# THE PATCH TOPOLOGY AND THE ULTRAFILTER TOPOLOGY ON THE PRIME SPECTRUM OF A COMMUTATIVE RING

MARCO FONTANA[1] AND K. ALAN LOPER[2]

[1] Dipartimento di Matematica
Università degli Studi Roma Tre
Largo San Leonardo Murialdo, 1
00146 Roma, Italy
fontana@mat.uniroma3.it

[2] Department of Mathematics
Ohio State University-Newark
Newark, Ohio 43055 USA
lopera@math.ohio-state.edu

Abstract. Let $R$ be a commutative ring and let $\mathrm{Spec}(R)$ denote the collection of prime ideals of $R$. We define a topology on $\mathrm{Spec}(R)$ by using ultrafilters and demonstrate that this topology is identical to the well known *patch* or *constructible* topology. The proof is accomplished by use of a von Neumann regular ring canonically associated with $R$.

Let $R$ be a commutative ring and let $\mathrm{Spec}(R)$ denote the collection of prime ideals of $R$. On $\mathrm{Spec}(R)$ we can define a topology known as *Zariski's topology*: the collection of all sets $V(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$ where $I$ is an ideal of $R$ constitutes the closed sets in this topology.

Zariski's topology has several attractive properties related to the geometric aspects of the study of the set of prime ideals [7, Chapter I]. For example, $\mathrm{Spec}(R)$ is always quasicompact (that is, every open covering has a finite refinement). On the other hand, this topology is very coarse. For example, $\mathrm{Spec}(R)$ is almost never Hausdorff (that is, two distinct points have nonintersecting neighborhoods).

Many authors have considered a finer topology, known as the *patch topology* [9] and as the *constructible topology* ([8, pages 337-339] or [1, Chapter 3, Exercises 27, 28 and 30]), which can be defined starting from Zariski's topology.

Consider two collections of subsets of $\mathrm{Spec}(R)$.

(1) The sets $V(I)$ defined above for $I$ an ideal of $R$.
(2) The sets $D(a) := \mathrm{Spec}(R) \setminus V(a)$ where $a \in R$ (where, as usual, $V(a)$ denotes the set $V(aR)$).

The patch topology is then the smallest topology in which both of the above classes consist of closed sets. The patch topology is a refinement of Zariski's topology which is always Hausdorff.

It is easy and natural to define another topology on $\operatorname{Spec}(R)$ by introducing the notion of an ultrafilter.

We start with definitions, some notation and some preliminary results.

Given an infinite set $S$ *an ultrafilter* $\mathcal{U}$ *on* $S$ is a collection of subsets of $S$ such that

(1) If $A \in \mathcal{U}$ and $A \subseteq B \subseteq S$ then $B \in \mathcal{U}$.
(2) If $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$.
(3) If $A \cup B \in \mathcal{U}$ and $A \cap B$ is empty then exactly one of $A$ and $B$ lies in $\mathcal{U}$.

Note that (1) implies that any nonempty ultrafilter on a set $S$ contains the set $S$ itself. It then follows from (3) that the empty set can never be a member of an ultrafilter. Hence an ultrafilter on a set $S$ is always a proper subset of the power set of $S$.

A principal ultrafilter on $S$ can be defined as follows. If $d \in S$ then *the principal ultrafilter of* $d$ *on* $S$ is the collection of all subsets of $S$ which contain $d$. Zorn's Lemma can be used to prove that nonprincipal ultrafilters exist on any infinite set.

Let $C$ be a subset of $\operatorname{Spec}(R)$, and let $\mathcal{U}$ be an ultrafilter on the set $C$. Set $P_{\mathcal{U}} := \{a \in R \mid V(a) \cap C \in \mathcal{U}\}$. By an argument similar to that used in [5, Lemma 2.4] it can be easily shown that $P_{\mathcal{U}}$ is a prime ideal of $R$. We call $P_{\mathcal{U}}$ an *ultrafilter limit point of* $C$. This notion of ultrafilter limits of collections of prime ideals has been used to great effect in several recent papers [5], [11], and [12]. If $\mathcal{U}$ is a principal ultrafilter then there is a prime $P \in C$ such that $\mathcal{U}$ consists of all subsets of $C$ which contain $P$. It is clear then that $P_{\mathcal{U}} = P \in C$. On the other hand, if $\mathcal{U}$ is nonprincipal, then it is not at all clear that $P_{\mathcal{U}}$ should lie in $C$. That motivates our definition.

