

# UPPERS TO ZERO AND SEMISTAR OPERATIONS IN POLYNOMIAL RINGS

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ABSTRACT. Given a stable semistar operation of finite type  $\star$  on an integral domain  $D$ , we show that it is possible to define in a canonical way a stable semistar operation of finite type  $[\star]$  on the polynomial ring  $D[X]$ , such that  $D$  is a  $\star$ -quasi-Prüfer domain if and only if each upper to zero in  $D[X]$  is a quasi- $[\star]$ -maximal ideal. This result completes the investigation initiated by Houston-Malik-Mott [18, Section 2] in the star operation setting. Moreover, we show that  $D$  is a Prüfer  $\star$ -multiplication (resp., a  $\star$ -Noetherian; a  $\star$ -Dedekind) domain if and only if  $D[X]$  is a Prüfer  $[\star]$ -multiplication (resp., a  $[\star]$ -Noetherian; a  $[\star]$ -Dedekind) domain. As an application of the techniques introduced here, we obtain a new interpretation of the Gabriel-Popescu localizing systems of finite type on an integral domain  $D$  (Problem 45 of [4]), in terms of multiplicatively closed sets of the polynomial ring  $D[X]$ .

## 1. INTRODUCTION AND BACKGROUND RESULTS

Let  $D$  be an integral domain with quotient field  $K$ . Let  $\overline{\mathbf{F}}(D)$  denote the set of all nonzero  $D$ -submodules of  $K$  and let  $\mathbf{F}(D)$  (resp.,  $\mathbf{f}(D)$ ) be the set of all nonzero fractional (resp., finitely generated fractional) ideals of  $D$ .

Following Okabe-Matsuda [22], a *semistar operation* on  $D$  is a map  $\star : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$ ,  $E \mapsto E^\star$ , such that, for all  $x \in K$ ,  $x \neq 0$ , and for all  $E, F \in \overline{\mathbf{F}}(D)$ , (a)  $(xE)^\star = xE^\star$ ; (b)  $E \subseteq F$  implies  $E^\star \subseteq F^\star$ ; (c)  $E \subseteq E^\star$  and  $E^{\star\star} := (E^\star)^\star = E^\star$ . A *(semi)star operation* is a semistar operation that, restricted to  $\mathbf{F}(D)$ , is a star operation (in the sense of [14, Section 32]). It is easy to see that a semistar operation  $\star$  on  $D$  is a (semi)star operation if and only if  $D^\star = D$ .

If  $\star$  is a semistar operation on  $D$ , then we can consider a map  $\star_f : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$  defined by  $E^{\star_f} := \bigcup \{F^\star \mid F \in \mathbf{f}(D) \text{ and } F \subseteq E\}$ , for each  $E \in \overline{\mathbf{F}}(D)$ . It is easy to see that  $\star_f$  is a semistar operation on  $D$ , called *the semistar operation of finite type associated to  $\star$* . A semistar operation  $\star$  is called a *semistar operation of finite type* if  $\star = \star_f$ . It is easy to see that  $(\star_f)_f = \star_f$  (that is,  $\star_f$  is of finite type).

If  $\star_1$  and  $\star_2$  are two semistar operations on  $D$ , we say that  $\star_1 \leq \star_2$  if  $E^{\star_1} \subseteq E^{\star_2}$ , for each  $E \in \overline{\mathbf{F}}(D)$ . Obviously, for each semistar operation  $\star$  defined on  $D$ , we have  $\star_f \leq \star$ . Let  $d_D$  (or, simply,  $d$ ) be the *identity (semi)star operation on  $D$* , clearly  $d \leq \star$ , for all semistar operation  $\star$  on  $D$ .

We say that a nonzero ideal  $I$  of  $D$  is a *quasi- $\star$ -ideal* if  $I^\star \cap D = I$ , a *quasi- $\star$ -prime* if it is a prime quasi- $\star$ -ideal, and a *quasi- $\star$ -maximal* if it is maximal in the set of all proper quasi- $\star$ -ideals. A quasi- $\star$ -maximal ideal is a prime ideal. It is possible to prove that each proper quasi- $\star_f$ -ideal is contained in a quasi- $\star_f$ -maximal ideal.

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More details can be found in [12, page 4781]. We will denote by  $\text{QMax}^*(D)$  (resp.,  $\text{QSpec}^*(D)$ ) the set of the quasi- $\star$ -maximal ideals (resp., quasi- $\star$ -prime ideals) of  $D$ . When  $\star$  is a (semi)star operation the notion of quasi- $\star$ -ideal coincides with the ‘‘classical’’ notion of  $\star$ -ideal (i.e., a nonzero ideal  $I$  such that  $I^\star = I$ ).

If  $\Delta$  is a nonempty set of prime ideals of an integral domain  $D$ , then the semistar operation  $\star_\Delta$  on  $D$  defined by  $E^{\star_\Delta} := \bigcap \{ED_P \mid P \in \Delta\}$ , for each  $E \in \overline{\mathbf{F}}(D)$ , is called *the spectral semistar operation associated to  $\Delta$* . A semistar operation  $\star$  on an integral domain  $D$  is called a *spectral semistar operation* if there exists a nonempty subset  $\Delta$  of the prime spectrum of  $D$ ,  $\text{Spec}(D)$ , such that  $\star = \star_\Delta$ .

When  $\Delta := \text{QMax}^{\star_f}(D)$ , we set  $\tilde{\star} := \star_\Delta$ , i.e.,  $E^{\tilde{\star}} := \bigcap \{ED_P \mid P \in \text{QMax}^{\star_f}(D)\}$ , for each  $E \in \overline{\mathbf{F}}(D)$ . A semistar operation  $\star$  is *stable* if  $(E \cap F)^\star = E^\star \cap F^\star$ , for each  $E, F \in \overline{\mathbf{F}}(D)$ . Spectral semistar operations are stable [7, Lemma 4.1 (3)]. In particular,  $\tilde{\star}$  is a semistar operation stable and of finite type [7, Corollary 3.9].

By  $v_D$  (or, simply, by  $v$ ) we denote the  $v$ -(semi)star operation defined as usual by  $E^v := (D : (D : E))$ , for each  $E \in \overline{\mathbf{F}}(D)$ . By  $t_D$  (or, simply, by  $t$ ) we denote  $(v_D)_f$  the  $t$ -(semi)star operation on  $D$  and by  $w_D$  (or just by  $w$ ) the stable semistar operation of finite type associated to  $v_D$  (or, equivalently, to  $t_D$ ), considered by F.G. Wang and R.L. McCasland in [27]; i.e.  $w_D := \widetilde{v_D} = \widetilde{t_D}$ . Clearly  $w_D \leq t_D \leq v_D$ . Moreover, it is easy to see that for each (semi)star operation  $\star$  of  $D$ , we have  $\star \leq v_D$  and  $\star_f \leq t_D$  (cf. also [14, Theorem 34.1 (4)]).

