# The amalgamated duplication of a ring along a multiplicative-canonical ideal

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**Abstract**

After recalling briefly the main properties of the amalgamated duplication of a ring $R$ along an ideal $I$, denoted by $R\bowtie I$ [5], we restrict our attention to the study of the properties of $R\bowtie I$, when $I$ is a multiplicative canonical ideal of $R$ [9]. In particular, we study when every regular fractional ideal of $R\bowtie I$ is divisorial.

## 1 Introduction

If $R$ is a commutative ring with unity and $E$ is an $R$-module, *the idealization* $R\ltimes E$ (also called *trivial extension*), introduced by Nagata in 1956 (cf. Nagata's book [14], page 2), is a new ring where the module $E$ can be viewed as an ideal such that its square is (0). This construction has been used in many contexts to produce examples of rings satisfying preassigned conditions (see e.g. Huckaba's book [11]). In particular, in [16, Theorem 7] Reiten proved that, if $R$ is a local Cohen Macaulay ring, then $R\ltimes E$ is Gorenstein if and only if $E$ is a canonical module of $R$ (cf. also [7, Theorem 5.6]).

Fossum, in [6], generalized the idealization defining *a commutative extension* of a ring $R$ by an $R$–module $E$ and proved that, if $R$ is a local Cohen-Macaulay ring and if $E$ is a canonical module of $R$, then any commutative extension $S$ of $R$ by $E$ is a Gorenstein ring [6, Theorem].

In this paper, we deal with some applications of a similar general construction, introduced recently in [5], called *the amalgamated duplication of a ring $R$ along an $R$–module $E$*, which is an ideal in some overring of $R$, and denoted by $R\bowtie E$. When $E^2 = 0$, the new construction $R\bowtie E$ coincides with the Nagata's idealization $R\ltimes E$. In general, however, $R\bowtie E$ is not a commutative extension in

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the sense of Fossum. One main difference of this construction, with respect to the idealization (or with respect to any commutative extension, in the sense of Fossum) is that the ring $R\bowtie E$ can be a reduced ring (and it is always reduced if $R$ is a domain).

As it happens for the idealization, one interesting application of this construction is the fact that it allows to produce rings satisfying (or not satisfying) preassigned conditions. Moreover, in many cases, the amalgamated duplication of a ring preserves the property of being reduced (see [4], [5]). Note also that this new construction has been already applied for studying questions concerning the diameter and girth of the zero-divisor graph of a ring (see [13]).

M. D'Anna [4] has studied this construction in case $E = I$ is a proper ideal of $R$, proving that, if $R$ is a local Cohen-Macaulay ring with canonical module $\boldsymbol{\omega}_R$, then $R\bowtie I$ is a Gorenstein ring if and only if $I \cong \boldsymbol{\omega}_R$.

Since in the one-dimensional local Cohen-Macaulay case the Gorenstein rings are characterized by the property that the regular ideals are divisorial, it is natural to ask in a general (non necessarily Noetherian) setting, when $I$ is a multiplicative canonical ideal of $R$, whether every regular fractional ideal of the ring $R\bowtie I$ is divisorial. Recall that the notion of multiplicative canonical ideal was introduced in the integral domain case by W. Heinzer, J. Huckaba and I. Papick [9], and it can be easily extended to any commutative ring: a regular ideal $I$ of a ring $R$ is a *multiplicative-canonical* (or, simply, *m–canonical*) *ideal of* $R$ if each regular fractional ideal $J$ of $R$ is $I$–reflexive, i.e. $J = (I : (I : J)) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(J, I), I)$.

It turns out that the previous question has a positive answer if we assume a stronger condition on $I$: for each $n \geq 1$, every regular $R$–submodule of $R^n$ is $I$–reflexive. Under this hypothesis we obtain that, for each $m \geq 1$, every regular $R\bowtie I$–submodule of $(R\bowtie I)^m$ is $\mathrm{Hom}_R(R\bowtie I, I)$–reflexive (see Proposition 3.2 and Corollary 3.3). Moreover, $\mathrm{Hom}_R(R\bowtie I, I)$ is isomorphic to $R\bowtie I$ as an $R\bowtie I$–module (see Theorem 4.1). In particular, every regular fractional ideal of $R\bowtie I$ is divisorial (see Corollary 4.2).

As a by-product, we obtain that, if $R$ is a Noetherian local integral domain with an m–canonical ideal $I$, then $R\bowtie I$ is a reduced Noetherian local ring such that every regular fractional ideal is divisorial (see Corollary 4.6).