**Definition 1.** Let $R$ and $C$ be as above. We say that $C$ is *ultrafilter closed* if it contains all of its ultrafilter limit points.

It is not hard to see that the ultrafilter closed subsets of $\operatorname{Spec}(R)$ define a topology on the set $\operatorname{Spec}(R)$, called *the ultrafilter topology on* $\operatorname{Spec}(R)$. In fact:

- Suppose that $C_1, C_2, \ldots, C_n$ are ultrafilter closed subsets of $\operatorname{Spec}(R)$. Let $C := C_1 \cup C_2 \cup \ldots \cup C_n$. Let $\mathcal{U}$ be an ultrafilter on $C$. Then $\mathcal{U}$ defines an ultrafilter limit prime $P_{\mathcal{U}}$. We want to show that $P_{\mathcal{U}} \in C$. Note that at least one of the sets $C_i$ lies in $\mathcal{U}$. Without loss of generality, suppose that $C_1 \in \mathcal{U}$. The collection of sets $\mathcal{U}_1 := \{C_1 \cap B \mid B \in \mathcal{U}\}$ is then an ultrafilter on $C_1$ and the ultrafilter prime it defines is indentical to $P_{\mathcal{U}}$. In particular,
  (1) Let $d \in P_{\mathcal{U}_1}$. We know that $V(d) \cap C_1 \in \mathcal{U}_1$. Since $C_1 \in \mathcal{U}$ then every set in $\mathcal{U}_1$ is also in $\mathcal{U}$. Hence $V(d) \cap C_1 \in \mathcal{U}$. Then note that $V(d) \cap C_1 \subseteq V(d) \cap C$ and so $V(d) \cap C \in \mathcal{U}$. Hence, $d \in P_{\mathcal{U}}$.
  (2) Let $d \in P_{\mathcal{U}}$. Then $V(d) \cap C \in \mathcal{U}$. The definition of $\mathcal{U}_1$ then implies that $V(d) \cap C_1 = (V(d) \cap C) \cap C_1 \in \mathcal{U}_1$. Hence, $d \in P_{\mathcal{U}_1}$.

  Hence, $P_{\mathcal{U}} \in C$ since it is in the ultrafilter closure of $C_1$ which is an ultrafilter closed set.
- Suppose that $\{C_\lambda \mid \lambda \in \Lambda\}$ is a collection of ultrafilter closed subsets of $\operatorname{Spec}(R)$. Let $C := \cap_{\lambda \in \Lambda} C_\lambda$. Let $\mathcal{U}$ be an ultrafilter on $C$ and $P_{\mathcal{U}}$ the ultrafilter limit prime associated to $\mathcal{U}$. We want to show that $P_{\mathcal{U}}$ lies in $C$.

For each $\lambda \in \Lambda$ the collection $\mathcal{U}_\lambda := \{B \subseteq C_\lambda \mid B \cap C \in \mathcal{U}\}$ is an ultrafilter on $C_\lambda$. Moreover, it defines the same limit prime $P_\mathcal{U}$ as the ultrafilter $\mathcal{U}$ defines on $C$ (using an argument similar to that given above for finite unions). Since $C_\lambda$ is ultrafilter closed then we have proven that $P_\mathcal{U} \in C_\lambda$, for each $\lambda \in \Lambda$, and hence $P_\mathcal{U} \in C$.

Note: The above discussion assumes that the sets are infinite and the ultrafilters are nonprincipal. The proofs are completely routine otherwise.

It is natural to ask then how this topology compares with the other topologies we have defined on $\mathrm{Spec}(R)$. The goal of this paper is to demonstrate that the ultrafilter topology coincides with the patch topology.

