R and F such that $PS \neq S$; it is not empty because $R \in \mathcal{R}$. If $\{S_\alpha\}$ is a chain in $\mathcal{R}$, then their union S too belongs to $\mathcal{R}$: if not, $PS = S$ and $1 = s_1p_1 + \cdots + s_np_n$ for some $s_i \in S$, $p_i \in P$; since every s_i belongs to all member of the chain after S_{α_i} (say), there is an $\overline{\alpha}$ such that every $s_i \in S_{\overline{\alpha}}$, and thus $1 \in PS_{\overline{\alpha}}$, contradicting the fact that $PS_{\overline{\alpha}} \neq S$.

By Zorn lemma, it follows that $\mathcal{R}$ has a maximal element V; suppose that there is an $x \in F$ such that neither x nor x^{-1} are in V. Since V is maximal, the rings $V[x]$ and $V[x^{-1}]$ are not in $\mathcal{R}$, and thus $PV[x] = V[x]$ and $PV[x^{-1}] = V[x^{-1}]$; therefore

$$\begin{cases} 1 = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n \\ 1 = q_0 + q_1x^{-1} + \cdots + q_mx^{-m} \end{cases} \tag{4.6}$$

for some elements $p_i, q_i \in PV$, and natural numbers $n \geq m$ (without loss of generality) which can be chosen to be the smallest possible. Multiplying by

x^n, we get $x^n = q_0x^n + q_1x^{n-1} + \cdots + q_mx^{n-m}$, i.e., $(1-q_0)x^n = q_1x^{n-1} + \cdots + q_mx^{n-m}$. Multiplying the first equality by $1-q_0$ and substituting we have that

$$\begin{gathered} p_n(1-q_0)x^n + \cdots + (1-q_0)p_1x + (1-q_0)p_0 = 1-q_0 \\ p_n(q_1x^{n-1} + \cdots + q_mx^{n-m}) + p_{n-1}(1-q_0)x^{n-1} + \cdots + (1-q_0)p_0 + q_0 = 1 \end{gathered} \tag{4.7}$$

is an equation of degree $n-1$, against the minimality of n. Hence $PV[x]$ and $PV[x^{-1}]$ cannot both be equal to the $V[x]$ and $V[x^{-1}]$ (respectively), and thus one of these should be in $\mathcal{R}$, against the maximality of V. Then V is a valuation ring with quotient field F. □

Corollary 4.9. *The set Δ of valuation rings between R and a field F is a b-set of R.*

Proof. The first two conditions are clearly satisfied. Let $S := R\left[\frac{I}{x}\right]$, and let $M \in \operatorname{Max}(S)$: since S has the same quotient field of R, if $R \subseteq F$ also $S \subseteq F$. By the previous theorem, there is a valuation ring between S and F such that $MV \neq V$; hence Δ satisfies the condition of Proposition 4.7. □

Proposition 4.10. *Let R be an integral domain. For every ideal I, $I^- = I^b$.*

Proof. Just take F to be the quotient field of R in the above corollary. □

We note that these proofs does not use in a fundamental way the intersection with R: so with the same reasoning we can deduce that, if I is an integral ideal (or a fractional ideal) of R, the intersection $\bigcap IV$ is equal to the set of elements of K (the quotient field of R) that are integral over I (i.e., $x \in K$ such that there is an equation $x^n + a_1x^{n_1} + \cdots + a_n = 0$ with $a_i \in I^i$). Moreover, this statement can be extended to the case where I is merely an R-submodule of K, and not a fractional ideal; however, the condition of Proposition 4.7 must be replaced by a stronger version, where, instead of the $R[\frac{I}{x}]$, we consider all the overrings of R (it would be sufficient to consider only the rings $R[\frac{H}{x}]$, but, for every overring S, we have that $S = R[\frac{xS}{x}]$ and so every overring is in this form). While the set of valuation overrings of R is still enough to get the equality, this is not true for every b-set: for example, if W is a valuation overring of R with maximal ideal M, there are no proper overrings V of W such that $MV \neq V$, and thus every valuation overring of R must appear among the V in the intersection $\bigcap IV$.

4.1.1 The Noetherian case: discrete b-sets

When R is Noetherian, not all the valuation rings whose existence is guaranteed by Theorem 4.8 are Noetherian: in fact, it could be proved that there are valuation overrings of R with dimension equal to $\dim R$ [22, Corollary 19.7], and thus every Noetherian domain of dimension greater than 1 has non-Noetherian valuation overrings. However, Noetherian valuation overrings are enough to obtain integral closure; the strategy is the same of the general case, but a different argument is needed to assure the existence of discrete valuation overrings.

Theorem 4.11. *Let R be a Noetherian domain and P a prime ideal. There is a discrete valuation overring V of R such that $PV \neq V$.*

Proof. [16, Lemme 2] It is enough to consider the case where R is local and $P = M$ is the maximal ideal of R.

Let $x_1, \ldots, x_d$ be a system of parameters in R (i.e., $d = \dim R$ and $(x_1, \ldots, x_d)$ is M-primary); let $I := (x_1, \ldots, x_d)$, $J := (x_1, \ldots, x_{d-1})$, and define $y_i := \frac{x_i}{x_d}$ ($x_d \neq 0$ because, otherwise, M would be minimal over J, and thus would have height $\leq d-1$) and $S := R[y_1, \ldots, y_{d-1}]$. Suppose $x_dS = S$: then $1 = x_dP(y_1, \ldots, y_{d-1})$, where P is a polynomial with coefficients in R with degree g and constant term a. Multiplying by x_d^s for a $s \geq g$, we get $x_d^s = x_d^{s+1}P'(x_1, \ldots, x_d)$, where P' has no terms only in x_d but the constant term a. Hence $x_d^s \equiv ax_d^{s+1} \bmod J$, and thus $x_d^s \in (x_d^{s+1}) + J$; by induction, $x_d^s \in (x_d^{s+n}) + J$ for every $n \in \mathbb{N}$. Hence, in particular,

$$x_d^g \in J + \bigcap_{n \geq 1}(x_d^{g+n}) = J, \tag{4.8}$$

because in a Noetherian ring the intersection of the powers of a principal ideal is (0). But this would imply that, if $M^m \in I$ and k is bigger than m and g, then $M^k \subseteq J$, against the hypothesis that M is not minimal over J. Thus $x_dS \neq S$.

Let Q be a prime ideal minimal over x_dS. By the Principal Ideal Theorem, Q has height 1, and thus S_Q is Noetherian of dimension 1; hence its integral closure T is again Noetherian and of dimension 1 (see e.g. [35, Theorem 93]), and thus is a discrete valuation ring with maximal ideal N. Moreover, N contains x_d (because $Q \subseteq N$) and also the other x_i (because $x_i = y_ix_d \in x_dS$), and thus $I \subseteq N \cap R$. But $N \cap R$ is prime, and M is the unique prime ideal over I; hence $M = N \cap R$ and $MT \neq T$. □

Corollary 4.12. *Let R be a Noetherian domain. The set of discrete valuation overrings of R satisfies the condition of Proposition 4.7; moreover, so does* *the set of discrete valuation overrings whose maximal ideal contracts to a maximal ideal of R.*

Proof. It is sufficient to note that every $S = R\left[\frac{I}{x}\right]$ is Noetherian because I is finitely generated, making S a finitely generated R-algebra, which is Noetherian by Hilbert Basis Theorem, and then apply the previous theorem. □

This is sufficient to prove that $I^b = \bigcap IV \cap R$, where the intersection ranges among DVRs; we make a small deviation to include some more cases.