Let  $R$  be an overring of an integral domain  $D$ , let  $\iota : D \hookrightarrow R$  be the canonical embedding and let  $\star$  be a semistar operation on  $D$ . We denote by  $\star_\iota$  the semistar operation on  $R$  defined by  $E^{\star_\iota} := E^\star$ , for each  $E \in \overline{\mathbf{F}}(R) (\subseteq \overline{\mathbf{F}}(D))$ . It is not difficult to see that if  $\star$  is a semistar operation of finite type (resp., a stable semistar operation) on  $D$  then  $\star_\iota$  is a semistar operation of finite type (resp., a stable semistar operation) on  $R$  (cf. for instance [11, Proposition 2.8] and [24, Propositions 2.11 and 2.13]).

A different approach to the stable semistar operation is possible by using the notion of localizing system [7]. Recall that a *localizing system of ideals*  $\mathcal{F}$  of  $D$  is a set of (integral) ideals of  $D$  verifying the following conditions (a) if  $I \in \mathcal{F}$  and if  $I \subseteq J$ , then  $J \in \mathcal{F}$ ; (b) if  $I \in \mathcal{F}$  and if  $J$  is an ideal of  $D$  such that  $(J :_D iD) \in \mathcal{F}$ , for each  $i \in I$ , then  $J \in \mathcal{F}$ . To avoid uninteresting cases, we assume that  $\mathcal{F}$  is *nontrivial*, i.e.,  $\mathcal{F}$  is not empty and  $(0) \notin \mathcal{F}$ .

The localizing systems, and the equivalent notions of Gabriel topologies (or, topologizing systems) and hereditary torsion theories, were introduced in the 60’s of the last century for the purpose of extending to non-commutative rings the theory of localization and for characterizing, from an ideal-theoretic point of view, the topologies associated to the hereditary torsion theories (cf. [13], [1, Ch. II, §2, Exercises 17-25, p. 157], [25], and [26, Ch. VI]).

For each nonempty subset  $\Delta$  of prime ideals of  $D$ , set  $\mathcal{F}(\Delta) := \{I \text{ ideal of } D \mid I \not\subseteq P \text{ for each } P \in \Delta\}$ . It is easy to verify that  $\mathcal{F}(\Delta)$  is a localizing system of  $D$  [8, Proposition 5.1.4]. If  $P$  is a prime ideal of  $D$ , we denote simply by  $\mathcal{F}(P)$  the localizing system  $\mathcal{F}(\{P\})$ . It is obvious that  $\mathcal{F}(\Delta) = \bigcap \{\mathcal{F}(P) \mid P \in \Delta\}$ .

A *spectral localizing system* is a localizing system  $\mathcal{F}$  such that  $\mathcal{F} = \mathcal{F}(\Delta)$ , for some subset  $\Delta$  of  $\text{Spec}(D)$ . A *localizing system of finite type* is a localizing system  $\mathcal{F}$  such that for each  $I \in \mathcal{F}$  there exists a finitely generated ideal  $J \in \mathcal{F}$  with  $J \subseteq I$ .

Let  $\mathcal{F}$  be a localizing system of ideals of  $D$ . It is easy to see that, if  $I, J \in \mathcal{F}$ , then  $IJ \in \mathcal{F}$ , thus  $\mathcal{F}$  is a multiplicative system of ideals and, inside the field of quotients  $K$  of  $D$ , it is possible to consider *the generalized ring of fractions of  $D$  with respect to  $\mathcal{F}$* , i.e.,  $D_{\mathcal{F}} := \bigcup \{(D : I) \mid I \in \mathcal{F}\} = \{z \in K \mid (D :_D zD) \in \mathcal{F}\}$ . It is easy to see that, for each  $E \in \overline{\mathbf{F}}(D)$ ,  $E_{\mathcal{F}} := \bigcup \{(E : I) \mid I \in \mathcal{F}\} = \{z \in K \mid (E :_D zD) \in \mathcal{F}\}$  belongs to  $\overline{\mathbf{F}}(D_{\mathcal{F}}) (\subseteq \overline{\mathbf{F}}(D))$ . We collect in the following lemma the main properties

of the localizing systems that we will need in the present paper (cf. [7, Proposition 2.4, Proposition 2.8, Theorem 2.10 (B) and Corollary 2.11] and [8, (5.1e), Lemma 5.1.5 (2), Propositions 5.1.4, 5.1.7 ((1) $\Leftrightarrow$ (4)) and 5.18]).

**Lemma 1.1.** *Let  $\mathcal{F}$  be a localizing system of ideals of an integral domain  $D$ .*

- (1) *For each  $E \in \overline{\mathbf{F}}(D)$ , the mapping  $E \mapsto E_{\mathcal{F}}$  defines a stable semistar operation on  $D$ , denoted by  $\star_{\mathcal{F}}$ .*
- (2) *If  $\Delta(\mathcal{F}) := \{Q \in \text{Spec}(D) \mid Q \notin \mathcal{F}\}$ , then  $\mathcal{F} \subseteq \mathcal{F}(\Delta(\mathcal{F}))$ .*
- (3) *If  $\mathcal{F}$  is a localizing system of finite type then  $\mathcal{F} = \mathcal{F}(\Delta(\mathcal{F}))$ .*
- (4) *If  $\mathcal{F} = \mathcal{F}(\Delta)$  is a spectral localizing system then  $\mathcal{F}(\Delta) = \mathcal{F}(\Delta(\mathcal{F}))$ . Moreover, for each  $E \in \overline{\mathbf{F}}(D)$ ,  $E_{\mathcal{F}(\Delta)} = \bigcap \{ED_P \mid P \in \Delta\}$ .*
- (5)  *$\mathcal{F}$  is a localizing system of finite type if and only if there exists a quasi-compact subspace  $\nabla$  of  $\text{Spec}(D)$  (endowed with the Zariski topology) such that  $\mathcal{F} = \mathcal{F}(\nabla)$ .*
- (6) *Let  $\star$  be a semistar operation on  $D$  and set  $\mathcal{F}^* := \{I \text{ nonzero ideal of } D \mid I^* = D^*\}$ . Then  $\mathcal{F}^*$  is a localizing system on  $D$  and  $\star_{\mathcal{F}^*} = \star$  if and only if  $\star$  is stable.*
- (7) *The mapping  $\mathcal{F} \mapsto \star_{\mathcal{F}}$  establishes a bijection between the set of the localizing systems (resp., the localizing systems of finite type) on  $D$  and the set of the stable semistar operations (resp., the stable semistar operations of finite type) on  $D$ .*