# 2 Background on $R\bowtie I$

Let $R$ be a commutative ring with unity, $T(R)$ ($:= \{\text{regular elements}\}^{-1}R$) its total ring of fractions. In this section we will give the definition of the ring $R\bowtie E$, where $E$ is an $R$-submodule of $T(R)$ such that $E\cdot E \subseteq E$ (note that this condition is equivalent to requiring that there exists a subring $S$ of $T(R)$ containing $R$ and $E$, such that $E$ is an ideal of $S$) and we will summarize some of its properties we will need in this paper. For the sake of simplicity, we will state these properties, for $E$ being a nonzero (integral) ideal of $R$. *Mutatis mutandis* the results hold in the general situation (cf. [5], where the interested reader can also find the details of the proofs).

Let $R \bowtie E$ be the following subring of $R \times T(R)$ (endowed with the usual componentwise operations):

$$R \bowtie E := \{(r, r+e) \mid r \in R,\ e \in E\}\ .$$

It is obvious that, if in the $R$-module direct sum $R \oplus E$ we introduce a multiplicative structure by setting $(r,e)(s,f) := (rs, rf + se + ef)$, where $r, s \in R$ and $e, f \in E$, then we get the ring isomorphism $R \bowtie E \cong R \oplus E$.

If $E = I$ is an ideal in $R$ (that we will assume to be proper and different from $(0)$, to avoid the trivial cases), then the ring $R \bowtie I$ is a subring of $R \times R$ and it is not difficult to see that both the diagonal embedding $R \hookrightarrow R \bowtie I$ and the inclusion $R \bowtie I \subset R \times R$ are integral. Moreover there exist two distinguished ideals in $R \bowtie I$, $\boldsymbol{\mathfrak{O}_1} := (0) \times I$ and $\boldsymbol{\mathfrak{O}_2} := I \times (0)$, such that $R \cong R \bowtie I / \boldsymbol{\mathfrak{O}_i}$, for $i = 1, 2$.

As consequences of the previous facts we have:

**Proposition 2.1** *Let $I$ be a nonzero ideal of a ring $R$.*

**(1)** *If $R$ is a domain then $R \bowtie I$ is reduced and $\boldsymbol{\mathfrak{O}_1}$ and $\boldsymbol{\mathfrak{O}_2}$ are the only minimal primes of $R \bowtie I$.*

**(2)** *$R$ is reduced if and only if $R \bowtie I$ is reduced.*

**(3)** $\dim(R \bowtie I) = \dim(R)$.

**(4)** *$R$ is Noetherian if and only if $R \bowtie I$ is Noetherian.*

Moreover it is possible to describe explicitly the prime spectrum of $R \bowtie I$.

**Proposition 2.2** *Let $P$ be a prime ideal of $R$ and set:*

$$\begin{aligned} \boldsymbol{\mathcal{P}} &:= \{(p, p+i) \mid p \in P,\ i \in I \cap P\}, \\ \boldsymbol{\mathcal{P}}_1 &:= \{(p, p+i) \mid p \in P,\ i \in I\}, \quad \textit{and} \\ \boldsymbol{\mathcal{P}}_2 &:= \{(p+i, p) \mid p \in P,\ i \in I\}. \end{aligned}$$

**(a)** *If $I \subseteq P$, then $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}_1 = \boldsymbol{\mathcal{P}}_2$ is a prime ideal of $R \bowtie I$ and it is the unique prime ideal of $R \bowtie I$ lying over $P$.*

**(b)** *If $I \nsubseteq P$, then $\boldsymbol{\mathcal{P}}_1 \neq \boldsymbol{\mathcal{P}}_2$, $\boldsymbol{\mathcal{P}}_1 \cap \boldsymbol{\mathcal{P}}_2 = \boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{P}}_1$ and $\boldsymbol{\mathcal{P}}_2$ are the only prime ideals of $R \bowtie I$ lying over $P$.*

**(c)** *The extension $P(R \bowtie I)$ of $P$ in $R \bowtie I$ coincides with $\{(p, p+i) \mid p \in P,\ i \in IP\}$ and, moreover, $\sqrt{P(R \bowtie I)} = \boldsymbol{\mathcal{P}}$.*

*Furthermore, in case* (a) *we have:*

$$R/P \cong (R \bowtie I)/\boldsymbol{\mathcal{P}} \quad \textit{and} \quad (R \bowtie I)_{\boldsymbol{\mathcal{P}}} \cong R_P \bowtie I_P\,;$$

*in case* (b) *we have:*

$$R/P \cong (R \bowtie I)/\boldsymbol{\mathcal{P}}_i \quad \textit{and} \quad R_P \cong (R \bowtie I)_{\boldsymbol{\mathcal{P}}_i}\,,\ \textit{for } i = 1, 2.$$

*In particular, $R$ is a local ring if and only if $R \bowtie I$ is a local ring.*

## 3 Remarks on reflexivity

We start this section recalling some definitions and results related to the notion of multiplicative canonical ideal of a domain, and giving the suitable generalizations for the non–domain case.