Lemma 4.13. *Let R be a domain; for every maximal ideal M, let Δ_M be a b-set for R_M. Then $\Delta := \bigcup_{M\in\mathrm{Max}(R)} \Delta_M$ is a b-set for R.*

Proof. We use the fact that $I^bR_M = (IR_M)^b$, which is proved in Proposition 4.27 below.

We have that

$$I^b = \bigcap_{M\in\mathrm{Max}(R)} I^bR_M = \bigcap_{M\in\mathrm{Max}(R)} (IR_M)^b = \bigcap_{M\in\mathrm{Max}(R)} \bigcap_{V\in\Delta_M} (IR_M)V = \\ = \bigcap_{M\in\mathrm{Max}(R)} \bigcap_{V\in\Delta_M} IV = \bigcap_{V\in\Delta} IV \quad (4.9)$$

and thus Δ is a b-set for R. □

Proposition 4.14. *Let R be a locally Noetherian domain (i.e., R_M is Noetherian for every maximal ideal M), and let $\Delta_R = \{V_\alpha\}$ be the set of discrete valuation overrings of R such that $MV_\alpha \neq V_\alpha$ for some $M \in \mathrm{Max}(R)$. Then*

$$I^b = \bigcap_{V_\alpha\in\Delta_R} IV_\alpha \cap R. \quad (4.10)$$

In particular, a locally Noetherian domain admits a discrete b-set.

Proof. For each $M \in \mathrm{Max}(R)$, the set Δ_M of discrete valuation overrings V of R_M such that $MV \neq V$ satisfies the condition of Proposition 4.7, and thus is a discrete b-set of R_M; by the previous lemma, so is the union $\Delta := \bigcup \Delta_M$, which clearly is discrete. □

Proposition 4.15. *Integral closure is a Nakayama closure.*

Proof. Let (R, M) be a Noetherian local ring, and suppose $J \subseteq I \subseteq (J + MI)^b$. Then, for every $V \in \Delta$ (where Δ is the set of the previous proposition),

$IV \subseteq (J + MI)V = JV + MIV$; since $MV \neq V$ by the choice of Δ, by Nakayama lemma we have that $IV = JV$ for every $V \in \Delta$. Thus

$$I^b = \bigcap_{V\in\Delta} IV = \bigcap_{V\in\Delta} JV = J^b \tag{4.11}$$

and integral closure is Nakayama. □

From this it follows that for every ideal I there are ideals J minimal with respect to the property that $J \subseteq I$ and $J^b = I^b$, and that, if the residue filed R/M is infinite, the minimal number of elements need to generate any such J is the same, and equal to the *analytic spread* $\ell(I)$ of I (see [40] and the discussion in Section 1.6).

We proceed to find another class of domains that are not Noetherian, but that admits discrete b-sets.

Proposition 4.16. *Let $R \subseteq S$ be an extension of domains. The following are equivalent:*

1. *$I^b = (IS)^b \cap R$ for every ideal $I \trianglelefteq R$;*
2. *every b-set of S is a b-set of R;*
3. *there is a b-set of S that is a b-set of R.*

Proof. $(1 \Longrightarrow 2)$. Let Δ be a b-set of S; then

$$I^b = (IS)^b \cap R = \bigcap_{V\in\Delta} ISV \cap S \cap R = \bigcap_{V\in\Delta} IV \cap R \tag{4.12}$$

and thus Δ is a b-set of R.

$(2 \Longrightarrow 3)$ is obvious.

$(3 \Longrightarrow 1)$. Suppose that Δ is a b-set of both R and S. Then $S \subseteq V$ for every $V \in \Delta$ and thus

$$I^b = \bigcap_{V\in\Delta} IV \cap R = \bigcap_{V\in\Delta} ISV \cap R = \bigcap_{V\in\Delta} ISV \cap S \cap R = (IS)^b \cap R. \tag{4.13}$$

□

Proposition 4.17. *Let $R \subseteq S$ be an integral extension of domains. Then $I^b = (IS)^b \cap R$ for every ideal $I \trianglelefteq R$.*

Proof. Suppose firstly that S is the integral closure of R in a field F. Then the valuation rings between R and F with quotient field F are precisely the valuation overrings of S; hence this set is a b-set of both R and S, and $I^b = (IS)^b \cap R$ by the previous proposition.

Suppose that $R \subseteq S \subseteq T$, where T is the integral closure of R in F; this imply that T is also the integral closure of S in F. By the first part of the proof and the previous proposition, every b-set of T is a b-set of R and S, and thus R and S share a common b-set. Hence $I^b = (IS)^b \cap R$. □

Corollary 4.18. *Let $R \subseteq S$ be an integral extension of domains, and suppose that S admits a discrete b-set (for example, if it is Noetherian). Then R admits a discrete b-set.*

Proof. By the previous proposition, R and S share a common b-set, and thus every b-set of S is also a b-set of R. By hypothesis, S has a discrete b-set, and thus so does R. □

Thus every domain with Noetherian integral closure admits a discrete b-set; for example, if $F \subseteq L$ is an infinite algebraic field extension, then $R = F + XL[X]$ and $S = F + XL[[X]]$ have Noetherian integral closure (it is equal, respectively, to $L[X]$ and $L[[X]]$), and so R and S admit a discrete b-set, although they are not Noetherian.

A consequence of the existence of discrete b-sets is given in Proposition 4.40.

If Δ is a b-set for R, the valuation rings of Δ need not to have the same quotient field; however, the quotient field of each $V \in \Delta$ contains the the quotient field K of R. We show that, if needed, we can always suppose that a b-set is composed by overrings.

Lemma 4.19. *Let V be a valuation ring with quotient field F, and let K be a subfield of F. Then $V \cap K$ is a valuation ring of K whose value group is a subgroup of the value group of V; in particular, if V is a DVR, $V \cap K$ is either a field or a DVR.*

Proof. Let $v : F \longrightarrow G$ be the valuation whose ring is V. The restriction $v|_K$ is still a valuation, and its associated ring is $\{x \in K \mid v|_K(x) \geq 0\} = \{x \in F \mid v(x) \geq 0\} \cap K = V \cap K$. Clearly $v|_K(K)$ is a subgroup of G; if V is a DVR, then $G \simeq \mathbb{Z}$ and thus $v|_K(K)$ is isomorphic either to $\mathbb{Z}$ or to the trivial group. □

Proposition 4.20. *Let Δ be a b-set for R, F a field containing R. The set $\Delta \cap F := \{V \cap F \mid V \in \Delta\}$ is a b-set for R, which is discrete if Δ is discrete.*

Proof. By persistence, $I^b \subseteq \bigcap_{W \in (\Delta \cap F)} IW \cap R$; moreover, by the previous lemma, each $W = V \cap F \in \Delta \cap F$ is a valuation ring. We have that $I(V \cap F) \subseteq IV \cap K$

and thus

$$\bigcap_{W\in(\Delta\cap F)} IW\cap R=\bigcap_{V\in\Delta} I(V\cap F)\cap R\subseteq\bigcap_{V\in\Delta} IV\cap F\cap R=\bigcap_{V\in\Delta} IV\cap R=I^b \tag{4.14}$$

so that $\Delta\cap F$ is a b-set; by the previous lemma, if each $V\in\Delta$ is discrete so are the intersections $V\cap K$. □

Corollary 4.21. *If R has a discrete b-set, then for each maximal ideal M there is a discrete valuation overring V such that $MV\neq V$.*

Corollary 4.22. *Let R be a Prüfer domain. R admits a discrete b-set if and only if R_M is a DVR for every maximal ideal M (Prüfer domains with this property are called* almost Dedekind domains*).*

We will prove slightly more in Corollary 4.42, so we delay the proof.