The notion of quasi-Prüfer domain (i.e., integral domain with Prüfer integral closure) has a semistar operation analog introduced in [3]. The starting point of the present work is [3, Corollary 2.4] where it is shown that the  $t$ -quasi-Prüfer domains coincide with the UMt-domains (i.e., the integral domains such that each upper to zero in  $D[X]$  is a maximal  $t_{D[X]}$ -ideal). There is no immediate extension to the semistar setting of the previous characterization, since in the general case we do not have the possibility to work at the same time with a semistar operation (like the  $t$ -operation) defined both on  $D$  and on  $D[X]$ . To overcome this difficulty, given a semistar operation of finite type  $\star$  on an integral domain  $D$ , we show that it is possible to define in a canonical way a semistar operation of finite type  $[\star]$  on  $D[X]$ , such that  $D$  is a  $\star$ -quasi-Prüfer domain if and only if each upper to zero in  $D[X]$  is a quasi- $[\star]$ -maximal ideal. Moreover, we show that  $D$  is a P $\star$ MD (resp., a  $\tilde{\star}$ -Noetherian domain; a  $\tilde{\star}$ -Dedekind domain) if and only if  $D[X]$  is a P $[\star]$ MD (resp., a  $[\star]$ -Noetherian domain; a  $[\star]$ -Dedekind domain).

As a by-product of the techniques introduced here, we obtain a new interpretation of the Gabriel localizing systems of finite type. More precisely, we give an explicit natural bijection between the set of localizing systems of finite type  $\mathcal{F}$  on an integral domain  $D$  and the set of extended saturated multiplicative sets  $\mathcal{S}$  of  $D[X]$ ; moreover,  $E_{\mathcal{F}} = E \cdot D[X]_{\mathcal{S}} \cap K$ , for all  $E \in \overline{\mathbf{F}}(D)$ .

## 2. STABLE SEMISTAR OPERATIONS AND POLYNOMIAL RINGS

Let  $D$  be an integral domain with quotient field  $K$ , and let  $X$  be an indeterminate over  $K$ . For each polynomial  $f \in K[X]$ , we denote by  $c_D(f)$  (or, simply,  $c(f)$ ) *the content on  $D$  of the polynomial  $f$* , i.e., the (fractional) ideal of  $D$  generated by the coefficients of  $f$ .

Let  $\star$  be a semistar operation on  $D$ , if  $\mathcal{N}^{\star} := \{g \in D[X] \mid g \neq 0 \text{ and } c_D(g)^{\star} = D^{\star}\}$ , then we set  $\text{Na}(D, \star) := D[X]_{\mathcal{N}^{\star}}$ . The ring of rational functions  $\text{Na}(D, \star)$  is called *the  $\star$ -Nagata domain of  $D$* . When  $\star = d$  the identity (semi)star operation on  $D$ , then  $\mathcal{N}^d = \mathcal{N} := \{g \in D[X] \mid c_D(g) = D\}$ . We set simply  $\text{Na}(D)$  instead of  $\text{Na}(D, d) = D[X]_{\mathcal{N}}$ . Note that  $\text{Na}(D)$  coincides with the classical Nagata domain  $D(X)$  (cf. for instance [23, Chapter I, §6 page 18] and [14, Section 33]).

Recall from [12, Propositions 3.1 and 3.4] that:

- (a)  $\mathcal{N}^\star = \mathcal{N}^{\star_f} = \mathcal{N}^{\tilde{\star}} = D[X] \setminus \bigcup\{P[X] \mid P \in \text{QMax}^{\star_f}(D)\}$  is a saturated multiplicatively closed subset of  $D[X]$ .
- (b)  $\text{Na}(D, \star) = \text{Na}(D, \star_f) = \text{Na}(D, \tilde{\star}) = \bigcap\{D_P(X) \mid P \in \text{QMax}^{\star_f}(D)\}$ .
- (c)  $\text{QMax}^{\star_f}(D) = \{M \cap D \mid M \in \text{Max}(\text{Na}(D, \star))\}$ .

Furthermore, the stable semistar operation of finite type  $\tilde{\star}$  on  $D$ , canonically associated to  $\star$ , has the following representation:

$$E^{\tilde{\star}} = E \cdot \text{Na}(D, \star) \cap K, \text{ for each } E \in \overline{\mathbf{F}}(D).$$

More generally, let  $R$  be an overring of  $D$ . We say that  $R$  is *t-linked* to  $(D, \star)$  if, for each nonzero finitely generated ideal  $I$  of  $D$ ,  $I^\star = D^\star$  implies  $(IR)^{tR} = R$  [5, Section 3]. It is known that  $R$  is a *t-linked* overring to  $(D, \star)$  if and only if  $R = R^{\tilde{\star}}$  [3, Lemma 2.9].

Let  $\iota : D \hookrightarrow R$  be the canonical embedding of  $D$  in its overring  $R$ . If  $R$  is a *t-linked* overring to  $(D, \star)$  then  $(\tilde{\star})_\iota$  is a stable (semi)star operation of finite type on  $R$  and

$$E^{(\tilde{\star})_\iota} = E \cdot \text{Na}(D, \star) \cap K = E \cdot D[X]_{\mathcal{N}^\star} \cap K, \text{ for each } E \in \overline{\mathbf{F}}(R)$$

(cf. [3, Lemma 2.9 ((i) $\Leftrightarrow$ (v))] and the last part of Section 1).

At this point, given an arbitrary multiplicative subset  $\mathcal{S}$  of  $D[X]$ , it is natural to ask whether the map  $E \mapsto ED[X]_{\mathcal{S}} \cap K$ , defined for all  $E \in \overline{\mathbf{F}}(D)$ , gives rise to a semistar operation  $\star$  on  $D$  (having the properties that  $D^\star = R$ , where  $R := D[X]_{\mathcal{S}} \cap K$ , and that  $R$  is *t-linked* to  $(D, \star)$ ). A complete answer to this question is given next. First we need a definition. Set:

$$\mathcal{S}^\sharp := D[X] \setminus \bigcup\{P[X] \mid P \in \text{Spec}(D) \text{ and } P[X] \cap \mathcal{S} = \emptyset\}.$$

It is clear that  $\mathcal{S}^\sharp$  is a saturated multiplicative set of  $D[X]$  and that  $\mathcal{S}^\sharp$  contains the saturation of  $\mathcal{S}$ , i.e.  $\mathcal{S}^\sharp \supseteq \overline{\mathcal{S}} = D[X] \setminus \bigcup\{Q \mid Q \in \text{Spec}(D[X]) \text{ and } Q \cap \mathcal{S} = \emptyset\}$ . We will call  $\mathcal{S}^\sharp$  *the extended saturation of  $\mathcal{S}$  in  $D[X]$*  and a multiplicative set  $\mathcal{S}$  of  $D[X]$  is called *extended saturated* if  $\mathcal{S} = \mathcal{S}^\sharp$ . Set

$$\Delta := \Delta(\mathcal{S}) := \{P \in \text{Spec}(D) \mid P[X] \cap \mathcal{S} = \emptyset\};$$

obviously,  $\Delta(\mathcal{S}) = \Delta(\mathcal{S}^\sharp)$ . Let  $\nabla := \nabla(\mathcal{S})$  be the set of the maximal elements of  $\Delta(\mathcal{S})$ .