It should be noted that related results were obtained recently in [3].

One direction is easy. We begin with that result.

**Proposition 2.** *Let $R$ be a ring and let $C \subseteq \mathrm{Spec}(R)$ be a collection of prime ideals. Suppose that $C$ is closed in the patch topology. Then $C$ is also closed in the ultrafilter topology.*

*Proof.* We consider each of the defining classes of closed sets for the patch topology separately.

Suppose that $C = V(I)$ for some ideal $I \subseteq R$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $C$ and construct the prime ideal $P_\mathcal{U}$. Let $a \in I$. Then $a \in P$ for every prime $P \in C$. Hence $C \subseteq V(a)$. Since our ultrafilter was defined on $C$ we know that $C \in \mathcal{U}$ and so $V(a) \cap C = C \in \mathcal{U}$. Hence, $I \subseteq P_\mathcal{U}$. Since $C$ is defined as the collection of all primes that contain $I$ this implies that $P_\mathcal{U} \in C$.

Now assume that $C = D(a)$ for some element $a \in R$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $C$ and construct the prime ideal $P_\mathcal{U}$. Then the set $V(a)$ contains none of the primes in $C$. Hence, $\emptyset = V(a) \cap C$ is not in $\mathcal{U}$ since, as noted earlier, no ultrafilter contains the empty set. Hence, $a$ is not an element of $P_\mathcal{U}$. It follows by definition that $P_\mathcal{U} \in C$.

Finally note that, in both of the above settings, the assertion that $P_\mathcal{U} \in C$ is trivial if the ultrafilter $\mathcal{U}$ is principal. Therefore, we have proven that all of the closed sets which generate the patch topology are also closed in the ultrafilter topology. The result follows immediately. □

The opposite direction is somewhat harder. We make use of the von Neuman regular ring $T(R)$ canonically associated with the ring $R$ (details below). We start by recalling a notation and some easy facts concerning the von Neumann regular rings. If $f : R \to S$ is a ring homomorphism, $f^a : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ denotes the map defined by $f^a(Q) := f^{-1}(Q)$, for each $Q \in \mathrm{Spec}(S)$.

**Lemma 3.** *The following are equivalent:*

- **(i)** *$R$ is von Neumann regular (i.e. for each $a \in R$ there exists $x \in R$ such that $a^2x = a$).*
- **(ii)** *for each $a \in R$ there exists a unique element $a^{(-1)} \in R$ (called* the punctual inverse of *$a$) such that $a^2a^{(-1)} = a$ and $\left(a^{(-1)}\right)^2 a = a^{(-1)}$.*

**Lemma 4.** *Let $R$ be a von Neumann regular ring and set $e(a) := aa^{(-1)}$, for each $a \in R$. Then the following hold:*

- **(1)** *$e(a)$ is idempotent.*

**(2)** *$aR = e(a)R$.*
**(3)** *If $a, b \in R$, then $aR + bR = e(a)R + e(b)R = (e(a) + e(b)(1 - e(a))R$.*

**Proposition 5.** *Let $R$ be any ring, let $\{X_a \mid a \in R\}$ be a family of indeterminates (one for each element $a \in R$) and let $I_R$ be the ideal generated by $\{a^2X_a - a, aX_a^2 - X_a \mid a \in R\}$ in the polynomial ring $R[X_a \mid a \in R]$. Set:*