4.1.2 Non-domains

We study how integral closure on a ring R is linked to integral closure of domains.

Proposition 4.23. *An element $r\in R$ is integral over an ideal I if and only if, for every minimal prime P of R, the image of r in R/P is integral over $(I+P)/P$.*

Proof. If r is integral, by persistence $\overline{r}$ is integral over $(I+P)/P$ for every P (and in particular for minimal primes).

Conversely, suppose $\overline{r}$ is integral over $(I+P)/P$ for every minimal P, and define $W:=\{r^n+a_1r^{n-1}+\cdots+a_n\mid n\in\mathbb{N},a_i\in I^i\}$. If $0\in W$, then r is integral over I; otherwise, we note that W is a multiplicatively closed set, because the product of two elements of W is still in W. Hence there is prime ideal Q disjoint from W; thus, for every minimal prime P contained in Q, $P\cap W=\varnothing$. But $\overline{r}$ is integral over $(I+P)/P$, and thus there is an equation of integral dependence $\overline{r}^n+\overline{a_1}\cdot\overline{r}^{n-1}+\cdots+\overline{a_{n-1}}\cdot\overline{r}+\overline{a_n}=\overline{0}$ with $\overline{a_i}\in((I+P)/P)^i=(I^i+P)/P$; hence there are $a_i\in I^i$ such that $w=r^n+a_1r^{n-1}+\cdots+a_{n-1}r+a_n\in P$; this w is both in P and in W, against the hypothesis. Then $r\in I^-$. □

Propositions 4.8 and 4.9 show that the intersection $\bigcap IV$ does not shrink if we consider only valuation overrings or every valuation ring that contains R. The same happens for arbitrary homomorphisms:

Proposition 4.24. *Let R be a ring and let $\mathcal{V}$ be the set of couples (V_α, ϕ_α) where V_α is a valuation domain and $\phi_\alpha : R \longrightarrow V_\alpha$ is a homomorphism. Then*

$$I^b := \bigcap_{(V_\alpha,\phi_\alpha)\in\mathcal{V}} \phi_\alpha^{-1}(\phi_\alpha(I)V_\alpha) \tag{4.15}$$

Proof. By persistence, $\phi_\alpha(I^b) \subseteq (\phi_\alpha(I)V_\alpha)^b$ (Lemma 4.5) and $(\phi_\alpha(I)V_\alpha)^b = \phi_\alpha(I)V_\alpha$ because every V_α is a valuation domain; hence $I^b \subseteq \phi_\alpha^{-1}(\phi_\alpha(I)V_\alpha)$ for every (V_α, ϕ_α).

Conversely, if R is a domain, then $\mathcal{V}$ contains its valuation overrings, and thus the intersection is contained in I^b. If R is not a domain, let $Q_\alpha = \ker(\phi_\alpha)$; Q_α contains a minimal prime P, and thus there is a map $\psi_\alpha : R/P \longrightarrow V_\alpha$; since every map from R/P to a valuation ring V can be extended to a map $R \longrightarrow V$ composing with the quotient map, $\mathcal{V}$ is the union of the sets $\mathcal{V}_P$ of couples $(V_\alpha, \psi_\alpha^{(P)})$ with $\psi_\alpha^{(P)} : R/P \longrightarrow V_\alpha$.

Hence ($\mathrm{Min}(R)$ is the set of minimal primes of R)

$$\bigcap_{(V_\alpha,\phi_\alpha)\in\mathcal{V}} \phi_\alpha^{-1}(\phi_\alpha(I)V_\alpha) = \bigcap_{P\in\mathrm{Min}(R)} \bigcap_{(V_\alpha,\psi_\alpha^{(P)})\in\mathcal{V}_P} (\psi_\alpha^{(P)})^{-1}(\psi_\alpha^{(P)}(\pi_P(I))V_\alpha) =$$
$$= \bigcap_{P\in\mathrm{Min}(R)} \bigcap_{(V_\alpha,\psi_\alpha^{(P)})\in\mathcal{V}_P} (\psi_\alpha^{(P)})^{-1}\left(\psi_\alpha^{(P)}\left(\left(\frac{I+P}{P}\right)V_\alpha\right)\right) = \bigcap_{P\in\mathrm{Min}(R)} \left(\frac{I+P}{P}\right)^b \tag{4.16}$$

and the last intersection is, by Proposition 4.23, exactly I^b. □

4.2 Properties of integral closure

Proposition 4.25. *Integral closure is a semi-prime closure operation of finite type; if R is an integrally closed domain, then it is a star operation. Moreover, $I^b \subseteq \mathrm{rad}(I)$.*

Proof. Integral closure is semi-prime by Propositions 3.1 and 1.13.

To see that it is of finite type, suppose $r^n + a_1r^{n-1} + \cdots + a_n = 0$. Then, by Lemma 4.3, $a_i \in J_i^i$ for finitely generated ideals $J_i \subseteq I$; hence $J := J_1 + \cdots + J_n$ is finitely generated and $a_i \in J^i$, so that $r \in J^b$.

If R is an integrally closed domain, then $R = \bigcap V_\alpha$, where the intersection runs among all the valuation rings between R and K; by the results in Section 3.2, b is a star operation.

For the last claim, it is sufficient to observe that, if $r^n + a_1r^{n-1} + \cdots + a_n = 0$, then $r^n = -a_1r^{n-1} - \cdots - a_n \in I$ and thus $r \in \mathrm{rad}(I)$. □

The above proposition implies, in particular, that if R is an integrally closed domain then $(a)^b = (a)$ for every $a \in R$. It is true also the converse:

Proposition 4.26. *Let R be a ring. R is integrally closed in its total ring of quotient Q if and only if every principal ideal generated by a regular element is integrally closed.*

Proof. We will prove that the principal ideal (a) is integrally closed if and only if R is integrally closed in the localization R_a (as a is regular, the map $R \longrightarrow R_a$ is injective); since every R_a is canonically included in Q and $Q = \bigcup\{R_a \mid a \text{ is a regular element}\}$, this implies that R is integrally closed in Q if and only if it is integrally closed in every R_a, that is, if and only if every principal ideal generated by a regular element is integrally closed.

Suppose R is integrally closed in R_a, and let $r \in (a)^-$; since $(a)^n = (a^n)$, there is an equation $r^n + \alpha_1 a r^{n-1} + \alpha_2 a^2 r^{n-2} + \cdots + \alpha_n a^n = 0$ for some $\alpha_i \in R$. In R_a, a becomes invertible, and hence, dividing the above equation by a^n we get

$$\left(\frac{r}{a}\right)^n + \alpha_1 \left(\frac{r}{a}\right)^{n-1} + \alpha_2 \left(\frac{r}{a}\right)^{n-2} + \cdots + \alpha_n = 0 \tag{4.17}$$

and $\frac{r}{a}$ is integral over R; if R is integrally closed in R_a, then $\frac{r}{a} \in R$, that is, $r \in (a)$.