**Theorem 2.1.** *Let  $\mathcal{S}$  be a multiplicative subset of the polynomial ring  $D[X]$  and set  $E^{\circ_S} := ED[X]_{\mathcal{S}} \cap K$ , for all  $E \in \overline{\mathbf{F}}(D)$ . Clearly  $E^{\circ_S} \in \overline{\mathbf{F}}(D)$  and  $ED[X]_{\mathcal{S}} = E^{\circ_S}D[X]_{\mathcal{S}}$ , for all  $E \in \overline{\mathbf{F}}(D)$ .*

- (a) *The mapping  $\circ_S : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$ ,  $E \mapsto E^{\circ_S}$  defines a semistar operation on  $D$ .*
- (b)  *$\circ_S$  is stable and of finite type, i.e.,  $\circ_S = \widetilde{\circ_S}$ .*
- (c) *The extended saturation  $\mathcal{S}^\sharp$  of  $\mathcal{S}$  coincides with  $\mathcal{N}^{\circ_S} := \{g \in D[X] \mid g \neq 0 \text{ and } \mathbf{c}_D(g)^{\circ_S} = D^{\circ_S}\}$  and  $\circ_S = \circ_{\mathcal{S}^\sharp}$ .*
- (d) *If  $\mathcal{S}$  is extended saturated then  $\text{Na}(D, \circ_S) = D[X]_{\mathcal{S}}$ .*
- (e)  *$\text{QMax}^{\circ_S}(D) = \nabla(\mathcal{S})$ . In particular,  $\circ_S$  coincides with the spectral semistar operation  $\star_{\nabla(\mathcal{S})}$ , i.e.,*

$$E^{\circ_S} = \bigcap\{ED_P \mid P \in \nabla(\mathcal{S})\}, \text{ for all } E \in \overline{\mathbf{F}}(D).$$

- (f)  *$\circ_S$  is a (semi)star operation on  $D$  if and only if  $\mathcal{S} \subseteq \mathcal{N}^{v_D}$  or, equivalently, if and only if  $D = \bigcap\{D_P \mid P \in \nabla(\mathcal{S})\}$ .*

- (g) The map  $\mathcal{S} \mapsto \circlearrowleft_{\mathcal{S}}$  establishes a 1-1 correspondence between the extended saturated multiplicative subsets of  $D[X]$  (resp., extended saturated multiplicative subsets of  $D[X]$  contained in  $\mathcal{N}^{v_D}$ ) and the set of the stable semistar (resp., (semi)star) operations of finite type on  $D$ .
- (h) Let  $\mathcal{S}$  be an extended saturated multiplicative set of  $D[X]$ . Then  $\text{Na}(D, v_D) = D[X]_{\mathcal{S}}$  if and only if  $\mathcal{S} = \mathcal{N}^{v_D}$ .
- (i) Let  $R := D^{\circlearrowleft_{\mathcal{S}}}$  and let  $\iota : D \rightarrow R$  be the canonical embedding. The overring  $R$  is  $t$ -linked to  $(D, \circlearrowleft_{\mathcal{S}})$  and  $\mathcal{S} \subseteq \mathcal{N}^{v_R}$  (i.e.,  $(\circlearrowleft_{\mathcal{S}})_{\iota}$  is a (semi)star operation on  $R$ ). Moreover  $(\circlearrowleft_{\mathcal{S}})_{\iota} = w_R$  if and only if the extended saturation  $\mathcal{S}^{\sharp_R}$  of the multiplicative set  $\mathcal{S}$  in  $R[X]$  coincides with  $\mathcal{N}^{v_R}$ .

*Proof.* For the simplicity of notation, set  $*$  :=  $\circlearrowleft_{\mathcal{S}}$ . Since  $E \subseteq E^*$  and  $E^* = ED[X]_{\mathcal{S}} \cap K \subseteq ED[X]_{\mathcal{S}}$ , then  $E^*D[X]_{\mathcal{S}} = ED[X]_{\mathcal{S}}$ .

(a) The proof is straightforward.

(b) It suffices to show that  $E^* \subseteq \widetilde{E^*}$  for each  $E \in \overline{\mathbf{F}}(D)$ . If  $0 \neq x \in E^*$ , then there exist  $0 \neq f \in ED[X]$  and  $0 \neq g \in \mathcal{S}$  such that  $x = \frac{f}{g} \in K$ . So  $xg = f$ , and thus  $x\mathbf{c}_D(g) = \mathbf{c}_D(f) \subseteq E$ . Note that  $\mathbf{c}_D(g)^* = D^*$ , since  $gD[X]_{\mathcal{S}} \subseteq \mathbf{c}_D(g)D[X]_{\mathcal{S}}$  and  $gD[X]_{\mathcal{S}} = D[X]_{\mathcal{S}}$ . Therefore  $g \in \mathcal{N}^* = \{h \in D[X] \mid h \neq 0 \text{ and } \mathbf{c}_D(h)^* = D^*\}$  and so  $x = \frac{f}{g} \in ED[X]_{\mathcal{N}^*} \cap K = E \cdot \text{Na}(D, *) \cap K = \widetilde{E^*}$ .

(c) We have already observed (in the proof of (b)) that  $\mathcal{S} \subseteq \mathcal{N}^*$ . Since the multiplicative set  $\mathcal{N}^*$  coincides with  $D[X] \setminus \bigcup\{P[X] \mid P \in \text{QMax}^*(D)\}$  [12, Proposition 3.1 (2)], then  $\mathcal{N}^*$  is extended saturated and so  $\mathcal{S}^{\sharp} \subseteq \mathcal{N}^*$ . If  $0 \neq g \in D[X]$  and  $g \in \mathcal{N}^* \setminus \mathcal{S}^{\sharp}$  then  $g \in Q[X]$ , for some prime ideal  $Q \in \text{Spec}(D) \setminus \text{QMax}^*(D)$  and  $Q[X] \cap \mathcal{S} = \emptyset$ . Note that  $Q^* \cap D \neq D$ , i.e.  $Q^* \neq D^*$ , since  $QD[X]_{\mathcal{S}} \neq D[X]_{\mathcal{S}}$ . Since  $*$  is a semistar operation of finite type, we can find a quasi- $*$ -maximal ideal  $P$  in  $D$  containing  $Q^* \cap D$  and hence also containing  $Q$ . Therefore  $g \in P[X]$ , contradicting the assumption that  $g \in \mathcal{N}^*$ . Finally, using (b), we have  $\circlearrowleft_{\mathcal{S}} = \widetilde{\circlearrowleft_{\mathcal{S}}} = \widetilde{*} = \circlearrowleft_{\mathcal{N}^*} = \circlearrowleft_{\mathcal{S}^{\sharp}}$ .