Given an $R$–module $H$, for each $R$–module $F$, we can consider the $R$–module $F^{*_H} := \mathrm{Hom}_R(F, H)$. We have the following canonical homomorphism:

$$\rho_F : F \to (F^{*_H})^{*_H}\,, \quad a \mapsto \rho_F(a)\,, \text{where } \rho_F(a)(f) := f(a)\,,$$

for all $f \in F^{*_H}$, $a \in F$. We say that the $R$–module $F$ is *$H$–reflexive* (respectively, *$H$–torsionless*) if $\rho_F$ is an isomorphism (respectively, monomorphism) of $R$–modules.

Let $F$ be a regular $R$–submodule of $T(R)$ (i.e. $F$ contains a $T(R)$–unit). It is not hard to prove that each $R$–homomorphism $F \to T(R)$ can be canonically extended to an $R$–homomorphism $T(R) \to T(R)$. Since $\mathrm{Hom}_R(T(R), T(R))$ is canonically isomorphic to $T(R)$, we have that each $R$–homomorphism from $F$ into $T(R)$ is achieved by a multiplication on $F$ by a unique element of $T(R)$.

Given a regular ideal $I$ of the ring $R$ and a $R$–submodule $F$ of $T(R)$, set $(I : F) := \{z \in T(R) \mid zF \subseteq I\} \cong \mathrm{Hom}_R(F, I)$. If $F = J$ is a regular fractional ideal of $R$ then $(I : J)$ is also a regular fractional ideal of $R$. Therefore, by the previous considerations, we have a canonical isomorphism $(I : (I : J)) \xrightarrow{\sim} (I : J)^{*_I} \xrightarrow{\sim} (J^{*_I})^{*_I}$. In this situation, we can identify the map $\rho_J : J \to (J^{*_I})^{*_I}$ with the inclusion $J \subseteq (I : (I : J))$, so $J$ is $I$–torsionless.

We say that a regular ideal $I$ of a ring $R$ is *a multiplicative-canonical ideal of $R$* (or simply a *m–canonical ideal)* if each regular fractional ideal $J$ of $R$ is $I$–reflexive, i.e. the map $\rho_J : J \to (J^{*_I})^{*_I}$ is an isomorphism or, equivalently, $J = (I : (I : J))$.

Note that this definition is a natural extension of the concept introduced in the integral domain case by W. Heinzer, J. Huckaba and I. Papick [9] and of the notion of canonical ideal given by J. Herzog and E. Kunz [10, Definition 2.4] and by E. Matlis [12, Chapter XV] for 1–dimensional Cohen-Macaulay rings. In general, given a Cohen-Macaulay local ring $(R, M, k)$ of dimension $d$, *a canonical module* of $R$ is an $R$–module $\boldsymbol{\omega}$ such that the $k$–dimension of $\mathrm{Ext}^i_R(k, \boldsymbol{\omega})$ is 1 for $i = d$ and 0 for $i \neq d$. If $R$ is not local, *a canonical module for $R$* is an $R$–module $\boldsymbol{\omega}$ such that all the localizations $\boldsymbol{\omega}_M$ at the maximal ideals $M$ of $R$ are canonical modules of $R_M$. When a canonical module $\boldsymbol{\omega}$ exists and it is isomorphic to an ideal $I$ of $R$, $I$ is called *a canonical ideal of $R$.* In [9, Proposition 4.3] it is shown that a Noetherian domain with dimension bigger than 1 does not admit a m–canonical ideal, while there exist (Noetherian) Cohen-Macaulay domains of dimension bigger than 1 with canonical ideal (e.g. a Noetherian factorial domain $D$ of dimension $\geq 2$; in this case, $D$ is a Gorenstein domain [3, Corollary 3.3.19]). Hence in higher dimension the notions of canonical ideal and m–canonical ideal do not coincide.

The following proposition extends outside of the integral domain setting some results proved in [9, Lemma 2.2 (a), (c) and Proposition 5.1]. The proof is standard and we omit the details.