Conversely, suppose $b = \frac{r}{a^k} \in R_a \setminus R$ is integral over R, for a $r \in R$; then also $c := a^{k-1}b = \frac{r}{a}$ is integral over R. The equation of integral dependence of c over R, when multiplied by a^d (where d is its degree), becomes an equation of integral dependence of r over the ideal (a); if $(a)^- = (a)$, then $r \in (a)$ and $c \in R$. Then $b = \frac{r_1}{a^{k-1}}$ with $r_1 \in R$; repeating the process k times, we obtain $b \in R$, against the hypothesis. Hence R is integrally closed in R_a. □

More generally, integral closure behaves well under localization:

Proposition 4.27. *Let R be a ring and I an ideal.*

1. *$S^{-1}(I^b) = (S^{-1}I)^b$ for every multiplicatively closed subset S.*
2. *The following are equivalent:*
   - *(a) I is integrally closed.*
   - *(b) $S^{-1}I$ is integrally closed for all multiplicatively closed subset S.*
   - *(c) IR_P is integrally closed for all $P \in \mathrm{Spec}(R)$.*
   - *(d) IR_M is integrally closed for all $M \in \mathrm{Max}(R)$.*

Proof. 1. By persistence, $S^{-1}(I^b) \subseteq (S^{-1}I)^b$; suppose that $x = \frac{y}{s} \in (S^{-1}I)^b$ for $y \in R$, $s \in S$, and take an equation of integral dependence of x over $S^{-1}I$:

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{4.18}$$

where $a_i \in (S^{-1}I)^i = S^{-1}I^i$; there is a $t \in S$ such that $b_i := ta_i \in I^i$ for every i, and thus

$$x^n + \frac{b_1}{t}x^{n-1} + \cdots + \frac{b_{n-1}}{t}x + \frac{b_n}{t} = 0 \tag{4.19}$$

and multiplying by t^n we get

$$(tx)^n + b_1(tx)^{n-1} + \cdots + b_{n-1}t^{n-2}(tx) + b_nt^{n-1} = 0 \tag{4.20}$$

and hence $tx \in I^b$; but then $x \in S^{-1}(I^b)$, $(S^{-1}I)^b \subseteq S^{-1}(I^b)$, and they are equal.

2. ($a \Longrightarrow b$) By the previous point, $(S^{-1}I)^b = S^{-1}I^b = S^{-1}I$ and thus $S^{-1}I$ is integrally closed.

($b \Longrightarrow c \Longrightarrow d$) is obvious.

($d \Longrightarrow a$). We have that $J = \bigcap_{M\in\mathrm{Max}(R)} JR_M$ for every ideal J; thus

$$I^b = \bigcap_{M\in\mathrm{Max}(R)} I^bR_M = \bigcap_{M\in\mathrm{Max}(R)} (IR_M)^b = \bigcap_{M\in\mathrm{Max}(R)} IR_M = I \tag{4.21}$$

because $(IR_M)^b = IR_M$ by hypothesis.

□

For two-generated ideals, however, the picture is more complex and more interesting, and it allows to characterize the domains where every ideal is integrally closed.

Lemma 4.28. *Let R be an integrally closed domain. If $xy \in (x^2, y^2)$ (that is, if $(x,y)^2 = (x^2, y^2)$; in particular, if (x^2, y^2) is integrally closed) then (x, y) is invertible.*

Proof. Write $xy = \alpha x^2 + \beta y^2$, with $\alpha, \beta \in R$, and set $\gamma := \frac{\beta y}{x}$; we claim that $(x) = (x, y)(\beta, 1 - \gamma)$.

We have

$$\frac{\beta y}{x} = \frac{\beta xy}{x^2} = \frac{\beta(\alpha x^2 + \beta y^2)}{x^2} = \beta\alpha + \frac{\beta^2y^2}{x^2} = \beta\alpha + \gamma^2 \tag{4.22}$$

and thus $\gamma^2 - \gamma + \beta\alpha = 0$, and $\gamma \in R$ because R is integrally closed; hence $x\beta, x(1-\gamma) \in (x)$, and $y\beta = x\gamma \in (x)$. For $y(1-\gamma)$ we have

$$\frac{y(1-\gamma)}{x} = \frac{y\left(1-\frac{\beta y}{x}\right)}{x} = \frac{y(x-\beta y)}{x^2} = \frac{\alpha x^2 + \beta y^2 - \beta y^2}{x^2} = \alpha \in R \quad (4.23)$$

and $y(1-\gamma) \in (x)$, so that $(x,y)(\beta, 1-\gamma) \subseteq (x)$.

For the reverse containment, we have $x(1-\gamma) + \beta y = x - y\beta + \beta y = x$ and $(x) \subseteq (x,y)(\beta, 1-\gamma)$.

Now (x) is principal, hence invertible; in particular (x,y), being its factor, is invertible. □

Proposition 4.29. *Let R be a domain. Every ideal is integrally closed if and only if R is a Prüfer domain.*

Proof. Principal ideals are integrally closed; thus, by Proposition 4.27, R is integrally closed.

Pick now an ideal (x,y) generated by two elements (if all the ideals are principal, then R is a principal ideal domain and thus is Prüfer); as (x^2,y^2) is integrally closed, (x,y) is invertible by the previous lemma. Since all two-generated ideals are invertible, R is a Prüfer domain ([22, Theorem 22.1]).

Conversely, if R is a Prüfer domain, the unique star operation of finite type on R is the identity; but b is a star operation of finite type (since R is integrally closed), and hence every ideal is integrally closed. □

Corollary 4.30. *Let R be an integrally closed domain. Then, R is a Prüfer domain if and only if the t-operation is equal to the identity.*

Proof. The "only if" part has been yet observed.

Conversely, suppose that every ideal is t-closed. Since integral closure is a closure operation of finite type, it is finer than the t-operation and thus every ideal is integrally closed; by the above proposition, R is Prüfer. □

Corollary 4.31. *Let R be an integrally closed domain. The integral closure distributes over finite intersections if and only if R is a Prüfer domain.*

Proof. Since b is of finite type, it distributes over finite intersections if and only if it is spectral; but, since $I^b \subseteq \mathrm{rad}(I)$ for each ideal I of R, $b - \mathrm{Spec}(R) = \mathrm{Spec}(R)$, and thus b_w is the identity. Hence b distributes over finite intersections if and only if b is the identity, i.e., if and only if R is Prüfer. □

Proposition 4.32. *Let I, J, L be ideals of R with I finitely generated. If $(IJ)^b \subseteq (IL)^b$, then $J^b \subseteq L^b$.*

Proof. For every homomorphism $\phi : R \longrightarrow V$, where V is a valuation domain, and every ideal H of R, we have $\phi(H^b)V = \phi(H)V$; hence $(IJ)^b \subseteq (IL)^b$ implies that $\phi(IJ)V \subseteq \phi(IL)V \Longrightarrow \phi(I)\phi(J)V \subseteq \phi(I)\phi(L)V$; but $\phi(I)V$ is principal because it is finitely generated, so $\phi(J)V \subseteq \phi(L)V$, and thus $J \subseteq \phi^{-1}(\phi(L)V)$. Intersecting on all such ϕ we have $J \subseteq L^b$ and thus $J^b \subseteq L^b$. $\square$

Star operations that satisfy this conditions are said to be *aritmetisch brauchbar*; this property is the basis of the constructions of *Kronecker functions rings* (see e.g. [20]).