(d) is a straightforward consequence of (c).

(e) By [12, Proposition 3.1 (5)] and by (c) we have  $\text{QMax}^{\circlearrowleft_{\mathcal{S}}}(D) = \{M \cap D \mid M \in \text{Max}(D[X]_{\mathcal{N}^*})\} = \nabla(\mathcal{S})$ . The remaining statement follows from (b).

(f) Suppose that  $*'$  is a (semi)star operation on  $D$ , and let  $g \in \mathcal{S}$ . If  $g \notin \mathcal{N}^{v_D}$ , then  $\mathbf{c}_D(g)^{-1} \neq D$ , we can choose  $x \in \mathbf{c}_D(g)^{-1} \setminus D$ , so  $x = \frac{xg}{g} \in D[X]_{\mathcal{S}} \cap K = D^{*'}$ . Since  $D = D^{*'}$  by assumption, we reach a contradiction. Thus  $g \in \mathcal{N}^{v_D}$ . Conversely, assume  $\mathcal{S} \subseteq \mathcal{N}^{v_D}$ , then  $D^{*'} = D[X]_{\mathcal{S}} \cap K \subseteq \text{Na}(D, v) \cap K = D^w = D$ . The second equivalence follows from (e).

(g) Let  $\star$  be a stable semistar operation of finite type on  $D$ . Then  $\star = \star_{\Delta}$ , where  $\Delta := \text{QMax}^*(D)$ . Set  $\mathcal{S}(\Delta) := D[X] \setminus \bigcup\{P[X] \mid P \in \Delta\}$ . Clearly,  $\mathcal{S}(\Delta)$  is an extended saturated multiplicative set of  $D[X]$  and  $\nabla(\mathcal{S}(\Delta)) = \Delta$ . Therefore  $\circlearrowleft_{\mathcal{S}(\Delta)} = \star_{\Delta} = \star$ . We easily conclude by using (b), (c) and (f).

(h) is a straightforward consequence of (g).

(i) A part of this statement is a consequence of (f) and (h), after remarking that  $(\circlearrowleft_{\mathcal{S}})_{\iota}$  is a (semi)star operation on  $R$  "of type  $\circlearrowleft$ " (defined by a multiplicative set of  $R[X]$ ). The fact that  $R$  is  $t$ -linked to  $(D, \circlearrowleft_{\mathcal{S}})$  is a consequence of (b) and of [3, Lemma 2.9 (i) $\Leftrightarrow$ (v)].  $\square$

The previous theorem leads to a new interpretation of the localizing systems of finite type on an integral domain  $D$  in terms of multiplicatively closed sets of the polynomial ring  $D[X]$ .

**Corollary 2.2.** *The map  $\mathcal{F} \mapsto \mathcal{S} := \mathcal{S}(\mathcal{F}) := D[X] \setminus \{Q[X] \mid Q \in \text{Spec}(D) \text{ and } Q \notin \mathcal{F}\}$  establishes a natural bijection between the set of localizing systems of finite type*

$\mathcal{F}$  on an integral domain  $D$  and the set of extended saturated multiplicative sets  $\mathcal{S}$  of  $D[X]$ . Moreover,  $E_{\mathcal{F}} = E \cdot D[X]_{\mathcal{S}(\mathcal{F})} \cap K (= E^{\circ_{\mathcal{S}(\mathcal{F})}})$ , for all  $E \in \overline{\mathbf{F}}(D)$ .

*Proof.* Let  $\Delta(\mathcal{F}) := \{Q \in \text{Spec}(D) \mid Q \notin \mathcal{F}\}$  and so  $\mathcal{S}(\mathcal{F}) := D[X] \setminus \{Q[X] \mid Q \in \Delta(\mathcal{F})\}$ . Conversely, given an extended saturated multiplicative set  $\mathcal{S}$  of  $D[X]$ , consider the set  $\Delta(\mathcal{S}) := \{P \in \text{Spec}(D) \mid P[X] \cap \mathcal{S} = \emptyset\}$  and define  $\mathcal{F}(\mathcal{S}) := \bigcap \{\mathcal{F}(P) \mid P \in \Delta(\mathcal{S})\}$ , where  $\mathcal{F}(P) := \{I \mid I \text{ is an ideal of } D, I \not\subseteq P\}$ . The map defined by  $\mathcal{F} \mapsto \mathcal{S}(\mathcal{F})$  is a bijection, having as inverse the map defined by  $\mathcal{S} \mapsto \mathcal{F}(\mathcal{S})$ . As a matter of fact, given a localizing systems of finite type  $\mathcal{F}$  on  $D$ , then  $\Delta(\mathcal{S}(\mathcal{F})) = \Delta(\mathcal{F})$  and thus  $\mathcal{F} = \mathcal{F}(\mathcal{S}(\mathcal{F}))$ , since for a localizing system of finite type we have  $\mathcal{F} = \bigcap \{\mathcal{F}(P) \mid P \in \Delta(\mathcal{F})\}$  [8, Lemma 5.1.5 (2)]. Conversely, given an extended saturated multiplicative set  $\mathcal{S}$  of  $D[X]$ , then it is easy to see that  $\Delta(\mathcal{S}) \subseteq \Delta(\mathcal{F}(\mathcal{S}))$ . On the other hand, if  $Q \in \Delta(\mathcal{F}(\mathcal{S}))$ , then  $Q \notin \mathcal{F}(\mathcal{S})$  and so  $Q \notin \mathcal{F}(P)$ , i.e.,  $Q \subseteq P$ , for some  $P \in \Delta(\mathcal{S})$ , hence  $Q[X] \cap \mathcal{S} = \emptyset$ , i.e.,  $Q \in \Delta(\mathcal{S})$ . From the fact that  $\Delta(\mathcal{S}) = \Delta(\mathcal{F}(\mathcal{S}))$  we have  $\mathcal{S}(\mathcal{F}(\mathcal{S})) = D[X] \setminus \{P[X] \mid P \in \Delta(\mathcal{S})\} = \mathcal{S}^{\#} = \mathcal{S}$ .

By Lemma 1.1 (7), the last statement follows by observing that  $\mathcal{F}$  coincides with  $\mathcal{F}^{\circ} := \{I \text{ nonzero ideal of } D \mid I^{\circ_{\mathcal{S}(\mathcal{F})}} \cap D = D\}$ .  $\square$

The notion of quasi-Prüfer domain has a semistar analog introduced in [3]. Recall that an integral domain  $D$  is a  $\star$ -quasi-Prüfer domain if for each prime ideal  $Q$  in  $D[X]$  such that  $Q \subseteq P[X]$ , for some  $P \in \text{QSpec}^*(D)$ , then  $Q = (Q \cap D)[X]$ .