**Proposition 3.1** *Let $I$ be a m–canonical ideal of a ring $R$; then we have:*

**(a)** $(I : I) = R \cong \mathrm{Hom}_R(I, I)$ *(the isomorphism is realized by the canonical multiplication map $R \to \mathrm{Hom}_R(I, I)$).*

**(b)** *If $L$ is an invertible ideal of $R$ (i.e. a regular ideal such that $LL^{-1} = R$, where $L^{-1} := (R : L)$), then $IL$ is also a m–canonical ideal of $R$; in particular, for each regular element $x \in R$, the ideal $xI$ is also a m–canonical ideal of $R$.*

**(c)** *Let $S$ be an overring of $R$, $R \subseteq S \subseteq T(R)$, such that $(R : S)$ is a regular ideal of $R$. Then $(I : S)$ $(\cong \mathrm{Hom}_R(S, I) = S^{*_I})$ is a m–canonical ideal of $S$.*

We recall that a regular fractional ideal $J$ of a ring $R$ is called a *divisorial* ideal of $R$ if $(R : (R : J)) = J$. Clearly, an invertible ideal of $R$ is a divisorial ideal. If every regular fractional ideal of $R$ is divisorial, then $R$ itself is an m–canonical ideal.

The goal of the remaining part of this paper is to study when every regular fractional ideal of $R \bowtie I$ is divisorial. We start by studying some reflexivity properties related to the notion of m–canonical ideal, in order to find an $R \bowtie I$–module $E$, with the property that every regular ideal of $R \bowtie I$ is $E$–reflexive.

Let $I$ be a regular ideal of a ring $R$ and set

$$\begin{aligned}\mathcal{F}_1 := \mathcal{F}_1(R) :=& \{F \mid F \text{ is a regular } (I\text{–torsionless}) \ R\text{–submodule of } R\} = \\ =& \{J \mid J \text{ is a regular ideal of } R\ \}\ ;\end{aligned}$$

then we say that *the ring* $R$ is $(I, \mathcal{F}_1(R))$*–reflexive* (or, simply, $(I, \mathcal{F}_1)$*–reflexive*) if each $F$ in $\mathcal{F}_1(R)$ is $I$–reflexive. It is obvious that $R$ is $(I, \mathcal{F}_1)$–reflexive if and only if $I$ is an m-canonical ideal of $R$. (Note that each regular fractional ideal $J$ is $I$–reflexive if and only if $dJ$ is $I$–reflexive, for each regular element $d \in R$ such that $dJ \subseteq R$.)

Let $I$ be a regular ideal of a ring $R$. We have already observed that every regular ideal of $R$ is $I$–torsionless. This property holds more generally for every regular $R$–submodule of $R^n$, for each $n \geq 1$. In other words, if $F$ is a regular $R$–submodule of $R^n$ and if $x \in F \setminus \{0\}$, then we can find $h \in \mathrm{Hom}_R(F, I)$ such that $h(x) \neq 0$. As a matter of fact, write $x = (x_1, x_2, \ldots, x_n)$ with $x_i \neq 0$, and let $\pi_i : R^n \to R$ be the projection on the $i$-th coordinate. Choose $y \in I$ such that $x_i y \neq 0$, and take $h \in \mathrm{Hom}_R(F, I)$ to be the composition $F \subseteq R^n \xrightarrow{\pi_i} R \xrightarrow{y} I$. Set:

$$\mathcal{F} := \mathcal{F}(R) := \{F \mid \begin{array}{l} F \text{ is a regular, } (I\text{–torsionless}) \ R\text{–submodule of } R^n, \\ \text{for some } n \geq 1\} \, . \end{array}$$

We say that *the ring $R$ is* $(I, \mathcal{F}(R))$*–reflexive* (or, simply, $(I, \mathcal{F})$*–reflexive*) if every $F \in \mathcal{F}(R)$ is $I$–reflexive (i.e. the canonical monomorphism $\rho_F : F \to \mathrm{Hom}_R(\mathrm{Hom}_R(F, I), I)$ is an isomorphism of $R$–modules).

Note that if $R$ is $(I, \mathcal{F})$–reflexive then $I$ is a m–canonical ideal of $R$, since each regular ideal $J$ of $R$ belongs to $\mathcal{F}(R)$.