4.3 Δ-closures and complete integral closure

The following lemma is the basis of the theory of *reductions*, introduced in [40], which can be used very effectively to avoid any use of valuations in the study of integral closure.

Lemma 4.33. *$x \in I^b$ if and only if there is a $n \in \mathbb{N}$ such that $(I + (x))^n = I(I + (x))^{n-1}$*

Proof. Since $I \subseteq I+(x)$, we have always $I(I+(x))^{n-1} \subseteq (I+(x))^n$; moreover,

$$I(I + (x))^{n-1} = I^n + xI^{n-1} + x^2I^{n-2} + \cdots + x^{n-1}I \quad \text{and} \tag{4.24}$$
$$(I + (x))^n = I^n + xI^{n-1} + x^2I^{n-2} + \cdots + x^{n-1}I + (x^n) \tag{4.25}$$

and thus we have to prove that $x^n \in I(I + (x))^{n-1} \iff x \in I^b$.

Suppose $x \in I^b$. Then $x^n + a_1x^{n-1} + \cdots + a_0 = n$ for some $a_i \in I^i$, i.e., $x^n = -a_1x^{n-1} - \cdots - a_n$; since $x^j \in (I + (x))^j$, for every $j > 0$ we have

$$a_ix^{n-i} \in I^i(I + (x))^{n-i} \subseteq I(I + (x))^{i-1}(I + (x))^{n-i} = I(I + (x))^{n-1} \tag{4.26}$$

and thus $-a_1x^{n-1} - \cdots - a_n \in I(I + (x))^{n-1}$, and $x^n \in I(I + (x))^{n-1}$.

Conversely, if $x^n \in I(I + (x))^{n-1}$, then $x^n \in I^n + xI^{n-1} + x^2I^{n-2} + \cdots + x^{n-1}I$, and so there are $a_i \in I^i$ such that $x^n = a_n + xa_{n-1} + \cdots + x^{n-2}a_2 + x^{n-1}a_1$, that is, x is integral over I. $\square$

Proposition 4.34. *Let Δ be the set of all nonzero finitely generated ideals of a domain R. Then*

$$I^b = \bigcup_{L\in\Delta} (IL :_R L). \tag{4.27}$$

Proof. ($\supseteq$): if $x \in (IH :_R H)$, then $xH \subseteq IH$, and thus $xHV \subseteq IHV$ for every valuation overring V of R. But, since H is finitely generated, $HV = hV$ for some $h \in V$, and thus $xhV \subseteq hIV \Longrightarrow xV \subseteq IV \Longrightarrow x \in IV$ for every valuation overring of R, and $x \in I^b$.

($\subseteq$): let $x \in I^b$. Since b is of finite type, there is a finitely generated ideal $J \subseteq I$ such that $x \in J^b$; let $L := (J, x)$. Then, by the previous lemma, $xL^{n-1} \subseteq LL^{n-1} = JL^{n-1}$, and thus $x \in (JL^{n-1} : L^{n-1}) \subseteq (IL^{n-1} :_R L^{n-1})$. $\square$

This is no longer true when R is not a domain: for example, if $R = K[[X,Y]]/(X,Y)$, let x, y be the respective images of X and Y in R and $I = (x)$. Then I is prime (and thus $I^b = I$), while $I = (0)$, and so $(0) = y(x^2) \subseteq (x^2)I$, that is, $y \in \bigcup(IL :_R L)$ (where the union ranges among finitely generated ideals), while $y \notin I = I^b$.

The set Δ used in this proposition is not the unique that can be used to obtain closure operations:

Proposition 4.35. *Let $\Delta \subseteq \mathcal{I}$ be a* multiplicatively closed *set of nonzero ideals of a ring R (i.e., if $I, J \in \Delta$, then $IJ \in \Delta$), and suppose every $L \in \Delta$ is finitely generated. Then the map*

$$I \mapsto I^{d_\Delta} = \bigcup_{L \in \Delta} (IL :_R L) \tag{4.28}$$

is a closure operation of R of finite type.

Closures of this type were introduced in [43] and called Δ-*closures.*

Proof. It is clear that $I \subseteq I^{d_\Delta}$ and that $I^{d_\Delta} \subseteq J^{d_\Delta}$ if $I \subseteq J$, because $(IL :_R L) \subseteq (JL :_R L)$. Thus we have to show that I^{d_Δ} is an ideal and that $I \mapsto I^{d_\Delta}$ is idempotent.

Suppose $x, y \in I^{d_\Delta}$: then $x \in (IL_1 :_R L_1)$, $y \in (IL_2 :_R L_2)$, and thus $(x + y)L_1L_2 = (xL_1)L_2 + (yL_2)L_1 \subseteq IL_1L_2 + IL_2L_1 = IL_1L_2$, that is, $x + y \in (I(L_1L_2) :_R L_1L_2) \subseteq I^{\Delta}$; if $a \in R$, then $axL_1 \subseteq aIL_1 \subseteq IL_1$ so that $ax \in (IL_1 :_R L_1) \subseteq I^{d_\Delta}$. In particular, $I^{d_\Delta} = \sum_{L \in \Delta}(IL :_R L)$.

Suppose $x \in (I^{d_\Delta})^{d_\Delta}$, and let $H \in \Delta$ such that $x \in (I^{d_\Delta}H :_R H)$. Let $H = (h_1, \ldots, h_m)$; we have that

$$xH \subseteq HI^{d_\Delta} = H\left(\sum_{L \in \Delta}(IL :_R L)\right) = \sum_{L \in \Delta}(IL :_R L)H \subseteq \sum_{L \in \Delta}(IHL :_R L) \tag{4.29}$$

and thus, for each i, $xh_i \in (IHL_i :_R L_i)$ for some $L_i \in \Delta$. If $L := L_1 \cdots L_m$, then $L \in \Delta$ (since Δ is multiplicatively closed) and $xH \subseteq (IHL :_R L)$; therefore $x \in (IHL :_R HL)$ and $x \in I^{d_\Delta}$.

For the last claim, suppose $x \in (IL :_R L)$ and let $L = (l_1, \ldots, l_n)$. Each xl_i is contained in J_iL for some $J_i \subseteq I$ finitely generated, and thus $x \in (JL :_R L)$, where $J := J_1 + \cdots + J_n$, and $x \in J^{d_\Delta}$. □

Proposition 4.36. *Let R be a domain. For every multiplicatively closed subset $\Delta \subseteq \mathcal{I}$ of finitely generated ideals, d_Δ is semi-prime; if $(L :_K L) = R$ for every $L \in \Delta$, then d_Δ is a star operation.*

Proof. For every ideal I and every $x \in R$, we have

$$xI^{d_\Delta} = x\sum_{L\in\Delta}(IL :_R L) = \sum_{L\in\Delta} x((IL :_K L) \cap R) = \sum_{L\in\Delta}((xIL :_K L) \cap xR) =$$
$$= \sum_{L\in\Delta}((xIL :_K L)\cap R\cap xR) = xR\cap\sum_{L\in\Delta}(xIL :_R L) = xR\cap(xI)^{d_\Delta} \subseteq (xI)^{d_\Delta}. \tag{4.30}$$

Suppose $(L :_K L) = R$ for every $L \in \Delta$, and pick $x \in R$. Then

$$(x)^{d_\Delta} = \sum_{L\in\Delta}(xL :_R L) = \sum_{L\in\Delta}((xL :_K L)\cap R) = R\cap\sum_{L\in\Delta} x(L :_K L) = R\cap xR = (x); \tag{4.31}$$

hence $(xI)^{d_\Delta} \subseteq (xR)^{d_\Delta} = xR$ and thus $xI^{d_\Delta} = xR \cap (xI)^{d_\Delta} = (xI)^{d_\Delta}$. □

Possible Δ are (a) the set of (nonzero) finitely generated ideals, (b) the set of principal ideals, (c) the set of invertible ideals, (d) the set $\{I^n\}$ of powers of a finitely generated ideal I.