As motivated in [3], the previous notion has particular interest in case of semistar operations of finite type. Note that the  $d$ -quasi-Prüfer domains coincide with the quasi-Prüfer domains [3, Theorem 1.1]. For  $\star = v$ , we have observed in [3, Corollary 2.4 (b)] that the  $t$ -quasi-Prüfer domains coincide with the UMT-domains, i.e., the domains such that each upper to zero in  $D[X]$  is a maximal  $t_{D[X]}$ -ideal. There is no immediate extension to the semistar setting of the previous characterization, since in the general case we do not have the possibility to work at the same time with a semistar operation (like the  $t$ -operation) defined both on  $D$  and on  $D[X]$ .

This motivated the following question posed in [3]: *Given a semistar operation of finite type  $\star$  on  $D$ , is it possible to define in a canonical way a semistar operation of finite type  $\star_{D[X]}$  on  $D[X]$ , such that  $D$  is a  $\star$ -quasi-Prüfer domain if and only if each upper to zero in  $D[X]$  is a quasi- $\star_{D[X]}$ -maximal ideal?*

In the next theorem and in the subsequent corollary we give a satisfactory answer to the previous question, using the techniques introduced in Theorem 2.1.

**Theorem 2.3.** *Let  $D$  be an integral domain with quotient field  $K$ , let  $X, Y$  be two indeterminates over  $D$  and let  $\star$  be a semistar operation on  $D$ . Set  $D_1 := D[X]$ ,  $K_1 := K(X)$  and take the following subset of  $\text{Spec}(D_1)$ :*

$$\Delta_1^{\star} := \{Q_1 \in \text{Spec}(D_1) \mid Q_1 \cap D = (0) \text{ or } Q_1 = (Q_1 \cap D)[X] \text{ and } (Q_1 \cap D)^{\star_f} \subsetneq D^{\star}\}.$$

Set  $\mathcal{S}_1^{\star} := \mathcal{S}(\Delta_1^{\star}) := D_1[Y] \setminus (\bigcup \{Q_1[Y] \mid Q_1 \in \Delta_1^{\star}\})$  and:

$$E^{\circ_{\mathcal{S}_1^{\star}}} := E[Y]_{\mathcal{S}_1^{\star}} \cap K_1, \quad \text{for all } E \in \overline{\mathbf{F}}(D_1).$$

- (a) The mapping  $[\star] := \circ_{\mathcal{S}_1^{\star}} : \overline{\mathbf{F}}(D[X]) \rightarrow \overline{\mathbf{F}}(D[X])$ ,  $E \mapsto E^{\circ_{\mathcal{S}_1^{\star}}}$  is a stable semistar operation of finite type on  $D[X]$ , i.e.,  $[\widetilde{\star}] = [\star]$ . Moreover, if  $\star$  is a (semi)star operation on  $D$ , then  $[\star]$  is a (semi)star operation on  $D[X]$ .
- (b)  $[\widetilde{\star}] = [\star_f] = [\star]$ .
- (c)  $(ED[X])^{[\star]} \cap K = ED_1[Y]_{\mathcal{S}_1^{\star}} \cap K = E^{\widetilde{\star}}$  for all  $E \in \overline{\mathbf{F}}(D)$ .
- (d)  $(ED[X])^{[\star]} = E^{\widetilde{\star}}D[X]$ , for all  $E \in \overline{\mathbf{F}}(D)$ .
- (e)  $\text{QMax}^{[\star]}(D_1) = \{Q_1 \mid Q_1 \in \text{Spec}(D_1) \text{ such that } Q_1 \cap D = (0) \text{ and } \mathbf{c}_D(Q_1)^{\star_f} = D^{\star}\} \cup \{P[X] \mid P \in \text{QMax}^{\star}(D)\}$ .

$$(f) [w_D] = [t_D] = [v_D] = \widetilde{v_{D_1}} = w_{D_1}.$$

*Proof.* Note that, if  $Q_1 \in \text{Spec}(D[X])$  is not an upper to zero and  $(Q_1 \cap D)^{\star_f} \subsetneq D^*$ , then the prime ideal  $Q_1 \cap D$  is contained in a quasi- $\star_f$ -maximal ideal of  $D$ . Moreover if  $Q_1 \cap D = (0)$  and  $\mathbf{c}_D(Q_1)^{\star_f} \subsetneq D^*$  then  $\mathbf{c}_D(Q_1)^{\star_f}$  is contained in a quasi- $\star_f$ -prime ideal  $P$  of  $D$  and hence  $Q_1 \subseteq P[X]$  with  $P^{\star_f} \subsetneq D^*$ . Set  $\nabla_1^* := \{Q_1 \in \text{Spec}(D_1) \mid \text{either } Q_1 \cap D = (0) \text{ and } \mathbf{c}_D(Q_1)^{\star_f} = D^* \text{ or } Q_1 = PD[X] \text{ and } P \in \text{QMax}^{\star_f}(D)\}$ .

It is easy to see that

$$\mathcal{S}_1^* := D_1[Y] \setminus \left( \bigcup \{Q_1[Y] \mid Q_1 \in \Delta_1^*\} \right) = D_1[Y] \setminus \left( \bigcup \{Q_1[Y] \mid Q_1 \in \nabla_1^*\} \right) = \mathcal{S}(\nabla_1^*).$$

(a) follows from Theorem 2.1 ((a), (b) and (f)).

(b) Since  $\text{QMax}^{\star_f}(D) = \text{QMax}^{\bar{\star}}(D)$ , the conclusion follows easily from the fact that  $\mathcal{S}_1^{\bar{\star}} = \mathcal{S}_1^{\star_f} = \mathcal{S}_1^*$ .

(c) Let  $\mathcal{N}^{[*]} := \{g \in D_1[Y] \mid g \neq 0 \text{ and } \mathbf{c}_{D_1}(g)^{[*]} = D_1^{[*]}\}$ . Since by construction  $\mathcal{S}_1^*$  is an extended saturated multiplicative set of  $D_1$  we know that  $\mathcal{S}_1^* = \mathcal{N}^{[*]}$  (Theorem 2.1 (c)). On the other hand, if  $h \in \mathcal{N}^* = D[X] \setminus \left( \bigcup \{P[X] \mid P \in \text{QMax}^{\star_f}(D)\} \right)$  then  $h \in D[X][Y] \setminus \left( \bigcup \{Q_1[Y] \mid Q_1 \in \nabla_1^*\} \right) = \mathcal{N}^{[*]}$ . Therefore, for all  $E \in \overline{\mathbf{F}}(D)$ ,  $E^{\bar{\star}} = ED[X]_{\mathcal{N}^*} \cap K \subseteq ED_1[Y]_{\mathcal{N}^{[*]}} \cap K = (ED_1[Y]_{\mathcal{N}^{[*]}} \cap K_1) \cap K = (ED_1)^{[*]} \cap K = (ED[X])^{[*]} \cap K$ .