**Proposition 3.2** *Let $R$ be a ring admitting a regular ideal $I$ such that $R$ is $(I, \mathcal{F}(R))$–reflexive, let $T$ be a subring of $R^m$, for some $m \geq 2$, containing the image of $R$ under the diagonal embedding and set $I^T := \mathrm{Hom}_R(T, I)$. Let $E$ be any $T$–module. Then the following canonical maps are isomorphisms of $T$–modules:*

$$\mathrm{Hom}_T(E, I^T) \cong \mathrm{Hom}_R(E, I)$$
$$\mathrm{Hom}_T(\mathrm{Hom}_T(E, I^T), I^T) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(E, I), I))\,.$$

**Proof.** Let $E$ be a $T$–module. We can consider $E$ as an $R$–module and so, by the "Hom-tensor adjointness", we have that the map $\mathrm{Hom}_R(E, I^T) \to \mathrm{Hom}_R(E \otimes_R T, I) = \mathrm{Hom}_R(E, I)$ defined by $h \mapsto h'$, where $h'(e) := h(e)(1)$, for all $e \in E$, establishes an isomorphism of $R$–modules. On the other hand, note that $I^T$ is endowed with a structure of $T$–module, by setting $z \cdot f(t) := f(zt)$, for each $f \in I^T$ and $z, t \in T$; similarly $\mathrm{Hom}_R(E, I)$ is a $T$–module, by setting $z \cdot h'(e) := h'(ze)$, for all $e \in E,\ z \in T,\ h' \in \mathrm{Hom}_R(E, I)$.

From the previous remarks it follows easily that the map:

$$\Phi : \mathrm{Hom}_T(E, I^T) \to \mathrm{Hom}_R(E, I)\,, \quad h \mapsto h'\,, \quad \text{where } h'(e) := h(e)(1)\,,\ e \in E$$

is bijective and preserves the sums. Moreover, $\Phi$ is $T$–linear, since $\Phi(zh)(e) = (zh(e))(1) = h(ze)(1) = \Phi(h)(ze) = (z\Phi(h))(e)$, for all $e \in E,\ z \in T,\ h \in \mathrm{Hom}_T(E, I^T)$. Therefore the map $\Phi$ establishes an isomorphism of $T$–modules.

By the previous isomorphism it follows that the canonical maps establish the following isomorphisms (as $T$– and $R$–modules):

$$\mathrm{Hom}_T(\mathrm{Hom}_T(E, I^T), I^T) \cong \mathrm{Hom}_R(\mathrm{Hom}_T(E, I^T), I) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(E, I), I))\,.$$

□

**Corollary 3.3** *Let $R$ be a ring admitting a regular ideal $I$ such that $R$ is $(I, \mathcal{F}(R))$–reflexive, let $T$ be a subring of $R^m$, for some $m \geq 2$, containing the image of $R$ under the diagonal embedding and set $I^T := \mathrm{Hom}_R(T, I)$. Then every regular $T$–submodule of $T^n$, for some $n \geq 1$, is $I^T$–reflexive.*

**Proof.** From the previous proposition, it follows that $E$ is a $T$–module $I^T$–torsionless (respectively, $I^T$–reflexive) if and only if $E$ is $I$–torsionless (respectively, $I$–reflexive) as a $R$–module. Moreover, if $E$ is a regular $T$–submodule of $T^n$, then clearly $E$ is regular $R$–submodule of $R^{mn}$. The conclusion is now straightforward. □

Notice that, in general, $I^T$ is not isomorphic to an ideal of $T$. However, we will see in the next section that $I^T$ is isomorphic to $T$ when $T = R \bowtie I$ and $I$ is m–canonical.

## 4 $R \bowtie I$ when $I$ is a m–canonical ideal

In this section we will investigate the construction $R \bowtie I$ in case $I$ is an m–canonical ideal. In particular we will extend, to not necessarily Noetherian rings, one of the main results obtained in [4, Theorem 11].

**Theorem 4.1** *Let $I$ be an ideal of a ring $R$ such that the canonical (multiplication) map $R \to \mathrm{Hom}_R(I, I)$ is an isomorphism (e.g. let $I$ be a m–canonical ideal of $R$; Proposition 3.1 (a)). Then $\mathrm{Hom}_R(R\bowtie I, I)$ is isomorphic as $R\bowtie I$-module to $R\bowtie I$.*

**Proof.** Since $I \cong \mathrm{Hom}_R(R, I)$ (under the map $\iota \mapsto \iota \cdot -$, for $\iota \in I$) and $R \cong \mathrm{Hom}_R(I, I)$ (under the map $x \mapsto x \cdot -$, for $x \in R$), we deduce immediately that there is a canonical isomorphism of $R$–modules $R \bowtie I \cong R \oplus I \cong \mathrm{Hom}_R(R, I) \oplus \mathrm{Hom}_R(I, I)$. Moreover we have the following canonical isomorphism of $R$–modules:

$$\mathrm{Hom}_R(R, I) \oplus \mathrm{Hom}_R(I, I) \to \mathrm{Hom}_R(R\bowtie I, I)\,, \;\; (g_1, g_2) \mapsto g\,,$$

(where $g : R\bowtie I \to I,\ (z, z + j) \mapsto g_1(z) + g_2(j)$, for each $z \in R$ and $i \in I$).