The set Δ induces naturally a ring

$$R^\Delta := \bigcup_{L\in\Delta}(L :_Q L) \tag{4.32}$$

where Q is the total ring of fractions of R; in particular, R^Δ is the integral closure of R if Δ is the family of finitely generated ideals. Proposition 4.36 can thus be restated as: if $R = R^\Delta$, then d_Δ is a star operation.

It is natural to ask what happens if we take Δ to be the set of all nonzero ideals of R, removing the condition that they must be finitely generated: in this case, the ring R^Δ defined above becomes the complete integral closure of R, and so we may ask for a definition of complete integral closure of ideals, and if it coincides with the map

$$I \mapsto I^d := \bigcup(IH :_R H), \tag{4.33}$$

where the union runs among all the nonzero ideals of R.

Definition 4.37. *Let R be a domain. We say that an element $x \in R$ is* almost integral over I *if there is a $\alpha \in R$, $\alpha \neq 0$, such that $\alpha x^n \in I^n$ for every $n \in \mathbb{N}$; the set of almost integral elements is the* complete integral closure *of I, and is denoted by I^{cic}.*

It is clear that I^{cic} is an ideal: if $\alpha x^n, \beta y^n \in I^n$, then $\alpha\beta(x+y)^n \in I^n$ and $\alpha(rx)^n \in I^n$ for every $r \in R$; moreover, $I \mapsto I^{cic}$ is extensive and order-preserving. However, I have not been able to show that it is idempotent (and thus that it is a closure operation); since complete integral closure *of rings* is not idempotent, it is possible that neither complete integral closure of ideals is.

It is straightforward to see that $I^{cic} = R \iff \bigcap_{n\geq 1} I^n \neq (0)$.

Lemma 4.38. *Let $I \mapsto I^d$ defined as in* (4.33)*. $I^d \subseteq I^{cic}$; in particular, $I^b \subseteq I^{cic}$.*

Proof. Let $x \in (IK :_R K)$; we say that $x^n \in (I^nK :_R K)$. By induction, if this is true for $n-1$, we have that

$$x^nK = xx^{n-1}K \subseteq xKI^{n-1} \subseteq IKI^{n-1} = KI^n. \tag{4.34}$$

Hence, for every $\alpha \in K$, $\alpha x^n \in I^nK \subseteq I^n$ and x is almost integral over I. The last claim follows from Proposition 4.34. □

This inclusion is enough to prove that, in Noetherian rings, the integral and the complete integral closure of an ideal coincide, just like it happens for rings. We consider first the case when R is a DVR.

Lemma 4.39. *In a discrete valuation ring, complete integral closure coincides with the identity.*

Proof. Let $M = (p)$ be the maximal ideal of R, $x \in I^{cic}$; let α such that $\alpha x^n \in I^n$. here are $i, s, t \in \mathbb{N}$ such that $I = (p^i)$, $x = p^s$ and $\alpha = p^t$ (up to units); hence

$$\alpha x^n = p^t p^{sn} = p^{sn+t} = up^{in} \tag{4.35}$$

where u is a unit of V. Thus $sn + t \geq in$ for every $n \in \mathbb{N}$ and $t \geq n(i-s)$ which can happen only if $i \geq s$, i.e., if $x \in I$. □

Proposition 4.40. *In a domain that admits a discrete b-set, integral closure coincides with complete integral closure.*

Proof. Suppose $x \in I^{cic}$, and let $V \in \Delta$, where Δ is a discrete b-set of R. Then $\alpha x^n \in I^n$, and thus $\alpha x^n \in I^nV = (IV)^n$; hence $x \in (IV)^{cic}$ and $x \in IV$ by the previous lemma. Therefore $I^{cic} \subseteq \bigcap\{IV \mid V \in \Delta\}$, and this intersection is exactly I^b by the definition of a b-set; hence $I^{cic} \subseteq I^b$. But by Lemma 4.38 we have that $I^b \subseteq I^{cic}$ for an arbitrary domain; thus $I^b = I^{cic}$. □

The Noetherian case is important enough to warrant an explicit statement:

Corollary 4.41. *In a Noetherian domain, integral closure coincides with complete integral closure.*

Proof. It follows directly from Propositions 4.14 (Noetherian domains admit a discrete b-set) and 4.40. □

Corollary 4.42. *Let R be a Prüfer domain. The following are equivalent:*

1. *R_M is a DVR for every maximal ideal M (i.e., R is almost Dedekind);*
2. *R admits a discrete b-set;*
3. *$I^b = I^{cic}$ for every $I \trianglelefteq R$.*

Proof. ($1 \Longrightarrow 2$). For each ideal I, since R is Prüfer, $I^b = I = \bigcap_{M\in\mathrm{Max}(R)} IR_M$; since each R_M is a valuation ring, $\mathrm{Max}(R)$ is a b-set, and is discrete because each R_M is a DVR.

($2 \Longrightarrow 3$). See Proposition 4.40.

($3 \Longrightarrow 1$) [22, Theorem 36.5]. Since $I^b = I^{cic}$, in particular $\bigcap_{n\geq 1} I^n = (0)$ for each proper ideal I; let $M \in \mathrm{Max}(R)$. Since M^n is M-primary for every $n \geq 1$, $M^n = M^nR_M \cap R$; hence

$$(0) = \bigcap_{n\geq 1} M^n = \bigcap_{n\geq 1}(M^nR_M \cap R) = R \cap \bigcap (MR_M)^n \tag{4.36}$$

and $\bigcap (MR_M)^n = (0)$. Since R_M is a valuation ring, this imply that it is a DVR. □

When R is not a domain, the example after Proposition 4.34 shows that we cannot extend carelessly the results above. (Moreover, even if R is Noetherian, it is not always true that $\bigcap_{n\geq 1} I^n = (0)$ when R is not a domain; for example, if I is generated by an idempotent element.) We must make a small change in the definition of complete integral closure:

Definition 4.43. *Let R be an arbitrary ring, I an ideal. An element $x \in R$ is* almost integral over I *if there is a $\alpha \in R$, α not contained in any minimal prime of R, such that $\alpha x^n \subseteq I^n$ for every $n \in \mathbb{N}$; the set of almost integral elements is the* complete integral closure *of I, and is denoted by I^{cic}.*

With this definition, it is possible to recover the result that $I^b = I^{cic}$ if R is Noetherian [34, Corollary 6.8.12].

