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Chang, Gyu Whan (KR-INHA); **Fontana, Marco** (I-ROME3)

Uppers to zero and semistar operations in polynomial rings. (English summary)

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A domain D is quasi-Prüfer if for each prime ideal P of D, if Q is a prime ideal of $D[X]$ such that $Q \subseteq P[X]$ then $Q = (Q \cap D)[X]$. This is well known to be equivalent to the fact that the integral closure of D is Prüfer.

The authors generalize the notion of quasi-Prüfer domains to arbitrary semistar operations. A domain D is $\star$-quasi Prüfer if every prime ideal Q of D included in the extension of a prime quasi-$\star$-ideal of D is extended from D. (Recall that if $\star$ is a semistar operation on D, a quasi-$\star$-ideal is an ideal I of D such that $I^\star \cap D = I$.)

In a previous work ["Uppers to zero in polynomial rings and Prüfer-like domains", Comm. Algebra, to appear], they have shown that a domain is t-Prüfer if and only if it is a UMt-domain, that is, a domain in which each upper to zero is a maximal t-ideal. The goal of the paper under review is to give a generalization to the case of an arbitrary semistar operation which is stable and of finite type of this result. The obstruction to doing this lies in the fact that, while in the particular case of the t-operation of D it is natural to consider the t-operation of $D[X]$, in the general case it is not clear which semistar operation on $D[X]$ should correspond to the semistar operation $\star$ of D.

So, given a semistar operation $\star$ on D, the authors define a semistar operation $[\star]$, stable and of finite type, on $D[X]$, essentially as the spectral semistar operation induced on $D[X]$ by the set of primes $\{Q \in \mathrm{Spec}(D[X]) \mid Q \cap D = (0)$ and $c_D(Q)^{\star_f} = D^\star\} \cup \{P[X] | P$ is a quasi-$\star_f$-maximal ideal of $D\}$. (Here $c_D(Q)$ is the content in D of the ideal Q of $D[X]$, $\star_f$ is the semistar operation of finite type canonically associated to $\star$, and a quasi-$\star_f$-maximal ideal is an ideal which is maximal in the set of quasi-$\star_f$-ideals.)

Then they show that a domain D is $\star_f$ quasi-Prüfer if and only if $D[X]$ is $[\star]$-quasi-Prüfer, if and only if each upper to zero is a quasi-$[\star]$-maximal ideal of $D[X]$.

Recall that a P$\star$MD (Prüfer semistar multiplication domain) is a domain in which $(II^{-1})^{\star_f} = D^\star$ for each finitely generated ideal I, a $\star$-Noetherian domain is a domain in which the ascending chain condition on quasi-$\star$-ideals holds, and a $\star$-Dedekind domain is a $\star$-Noetherian P$\star$MD.

To finish, the authors show that a domain D is $\star$-Noetherian if and only if $D[X]$ is $[\star]$-Noetherian, it is a P$\star$MD if and only if $D[X]$ is a P$[\star]$MD, and it is $\star$-Dedekind if and only if $D[X]$ is $[\star]$-Dedekind.

Reviewed by *Giampaolo Picozza*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.