Note that the composition map

$$R\bowtie I \to \mathrm{Hom}_R(R\bowtie I, I)\,, \;\; (x, x + i) \mapsto g_{(x,i)},$$

where $g_{(x,i)}((z, z + j)) := iz + xj$ (for all $x, z \in R$ and $i, j \in I$) is obviously an $R$–isomorphism, but it is not an $R\bowtie I$–isomorphism.

In order to get an $R\bowtie I$–isomorphism we consider the following map:

$$\sigma : R\bowtie I \to \mathrm{Hom}_R(R\bowtie I, I)\,, \;\; (x, x + i) \mapsto f_{(x,i)}$$

where $f_{(x,i)}(z, z + j) := xj + (z + j)i$, for all $x, z \in R$ and $i, j \in I$.

It is not difficult to check that $\sigma$ is an injective $R\bowtie I$–homomorphism (recall that the natural structure of $R\bowtie I$–module on $\mathrm{Hom}_R(R\bowtie I, I)$, is defined by the scalar multiplication by $(x', x' + i') \cdot f((z, z + j)) := f((x', x' + i')(z, z + j)))$.

It remains to prove that $\sigma$ is surjective, that is: for each $f \in \mathrm{Hom}_R(R\bowtie I, I)$, there exists $(\bar{x}, \bar{x} + \bar{\iota}) \in R \bowtie I$, such that $f = \sigma((\bar{x}, \bar{x} + \bar{\iota})) = f_{(\bar{x},\bar{\iota})}$. Let $\bar{\iota} := f((1, 1))$ and let $\bar{x} := (f((-\bar{j}, 0))/\bar{j})$, for some nonzero (regular) element $\bar{j} \in I$. Note that $(f((-\bar{j}, 0))/\bar{j})$ does not depend on the choice of $\bar{j}$, since $j'(f((-j, 0)) = f((-jj', 0)) = j(f((-j', 0))$, i.e. $(f((-j, 0))/j) = (f((-j', 0))/j')$, for any two nonzero (regular) elements $j, j' \in I$. The previous relation shows also that $(f((-\bar{j}, 0))/\bar{j})I \subseteq I$, therefore from the canonical isomorphism of $R$-modules $\mathrm{Hom}_R(I, I) \cong R$, we deduce that $(f((-\bar{j}, 0))/\bar{j})$ (which a priori is an element of $T(R)$) belongs to $R$. Note also that, for each $j \in I$, $f((0, j)) = f((-j, 0)) + jf((1, 1)) = (\bar{x} + \bar{\iota})j$. Therefore, for each $z \in R$ and $j \in I$, we have:

$$\begin{aligned} f((z, z + j)) &= f((z, z)) + f((0, j)) = \\ &= zf((1, 1)) + jf((1, 1)) + f((-j, 0)) = \\ &= (z + j)\bar{\iota} + \bar{x}j = \\ &= f_{(\bar{x},\bar{\iota})}((z, z + j))\,. \end{aligned}$$

Hence we can conclude that the map $\sigma : R \bowtie I \to \mathrm{Hom}_R(R \bowtie I, I)$ is an isomorphism of $R \bowtie I$-modules. $\square$

We remark that an alternative proof of the previous result can be given by showing that $\mathrm{Hom}_R(R \bowtie I, I)$ is a free $R \bowtie I$–module of rank one: if we denote by $\pi : R \bowtie I \to I$ the canonical projection, $(r, r+i) \mapsto i$ $(= (r+i) - r)$, then it is possible to show that $\{\pi\}$ is a basis for $\mathrm{Hom}_R(R \bowtie I, I)$ as a $R \bowtie I$–module.

**Corollary 4.2** *Let $R$ be a ring admitting a regular ideal $I$ such that $R$ is $(I, \mathcal{F}(R))$–reflexive. Then $R \bowtie I$ is $(R \bowtie I, \mathcal{F}(R \bowtie I))$–reflexive. In particular, every regular fractional ideal of $R \bowtie I$ is divisorial.*

**Proof.** If we set $T := R \bowtie I$, by Theorem 4.1 we have $I^T \cong T$; moreover by Proposition 3.3, $T$ is $(I^T, \mathcal{F}(T))$–reflexive. $\square$

It is natural to ask whether the last statement of the previous Corollary 4.2 holds by assuming that $I$ is an m–canonical ideal if $R$. A related problem is to find conditions on $R$ so that if $I$ is a m–canonical ideal of $R$, then $R$ is $(I, \mathcal{F})$–reflexive. The remaining part of this section is an investigation in this direction.