For the reverse containment, let  $0 \neq z = \frac{f}{g} \in ED[X][Y]_{\mathcal{S}_1^*} \cap K$ , where  $z \in K$  and  $f, g \in K[X, Y]$  are nonzero polynomials such that  $f \in ED[X][Y]$  and  $g \in \mathcal{S}_1^* = \mathcal{N}^{[*]}$ . Set  $g = g_0 + g_1Y + \cdots + g_nY^n$ , where  $g_i \in D_1$  with  $g_n \neq 0$  and  $n \geq 0$ ; then  $\mathbf{c}_{D_1}(g) = (g_0, g_1, \dots, g_n)$  and  $\mathbf{c}_D(g) = \mathbf{c}_D(g_0) + \mathbf{c}_D(g_1) + \cdots + \mathbf{c}_D(g_n)$ . Let  $Q_1 \in \Delta_1^*$ . Since  $\mathbf{c}_{D_1}(g)^{[*]} = D_1^{[*]}$ , then  $g \notin Q_1[Y]$ , and hence  $(g_0, g_1, \dots, g_n) \not\subseteq Q_1$ . So at least one among the  $g_i$ 's is not contained in  $Q_1$ , and thus  $\mathbf{c}_D(g) \not\subseteq Q_1 \cap D$ . In particular  $\mathbf{c}_D(g) \not\subseteq P$ , for all  $P \in \text{QMax}^{\star_f}(D)$ , i.e.,  $\mathbf{c}_D(g)^{\bar{\star}} = D^{\bar{\star}}$ . On the other hand,  $z\mathbf{c}_D(g) = \mathbf{c}_D(zg) = \mathbf{c}_D(f) \subseteq E$ . Hence  $z \in z\mathbf{c}_D(g)^{\bar{\star}} \subseteq E^{\bar{\star}}$ . Therefore we conclude that  $ED[X]_{\mathcal{N}^*} \cap K = E^{\bar{\star}}$ .

(d) By (c),  $(ED[X])^{[*]} \cap K = E^{\bar{\star}}$ , and thus  $E^{\bar{\star}}D[X] \subseteq (ED[X])^{[*]}$ , for all  $E \in \overline{\mathbf{F}}(D)$ .

For the converse, let  $0 \neq \frac{h}{\ell} = \frac{f}{g} \in (ED[X])^{[*]} = ED[X][Y]_{\mathcal{S}_1^*} \cap K_1$ , where  $h, \ell \in K[X]$  are nonzero polynomials such that  $\text{GCD}(h, \ell) = 1$  in  $K[X]$ ,  $0 \neq f \in ED[X][Y]$ , and  $0 \neq g \in \mathcal{S}_1^*$ . Then  $\ell f = hg$ , and since  $K[X, Y]$  is a UFD and  $\text{GCD}(h, \ell) = 1$ , we have  $\ell \mid g$  in  $K[X, Y]$ , i.e.,  $g = \ell \cdot \gamma$  for some  $\gamma \in K[X, Y]$ .

We claim that  $\ell \in K$ . Assume that  $\ell \in K[X] \setminus K$ . Choose a prime ideal  $Q_1$  of  $D_1 = D[X]$  such that  $\ell K[X] \cap D[X] \subseteq Q_1$  and  $Q_1 \cap D = (0)$ . Then  $g = \ell \cdot \gamma \in Q_1 K[X, Y] \cap D[X, Y] = Q_1[Y] \in \Delta_1^*$  and so  $g \notin \mathcal{S}_1^*$ , which is a contradiction.

Since  $0 \neq \ell \in K$ , set  $h' := \frac{h}{\ell} \in K[X]$ . Then  $h' = \frac{f}{g} \in ED[X][Y]_{\mathcal{S}_1^*} \cap K[X]$  and so  $h' \cdot g = f$ . Since  $g \in \mathcal{S}_1^*$ , by the proof of (c) above, we have  $\mathbf{c}_D(g)^{\bar{\star}} = D^{\bar{\star}}$ , and hence  $\mathbf{c}_D(h') \subseteq \mathbf{c}_D(h')\mathbf{c}_D(g)^{\bar{\star}} \subseteq (\mathbf{c}_D(h')\mathbf{c}_D(g)^{\bar{\star}})^{\bar{\star}} = (\mathbf{c}_D(h')\mathbf{c}_D(g))^{\bar{\star}} = \mathbf{c}_D(h' \cdot g)^{\bar{\star}} = \mathbf{c}_D(f)^{\bar{\star}} \subseteq E^{\bar{\star}}$  (cf. [14, Corollary 28.3] for the fourth equality). We conclude that  $h' = \frac{f}{g} \in E^{\bar{\star}}D[X]$ .

(e) By [12, Proposition 3.1 (5)] we know that  $\text{QMax}^{[*]}(D_1) = \{M \cap D_1 \mid M \in \text{Max}(D_1[Y]_{\mathcal{N}^{[*]}})\}$  and it is easy to verify that this last set coincides with  $\nabla_1^*$ .

(f) If  $\star_f = t$ , then by (e)  $\text{QMax}^{[*]}(D[X]) = \{Q_1 \mid Q_1 \in \text{Spec}(D_1) \text{ such that } Q_1 \cap D = (0) \text{ and } \mathbf{c}_D(Q_1)^t = D\} \cup \{P[X] \mid P \in \text{Max}^t(D)\} = \text{Max}^t_{D[X]}(D[X])$ .

The last equality holds because it is wellknown that if  $P \in \text{Max}^t(D)$  then  $P[X] \in \text{Max}^{t_{D[X]}}(D[X])$  [16, Proposition 4.3] and [20, Proposition 1.1]; moreover, if  $Q_1 \in \text{Spec}(D_1)$  is such that  $Q_1 \cap D = (0)$ , then  $Q_1$  is a  $t_{D_1}$ -maximal ideal if and only if  $\mathfrak{c}_D(Q_1)^t = D$  [20, Theorem 1.4]. Thus, by (a) and (b) and by the fact that  $\text{QMax}^{[\star]}(D[X]) = \text{Max}^{t_{D[X]}}(D[X])$ , we have  $[v_D] = [t_D] = [w_D] = \widetilde{[w_D]} = \widetilde{v_{D[X]}} = w_{D[X]}$ .  $\square$

**Corollary 2.4.** *Let  $\star$  be a semistar on an integral domain  $D$  and let  $[\star]$  be the stable semistar operation of finite type on  $D[X]$  canonically associated to  $\star$  as in Theorem 2.3 (a). The following statements are equivalent:*

- (i)  $D$  is a  $\star_f$ -quasi-Prüfer domain.
- (ii)  $D[X]$  is a  $[\star]$ -quasi-Prüfer domain.
- (iii) Each upper to zero is a quasi- $[\star]$ -maximal ideal of  $D[X]$ .