$$T(R) := \frac{R[X_a \mid a \in R]}{I_R}.$$

*Then*

**(1)** *$T(R)$ is von Neumann regular (thus, in particular, every finitely generated ideal of $T(R)$ is principal).*
**(2)** *The canonical embedding $\iota : R \to T(R)$ is an epimorphism.*
**(3)** *$\iota : R \to T(R)$ is an isomorphism if and only if $R$ is von Neumann regular.*
**(4)** *Let $\mathrm{Spec}(R)_{\boldsymbol{Z}}$ [respectively, $\mathrm{Spec}(T(R))_{\boldsymbol{Z}}$] be $\mathrm{Spec}(R)$ [respectively, $\mathrm{Spec}(T(R))$] endowed with the Zariski topology and let $\iota^a : \mathrm{Spec}(T(R)) \to \mathrm{Spec}(R)$ be the canonical map associated to $\iota : R \to T(R)$. Then $\iota^a : \mathrm{Spec}(T(R))_{\boldsymbol{Z}} \to \mathrm{Spec}(R)_{\boldsymbol{Z}}$ is continuous and bijective.*
**(5)** *Let $\mathrm{Spec}(R)_{\boldsymbol{C}}$ [respectively, $\mathrm{Spec}(T(R))_{\boldsymbol{C}}$] be $\mathrm{Spec}(R)$ [respectively, $\mathrm{Spec}(T(R))$] endowed with the patch topology. Then $\iota^a : \mathrm{Spec}(T(R))_{\boldsymbol{Z}} \to \mathrm{Spec}(R)_{\boldsymbol{C}}$ is an homeomorphism. In particular, $\mathrm{Spec}(T(R))_{\boldsymbol{Z}}$ coincides with $\mathrm{Spec}(T(R))_{\boldsymbol{C}}$.*

Details concerning the von Neumann regular ring $T(R)$ canonically associated with the ring $R$ and the proof of the previous (and related) results can be found in [13], [14], [1, Chapter 3, Exercise 30], [2, Chapitre 1, §1, Exercices 16, 17, 18; Chapitre 2 §4, Exercice 16], [10, pages 99, Exercice 8 page 119] and [6].

For the sake of simplicity, from now on, we identify $R$ with its canonical image in $T(R)$. Consider a subset $C$ of $\mathrm{Spec}(T(R))$. Then consider also the corresponding collection $C_R := \{P \cap R \mid P \in C\}$ of primes in $R$. Note that by Proposition 5 (4), we can identify the sets of primes of $R$ and $T(R)$ by using just the elements that lie in $R$. Define an ultrafilter $\mathcal{U}$ on $C$. Then we can define a corresponding ultrafilter $\mathcal{U}_R$ on $C_R$. Consider the prime $P_{\mathcal{U}}$ of $T(R)$ defined by $\mathcal{U}$. Consider the prime $P_{\mathcal{U}} \cap R$ of $R$. Choose an element $d \in R \subseteq T(R)$. Define $V_R(d)$ to be all the primes of $R$ which contain $d$. Recall that $P_{\mathcal{U}} = \{d \in T(R) \mid V(d) \cap C \in \mathcal{U}\}$. Then since we are assuming that $d \in R$ we have $d \in P_{\mathcal{U}}$ exactly when $V_R(d) \cap C_R \in \mathcal{U}_R$. But this is exactly the condition necessary for $d$ to lie in $P_{\mathcal{U}}$. The point of this is that the ultrafilter limit primes of a collection of primes of $T(R)$ will correspond precisely to the ultrafilter limit primes of the corresponding collection of primes of $R$. This implies that the contraction map $\iota^a : \mathrm{Spec}(T(R)) \to \mathrm{Spec}(R)$ and its inverse send closed sets to closed sets with respect to the ultrafilter topologies. The following result is then clear.

**Proposition 6.** *The contraction map $\iota^a : \mathrm{Spec}(T(R)) \to \mathrm{Spec}(R)$ is a homeomorphism with respect to the ultrafilter topologies.*

So we have homeomorphisms between $\mathrm{Spec}(T(R))$ and $\mathrm{Spec}(R)$ with respect to both the patch topology and the ultrafilter topology. All that remains is to show that the ultrafilter and patch topologies coincide on $\mathrm{Spec}(T(R))$.

We can use the result of Proposition 2 to show that a subset of $\mathrm{Spec}(T(R))$ which is closed in the patch topology (or, in the Zariski topology by Proposition 5 (5)) is also closed in the ultrafilter topology. We prove the converse.