Recall that *a Marot ring* is a ring such that each regular ideal is generated by its set of regular elements and a *ring has few zero divisors* if the set of zero divisors is a finite union of prime ideals [11, page 31]. Recall that a Noetherian ring is always a ring with few zero divisors and a ring with few zero divisors is a Marot ring; moreover an overring of a Marot ring is a Marot ring [11, Theorem 7.2 and Corollary 7.3].

The following result extends [9, Proposition 3.6] to the non integral domain case and shows that the conclusion of the previous Corollary 4.2 can be characterized in several ways in the context of Marot rings.

**Proposition 4.3** *Given a Marot ring $R$, the following statements are equivalent:*

**(i)** *$R$ has a principal m–canonical ideal;*

**(ii)** *$R$ has an invertible m–canonical ideal;*

**(iii)** *$R$ has a divisorial m–canonical ideal;*

**(iv)** *each regular fractional ideal of $R$ is divisorial.*

**Proof.** It is obvious that (iv)⇒(i)⇒(ii)⇒(iii).

(iii)⇒(iv). We start by showing a generalization of [9, Lemma 3.1]:

**Claim 1.** *Given two regular fractional ideals $I, J$ of $R$, then:*

$$(I : (I : J)) = \bigcap \{zI \mid z \text{ is a regular element of } T(R) \text{ with } J \subseteq zI\}.$$

Let $x \in (I : (I : J))$ and let $z \in T(R)$ be a regular element such that $J \subseteq zI$. Then, clearly, $x(I : J) \subseteq I$ and $z^{-1}J \subseteq I$. Therefore $xz^{-1} \in I$, i.e. $x \in zI$, and so $(I:(I:J)) \subseteq \bigcap\{zI \mid z$ is a regular element of $T(R)$ with $J \subseteq zI\}$. On the other side, let $x \in \bigcap\{zI \mid z$ is a regular element of $T(R)$ with $J \subseteq zI\}$. If $u$ is a regular element of $(I:J)$, then $J \subseteq u^{-1}I$. Therefore, $x \in u^{-1}I$, i.e. $xu \in I$. Since $(I:J)$ is a regular fractional ideal of the Marot ring $R$, it follows that $x \in (I:(I:J))$.

Since $(I:J)$ is a regular fractional ideal of the Marot ring $R$, if $u$ is a regular generator of $(I:J)$, then we have that $J \subseteq u^{-1}I$. Therefore, $x \in u^{-1}I$, i.e. $xu \in I$, and so $\bigcap\{zI \mid z$ is a regular element of $T(R)$ with $J \subseteq zI\} \subseteq (I:(I:J))$.

**Claim 2.** *If $\{J_\alpha \mid \alpha \in A\}$ is a family of divisorial ideals of $R$ such that $\bigcap_\alpha J_\alpha \neq (0)$, then $\bigcap_\alpha J_\alpha$ is divisorial.*

Let $\{z_\beta \mid \beta \in B\}$ be a family of regular elements in $T(R)$ such that $\bigcap_\beta z_\beta R \neq (0)$. Note that, for each $z_\beta$, we have $(R : z_\beta R) \subseteq (R : (\bigcap_\beta z_\beta R))$ and so $\sum_\beta (R : z_\beta R) \subseteq (R : (\bigcap_\beta z_\beta R))$. Moreover, it is easy to see that, given a family of regular fractional ideals $\{L_\beta \mid \beta \in B\}$ of $R$:

$$(R : \sum_\beta L_\beta) = \bigcap_\beta (R : L_\beta)\,.$$

Therefore:

$$\begin{aligned} \textstyle\bigcap_\beta z_\beta R \subseteq \quad & (R:(R:(\textstyle\bigcap_\beta z_\beta R))) \subseteq (R : \textstyle\sum_\beta (R : z_\beta R)) = \\ = \quad & \textstyle\bigcap_\beta (R : ((R : z_\beta R))) = \bigcap_\beta z_\beta R\,. \end{aligned}$$

Hence $\bigcap_\beta z_\beta R$ is a divisorial ideal of $R$.

The conclusion follows easily, since by Claim 1 (for $I = R$), a divisorial ideal in a Marot ring is the intersection of a family of principal regular fractional ideals.