When R is not Noetherian, the behaviour of complete integral closure can be far from that of integral closure: for example, it is not true, in general, that $I^{cic} \subseteq \mathrm{rad}(R)$, because $\bigcap_{n\geq 1} I^n$ can be bigger than zero. However, some properties can be recovered:

Proposition 4.44. *Let R be a domain, $a \in R$.*

1. *Complete integral closure is persistent.*

2. *The principal ideal (a) is completely integrally closed if and only if R is integrally closed in the localization R_a (i.e., if $\tilde{R} \cap R_a = R$).*

3. *$aI^{cic} \subseteq (aI)^{cic}$; if R is completely integrally closed, then $aI^{cic} = (aI)^{cic}$.*

4. *If complete integral closure is idempotent in R, then it is a semi-prime closure operation; if R is completely integrally closed, it is a star operation.*

Proof. 1. Let $\phi : R \longrightarrow S$ be a homomorphism of domains, and suppose $x \in I^{cic}$. Then $\alpha x^n \in I^n$ for some $\alpha \in R$, and thus $\phi(\alpha x^n) = \phi(\alpha)\phi(x)^n \in \phi(I^n)S = (\phi(I)S)^n$, and $\phi(x) \in (\phi(I)S)^{cic}$, i.e., $\phi(I^{cic})S \subseteq (\phi(I)S)^{cic}$.

2. $x \in (a)^{cic} \iff$ there is $\alpha \in R$ such that $\alpha x^n \in (a^n) \iff \alpha\frac{x^n}{a^n} \in R \iff \frac{x}{a}$ is almost integral over R. But $\frac{x}{a} \in R_a$, and thus $x \in (a)^{cic} \iff R = \tilde{R} \cap R_a$, i.e., if R is completely integrally closed in R_a.

3. $x \in aI^{cic} \Longrightarrow x = ay$ with $y \in I^{cic} \Longrightarrow$ there is a $\alpha \in R$ such that $\alpha y^n \in I^n$, and

$$\alpha x^n = \alpha(ay)^n = a^n \alpha y^n \in a^n I^n = (aI)^n \tag{4.37}$$

so that $x \in (aI)^{cic}$. If R is completely integrally closed, and $x \in (aI)^{cic}$, then $\alpha x^n \in (aI)^n = a^n I^n$ for some $\alpha \in R$; thus $\alpha\left(\frac{x}{a}\right)^n \in I^n \subseteq R$ and, in particular, $\frac{x}{a}$ is almost integral over R, and hence is in R. Thus $x = ay$, and $\alpha(ay)^n \in (aI)^n \Longrightarrow \alpha y^n \in I^n$, so that $y \in I^{cic}$.

4. This follows directly from the previous point.

$\square$

4.4 Tight closure

Throughout this section, all rings will be Noetherian and of prime characteristic p, but will not be assumed to be domains; $q = p^e$ will represent a power of p. We recall that the *bracket power* $I^{[n]}$ of I is the ideal generated by the nth powers of the elements of I.

Definition 4.45. *Let R be a domain and I an ideal of R. An element $x \in R$ is in the* tight closure *$I^\star$ of I if there is an element $c \in R$, $c \neq 0$, such that $cx^q \in I^{[q]}$ for all sufficiently large q.*

It is evident the similarity between the definition of tight closure and that of complete integral closure (Definition 4.37). Since we are dealing with Noetherian rings, and since $I^{[q]} \subseteq I^q$, by Corollary 4.41 we have that $I^\star \subseteq I^b$; it follows that the tight closure is always contained in the radical, and that prime ideals are tightly closed.

When R is not a domain, we have to restrict the choice of the element c:

Definition 4.46. *Let R be a ring and I an ideal of R. $x \in I^\star$ if there is an element $c \in R$, c not contained in any minimal prime of R, such that $cx^q \in I^{[q]}$ for all sufficiently large q.*

Proposition 4.47. *$x \in I^\star$ if and only if, for every minimal prime P of R, the image of x in R/P is contained in the tight closure of $I\frac{R}{P} = \frac{I+P}{P}$.*

Note the similarity between this characterization and Proposition 4.23.

Proof. If $x \in I^\star$ and $cx^q \in I^{[q]}$, then the image of c in R/P (where P is a minimal prime) is different from 0; hence $\overline{c} \cdot \overline{x}^q \in \left(\frac{I+P}{P}\right)^{[q]}$, and $\overline{x} \in \left(\frac{I+P}{P}\right)^\star$.

Conversely, let $P_1, \ldots, P_n$ be the minimal primes of R (they are finite because R is Noetherian), and let $\pi_i : R \longrightarrow R/P_i$ be the projections. Take $c_1, \ldots, c_n$ not contained in any P_j such that $\pi_i(c_i)\pi_i(x)^q \in \left(\frac{I+P}{P}\right)^{[q]}$ for every i; then $c := c_1 \cdots c_n$ is not contained in any minimal prime and $cx^q \in I^{[q]}$. $\square$

Proposition 4.48. *Tight closure is a semi-prime closure operation; if R is an integrally closed domain, it is a star operation.*

Proof. It is sufficient to consider the case when R is a domain.

Since $\operatorname{char} R = p$, $(x+y)^q = x^q + y^q$ for every $q = p^e$; thus, $I^\star$ is an ideal. It is clear that $I \subseteq I^\star$ and $I^\star \subseteq J^\star$ whenever $I \subseteq J$; for idempotence, suppose $x \in (I^\star)^\star$, and let $I^\star = (y_1, \ldots, y_n)$; there are $c, c_1, \ldots, c_n$ such that $c_i y_i^q \in I^q$ and

$$cx^q \in (I^\star)^{[q]} = (y_1^q, \ldots, y_n^q) \tag{4.38}$$

for all sufficiently large q (the last equality is true because q is a power of the characteristic of the ring); in particular, $d := c_1 \cdots c_n$ verify $dy_i^q \in I^q$ for every $i \in \{1, \dots, n\}$. Then

$$dcx^q \in d(I^\star)^q = (dy_1^q, \dots, dy_n^q) \subseteq I^{[q]} \tag{4.39}$$

and $x \in I^\star$.

Tight closure is semi-prime: if $y \in xI^\star$, then $y = xz$ with $z \in I^\star$, and there is a $c \neq 0$ such that $cz^q \in I^{[q]}$; thus $cy^q = cx^q z^q \in x^q I^{[q]} = (xI)^{[q]}$ and $y \in (xI)^\star$.

Suppose now that R is integrally closed. If $x \in (a)^\star$, then $cx^q \in (a)^{[q]} = (a^q)$ for a $c \neq 0$; hence $c\left(\frac{x}{a}\right)^q \in R$, that is, $\frac{x}{a}$ is almost integral over R. Since R is Noetherian, $\frac{x}{a}$ is integral over R, and thus $\frac{x}{a} \in R$, that is, $x \in (a)$ and $(a)^\star = (a)$.

If $y \in (xI)^\star$, then there is a c such that $cy^q \in (xI)^{[q]} = x^q I^{[q]} \subseteq (x^q)$, and thus $c\left(\frac{y}{x}\right)^q \in R$; again, $\frac{y}{x}$ is almost integral over R and thus $y \in (x)$. Then $y = xz$ and $cx^q z^q \in x^q I^{[q]} \Longrightarrow cz^q \in I^{[q]} \Longrightarrow z \in I^\star \Longrightarrow y \in xI^\star$. □

If R is not Noetherian, we could define tight closure in the same way; however, the proof above relies heavily on the fact that $I^\star$ is finitely generated, so it might be that, in general, tight closure is not idempotent. Even if it does, it suffers the same "problems" of complete integral closure: for example, it can be that $\bigcap_{e \geq 1} I^{[p^e]} \neq (0)$, and thus $I^\star$ need not to be contained in $\mathrm{rad}(I)$. However, we have always $I^\star \subseteq I^{cic}$.