**Proposition 7.** *Let $R$ be a ring and let $C$ be a collection of prime ideals in $\mathrm{Spec}(T(R))$. Suppose that $C$ is closed in the ultrafilter topology. Then $C$ is also closed in the patch topology.*

*Proof.* We prove the contrapositive.

Suppose that $C$ is a collection of prime ideals in $\mathrm{Spec}(T(R))$which is not closed in the patch topology. Let $I := \cap_{P \in C} P$ $(\subseteq T(R))$. Since $C$ is not closed in the patch topology (or, equivalently, in the Zariski topology) of $\mathrm{Spec}(T(R))$ then $V(I)$ is properly larger than $C$. Let $\overline{P}$ be a prime ideal in $T(R)$ such that $I \subseteq \overline{P}$ but $\overline{P} \notin C$. Choose a nonzero element $a \in \overline{P}$. We claim that $a \in Q$, for some $Q \in C$. Suppose not. We can find an element $x \in T(R)$ such that $a^2x - a = 0$. Note then that $a(ax-1) = 0$. It follows that any prime which does not contain $a$ must contain $ax - 1$. Hence, $ax - 1$ is contained in every prime in $C$. This implies that $ax - 1$ is contained in $I$. This then implies that $ax - 1$ is contained in $\overline{P}$. Since $a$ and $ax - 1$ are relatively prime and are both contained in $\overline{P}$ we have a contradiction.

For each element $a \in \overline{P}$ let $V_C(a)$ represent all of the primes of $C$ which contain $a$ (i.e., $V_C(a) = V(a) \cap C$). (Note that we have proven in the preceding paragraph that $V_C(a)$ is not empty.) Our goal is to build an ultrafilter on $C$ such that each set $V_C(a)$ is in the ultrafilter. If we can show that a collection of nonempty subsets of a set is closed under finite intersection then we will have shown that all subsets in the collection lie in a filter. Then note that the collection of all filters on a set can be partially ordered under inclusion. Zorn's Lemma implies that maximal filters exist and that every filter is contained in a maximal filter. The maximal filters correspond precisely to the ultrafilters. So our goal is to prove that the collection of sets of the form $V_C(a)$, with $a \in \overline{P}$, is closed under finite intersections. This is equivalent to showing that if $J$ is a nonzero finitely generated ideal contained in $\overline{P}$ then there is some prime $Q \in C$ such that $J \subseteq Q$. Recall however, that in $T(R)$ all finitely generated ideals are principal (Proposition 5 (1)). It follows that such an ideal $J$ is actually principal and since the generator must lie in $\overline{P}$ we have already proven what we need. (Note that we have proven that a finite intersection of ideals of the form $V_C(a)$ has that same form and hence is not empty.) So we let $\mathcal{U} := \mathcal{U}(\overline{P})$ be an ultrafilter which contains all of the sets $V_C(a)$ for $a \in \overline{P}$. Then we construct the ultrafilter limit prime $P_{\mathcal{U}}$. The construction was designed so that $\overline{P} \subseteq P_{\mathcal{U}}$. However, $T(R)$ is a zero-dimensional ring so we have actually proven that $\overline{P} = P_{\mathcal{U}}$. Since we assumed that $\overline{P}$ is not in $C$ this implies that $C$ is not ultrafilter closed. □

The preceding result finishes the last step in our main theorem.

**Theorem 8.** *Let $R$ be a ring. Then the patch topology and the ultrafilter topology on the collection* $\mathrm{Spec}(R)$ *of prime ideals of $R$ are identical.*

**Added in proof.** The authors have become aware recently that Gabriel Picavet obtained some results regarding ultrafilters and spectral topologies which appeared in: *Ultrafiltres sur un espace spectral, anneaux de Baer, anneaux a spectre minimal compact*, Math. Scand, **46**, (1980), 23-53. The authors also wish to thank Picavet for pointing out that the results of J.-P. Olivier cited in this paper can also be found in the paper by Olivier published as *L'anneau absolument plat universel, les epimorphismes et les parties constructibles* Boletin de la Sociedad Matematica Mexicana, **23** (1978), 68-74.

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