**Claim 3.** *Given two regular fractional ideals $I, J$ of $R$, assume that $I$ is divisorial. Then $(I:(I:J))$ is divisorial.*

This is an easy consequence of Claims 1 and 2, since $I$ divisorial implies that $zI$ is divisorial, for each regular element $z \in T(R)$.

Now we can easily conclude the proof since, if $J$ is a regular fractional ideal of $R$ and $I$ is the divisorial m–canonical ideal of $R$, then $J = (I:(I:J))$ and, by Claim 3, $(I:(I:J))$ is divisorial. $\square$

**Remark 4.4 (a)** Note that the hypothesis that $R$ is a Marot ring is essential in Claim 1 of the proof of Proposition 4.3, see [11, Theorem 8.3 and Section 27, Example 11]. Several classes of examples of Marot rings are given in [11, Section 7].
**(b)** Note that if $R$ is a ring with few divisors and $I$ is an ideal of $R$ then $R \bowtie I$ has few zero divisors and so is a Marot ring.

Let $R$ be an integral domain and $I$ a nonzero ideal of $R$. Recall that $R$ is said to be *$I$–reflexive* (respectively, *$I$–divisorial*) in the sense of Bazzoni and Salce [2] (cf. also [15]), if each $I$–torsionless $\mathrm{Hom}_R(I, I)$–module of finite rank (respectively, of rank 1) is $I$–reflexive.

**Proposition 4.5** *Let $R$ be an integral domain and $I$ a nonzero ideal of $R$. Then $R$ is $(I, \mathcal{F})$–reflexive (respectively, $(I, \mathcal{F}_1)$–reflexive) if and only if $R$ is $I$–reflexive (respectively, $I$–divisorial) and $R = (I : I)$.*

**Proof.** First note that, if $I$ is a m–canonical ideal of $R$ (i.e. if $(I, \mathcal{F}_1)$–reflexive; this happens when $R$ is $(I, \mathcal{F})$–reflexive), then $\mathrm{Hom}_R(I, I) \cong R$ (Proposition 3.1 (a)).

If $R$ is $(I, \mathcal{F})$–reflexive (respectively, $(I, \mathcal{F}_1)$–reflexive) we need to verify that each $I$–torsionless $R$–module $G$ of finite rank (respectively, of rank 1) is $I$–reflexive. By [8, Lemma 5.1], such a $G$ can be embedded in $I^n$, where $n$ is the rank of $G$, hence $G$ belongs to $\mathcal{F}$ (respectively, $\mathcal{F}_1$) and so $G$ is $I$–reflexive.

Conversely, let $R = (I : I)$ and let $F \in \mathcal{F}$ (respectively, $F \in \mathcal{F}_1$). For each nonzero element $i \in I$, then $iF$ is a $I$–torsionless $R$–module of finite rank (respectively, of rank 1) by [8, Lemma 5.1], since $iF \subseteq iR^n \subseteq I^n$ (respectively, $iF \subseteq iR \subseteq I$). Therefore $iF$ is $I$–reflexive and so $F$ is $I$–reflexive. □

**Corollary 4.6** *Let $R$ be a Noetherian local integral domain and let $I$ be a m–canonical ideal of $R$ and set $T := R \bowtie I$. Then $T$ is a Noetherian local reduced ring, with $\dim(T) = \dim(R)$, such that every regular fractional ideal of $T$ is divisorial.*

**Proof.** By Propositions 2.1 and 2.2 we know that $T$ is a Noetherian local reduced ring with $\dim(T) = \dim(R)$. Note that $I$ is a m–canonical ideal of an integral domain $R$ if and only if $I$–divisorial and $(I : I) = R$ (Proposition 4.5). Moreover, in the Noetherian local integral domain case, if $(I : I) = R$, then $R$ is $I$–divisorial if and only if $R$ is $I$–reflexive, by Bazzoni's generalization of Matlis' 1-dimensional theorem [1, Theorem 3.2]. By reapplying Proposition 4.5 we know, in this case, that $R$ is $I$–reflexive and $(I : I) = R$ if and only if $R$ is $(I, \mathcal{F}(R))$–reflexive. The conclusion follows immediately from Corollary 4.2. □

Note that the assumption that a Noetherian domain $R$ admits an m-canonical ideal implies $\dim(R) \leq 1$ (by [9, Proposition 4.3]). Therefore, under the assuntions of Corollary 4.6, we can conclude that $R \bowtie I$ is a one-dimensional reduced Gorenstein local ring [10, Korollar 3.4].

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