The c involved in the definition is in general dependent on both x and I; an element c such that $x \in I^\star$ if and only if $cx^q \in I^{[q]}$ for every q is called a *test element*, because it can be used to test if x is in the tight closure of I or not. Their set (plus 0) forms an ideal, called the *test ideal*. Test elements are known to exist for a wide class of rings: if R is a finitely generated module over the ring R^p of the pth powers of elements of R (such rings are called *F-finite*), or if R is a domain which is a finite A-module, where A is a regular domain [33, Chapter 2].

Usually, integral closure is really bigger than the tight closure of an ideal, although (since $(a)^n = (a^n) = (a)^{[n]}$ for every $n \in \mathbb{N}$ and every $a \in R$) they agree for principal ideals. However, the following theorem shows that we can compare integral closure of powers of the ideal with the tight closure:

Theorem 4.49 (Briançon-Skoda theorem). *Let I be an ideal generated by n elements. Then, for every $k \in \mathbb{N}$, $(I^{n+k})^b \subseteq (I^{k+1})^\star$.*

This result was proved firstly in [45] for ideals in the ring of convergent power series in n variables over $\mathbb{C}$, and subsequently generalized to arbitrary regular rings [38]; the tight closure version (which strictly speaking is not a generalization, because it covers only the case of positive characteristic) appeared in [30, Theorem 5.4].

Proof. Suppose $u \in (I^{n+k})^b$; then there is a $c \in R$, not contained in any minimal prime, such that $cu^m \in (I^{n+k})^m = I^{(n+k)m}$ for every m. We claim that, if $m = q = p^e$, $I^{(n+k)q} \subseteq (I^{k+1})^{[q]}$; taken this for granted, $cu^q \in (I^{k+1})^{[q]}$ and $u \in (I^{k+1})^\star$.

Let $I = (x_1, \dots, x_n)$. Then $I^{(n+k)q}$ is generated by the monomials of degree $(n+k)q$ in $x_1, \dots, x_n$; take one of them, say $\pi = x_1^{a_1} \cdots x_n^{a_n}$, and put $a_i := qb_i + r_i$, with $0 \le r_i < q$. Then

$$\pi = x_1^{qb_1+r_1} \cdots x_n^{qb_n+r_n} = (x_1^{b_1} \cdots x_n^{b_n})^q \cdot x_1^{r_1} \cdots x_n^{r_n} = \sigma^q \rho \tag{4.40}$$

The degree of ρ is strictly less than nq; hence the degree of σ is

$$> \frac{(n+k)q - nq}{q} = \frac{kq}{q} = k \tag{4.41}$$

and hence, being an integer, is $\ge k+1$. Thus $\sigma \in (I^{k+1})^{[q]}$, and every generator of $I^{(n+k)q}$ is contained in $(I^{k+1})^{[q]}$. $\square$

The tight closure cannot be dropped, as it could be that $(I^{n+k})^- \not\subseteq I^{k+1}$ [34, Section 13.4]. However, it can (and so the theorem is especially useful) when all the ideals are tightly closed: if this happens, the ring is said to be *weakly F-regular*. On this, the most important result is:

Theorem 4.50. *Regular local rings are weakly F-regular.*

Sketch of proof. It is sufficient to suppose that R is a domain. Let

$$\begin{aligned} F_e : R &\longrightarrow R \\ x &\mapsto x^{p^e} \end{aligned} \tag{4.42}$$

be the Frobenius map, and set $S_e := F_e(R)$. Since R is regular, the F_e are flat homomorphisms (i.e., the S_e are flat R-modules); hence, for any ideal I and every $x \in R$,

$$(I :_R x)^{[p^e]} = (I :_R x)S_e = (IS_e :_R F_e(x)) = (I^{[p^e]} :_R x^{p^e}). \tag{4.43}$$

If $x \in I^\star$, then there is a $c \neq 0$ such that $c \in \bigcap (I^{[p^e]} :_R x^{p^e})$; but

$$\bigcap_{e \ge 1} (I^{[p^e]} :_R x^{p^e}) = \bigcap_{e \ge 1} (I :_R x)^{[p^e]} \subseteq \bigcap_{n \ge 1} (I :_R x)^n \tag{4.44}$$

which is (0) unless $(I :_R x) = R$, i.e., $x \in I$. Hence $I^\star = I$. $\square$

If every localization of R is weakly F-regular, R is said to be *F-regular*. While an F-regular ring is weakly F-regular, it is an open question if the converse holds. Moreover, unlike integral closure, tight closure does not commute with localization [14].

Beyond regular rings, tight closure is also linked to regular sequences:

Theorem 4.51. *[29, Proposition 4.2] Let R, S be Noetherian domains such that S is a finite R-module, and let $x_1, \ldots, x_d$ be a regular sequence in R. If $I_k := (x_1, \ldots, x_k)$, then $(I_kS :_S x_{k+1}) \subseteq (I_kS)^\star$. In particular, if $S = R$, $(I_k :_R x_{k+1}) \subseteq I_k^\star$; if S is weakly S-regular, then every regular sequence in R is a regular sequence in S.*

Tight closure is a powerful method to tackle the so-called *homological conjectures*, a broad set of statements about Noetherian rings. Two examples are:

1. Let R be a regular local ring, S a module-finite extension ring of R. As R-modules, R is a direct summand of S.

2. Let R and S be local rings such that S is regular and R is a direct summand of S as R-modules. Then R is Cohen-Macaulay.

If R and S has positive characteristic p, both can be proved using tight closure; by an extension of tight closure, they have have been proved in the case when R and S have characteristic 0 along with their residue fields. The general case, when the characteristics of R and S are different from those of the respective quotient fields, is still open for rings of dimension ≥ 4 [19, 29].

4.5 Historical and bibliographical note

The two definitions 4.1 and 4.2 correspond to the two different fields that have used this closure. Both have their origin in Krull's *Idealtheorie* [37].

The former, using valuation rings, is used in multiplicative ideal theory, and has been called (in the past) *completion*. As said, the definition used in this context is different from the one there used.

The latter, using the equational definition, has been the main route used in the theory of Noetherian rings; the theory of reductions, first developed in [40] (and here present with Lemma 4.33, its most basic result), is a fundamental tool in this context.

Propositions 4.10 and 4.14 are classical results; proofs analogous to the the one presented, but more direct, can be found in [34, Chapter 6] or in [51, Appendix 4]. Likewise, results in Sections 4.1.2 and 4.2 are well known: see e.g. [34, Chapter 1] for the former and for localization properties, and [22] for the characterization of Prüfer domains in term of integral closure and the aritmetisch brauchbar property.

Results about Δ-closures are taken from [43]; Proposition 4.41 is another classical result, that allows to link integral closure with tight closure.

Tight closure has been introduced in [30], where the main features of the theory (test elements, connections to regular and Cohen-Macaulay ring, Briançon-Skoda theorem) are developed. The discussion of homological conjectures follows [19].

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