*Proof.* The equivalence (i) $\Leftrightarrow$ (iii) follows easily from Theorem 2.3 (e) and from the fact that  $D$  is a  $\star_f$ -quasi-Prüfer domain if and only if, for each upper to zero  $Q$  in  $D[X]$ ,  $\mathfrak{c}(Q)^{\star_f} = D^\star$  [3, Lemma 2.3].

For the equivalence between (i) and (ii), recall that  $D$  is a  $\star_f$ -quasi-Prüfer domain if and only if  $D_P$  is a quasi-Prüfer domain, for each quasi- $\star_f$ -maximal ideal  $P$  of  $D$  [3, Theorem 2.16 ((1) $\Leftrightarrow$ (11))]. Moreover, for each prime ideal  $P$  of  $D$ ,  $D[X]_{P[X]}$  coincides with the Nagata ring  $D_P(X)$  and this is a quasi-Prüfer domain if and only if  $D_P$  is a quasi-Prüfer domain [3, Theorem 1.1((1) $\Leftrightarrow$ (9))].

(i) $\Rightarrow$ (ii) Since we know already that (i) $\Leftrightarrow$ (iii), in the present situation we have  $\text{QMax}^{[\star]}(D[X]) = \{Q_1 \mid Q_1 \in \text{Spec}(D_1) \text{ such that } Q_1 \cap D = (0)\} \cup \{P[X] \mid P \in \text{QMax}^{\star_f}(D)\}$ . The conclusion follows from the fact that  $D[X]_{Q_1}$  is clearly a DVR for each upper to zero  $Q_1$  of  $D[X]$  and  $D[X]_{P[X]} = D_P(X)$  is a quasi-Prüfer domain, since  $D_P$  is a quasi-Prüfer domain, for each for each quasi- $\star_f$ -maximal ideal  $P$  of  $D$ .  $\square$

(ii) $\Rightarrow$ (i) is obvious by the previous argument.  $\square$

From the previous corollary and from [3, Corollary 2.4 (b)], we re-obtain that an integral domain  $D$  is a UMt-domain if and only if the polynomial ring  $D[X]$  is a UMt-domain [10, Theorem 2.4], since by Theorem 2.3 (f), the semistar operation  $[t_D]$  on  $D[X]$  coincides with  $w_{D[X]}$  and the notions of  $w$ -quasi-Prüfer domain and  $t$ -quasi-Prüfer domain coincide.

Let  $\star$  be a semistar operation on an integral domain  $D$ . We say that  $D$  is a  $\star$ -Noetherian domain if  $D$  has the ascending chain condition on quasi- $\star$ -ideals of  $D$ . It is easy to show that  $D$  is  $\star$ -Noetherian if and only if each nonzero ideal  $I$  of  $D$  is  $\star_f$ -type, i.e.,  $I^{\star_f} = J^{\star_f}$  for some  $J \in \mathbf{f}(D)$  and  $J \subseteq I$ . It is known that  $D$  is  $\tilde{\star}$ -Noetherian if and only if  $\text{Na}(D, \star) = D[X]_{\mathcal{N}^\star}$  is Noetherian, [24, Theorem 4..36] (cf. [2, Theorem 2.6] for the star operation case). An  $I \in \overline{\mathbf{F}}(D)$  is said to be *quasi- $\star$ -invertible* (resp.,  *$\star$ -invertible*) if  $(I : (D^\star : I))^\star = D^\star$  (resp.,  $(I : (D : I))^\star = D^\star$ ). Recall that  $D$  is a  $\star$ -Dedekind domain if each nonzero (integral) ideal of  $D$  is quasi- $\star_f$ -invertible and  $D$  is a *Prüfer  $\star$ -multiplication domain* (for short,  $P^\star\text{MD}$ ) if every nonzero finitely generated (integral) ideal of  $D$  is  $\star_f$ -invertible (cf. for instance [9]). It is known that  $D$  is a  $\star$ -Dedekind domain if and only if  $D$  is a  $P^\star\text{MD}$  and a  $\star$ -Noetherian domain [6, Proposition 4.1].

**Corollary 2.5.** *Let  $\star$  be a semistar on an integral domain  $D$  and let  $[\star]$  be the stable semistar operation of finite type on  $D[X]$  canonically associated to  $\star$  as in Theorem 2.3 (a). Then*

- (1)  $D$  is a  $P^\star\text{MD}$  if and only if  $D[X]$  is a  $P[\star]\text{MD}$ .



- (2)  $D$  is a  $\tilde{\star}$ -Noetherian domain if and only if  $D[X]$  is a  $[\star]$ -Noetherian domain.  
 (3)  $D$  is a  $\tilde{\star}$ -Dedekind domain if and only if  $D[X]$  is a  $[\star]$ -Dedekind domain.

*Proof.* (1) By Theorem 2.3(d), we have  $(D[X])^{[\star]} = D^{\tilde{\star}}[X]$ , and hence  $(D[X])^{[\star]}$  is integrally closed if and only if  $D^{\tilde{\star}}$  is integrally closed. Thus the result follows directly from Corollary 2.4 and [3, Corollary 2.17].

(2) Assume that  $D$  is a  $\tilde{\star}$ -Noetherian domain. Then  $D[X]_{\mathcal{N}^{\star}}$  is Noetherian and so  $(D[X]_{\mathcal{N}^{\star}})[Y] = (D[X][Y])_{\mathcal{N}^{\star}}$  is also Noetherian. On the other hand, recall that  $\mathcal{N}^{\star} \subseteq \mathcal{N}^{[\star]}$  (cf. the proof of Theorem 2.3(c)), and so  $(D[X][Y])_{\mathcal{N}^{[\star]}} = ((D[X][Y])_{\mathcal{N}^{\star}})_{\mathcal{N}^{[\star]}}$  is Noetherian. Hence,  $D[X]$  is  $[\star]$ -Noetherian.

For the converse, let  $I$  be a nonzero ideal of  $D$ . Since  $D[X]$  is  $[\star]$ -Noetherian, then  $(ID[X])^{[\star]} = (f_1, f_2, \dots, f_n)^{[\star]}$ , for a finite family of polynomials  $f_1, f_2, \dots, f_n \in ID[X]$ . Set  $J = \mathbf{c}_D(f_1) + \mathbf{c}_D(f_2) + \dots + \mathbf{c}_D(f_n)$ . Clearly  $(f_1, f_2, \dots, f_n) \subseteq JD[X]$  and thus  $(ID[X])^{[\star]} = (JD[X])^{[\star]}$ . Therefore, by Theorem 2.3(c), we have  $I^{\tilde{\star}} = (ID[X])^{[\star]} \cap K = (JD[X])^{[\star]} \cap K = J^{\tilde{\star}}$  and so we conclude that  $D$  is  $\tilde{\star}$ -Noetherian.

(3) This is an immediate consequence of (1), (2) and [6, Proposition 4.1].  $\square$

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