Ann. Univ. Ferrara - Sez. VII - Sc. Mat.
Vol. XXXVI, 1-13 (1990)

Characterizing Kronecker Function Rings (*).

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1. - Introduction.

Throughout this article, K denotes a field and X denotes an indeterminate over K. We shall say that a domain S having quotient field $K(X)$ is a *Kronecker function ring* (*with respect to K and X*), and write that S is a *KFR*, in case there exist a domain R with quotient field K and an *endlich arithmetisch brauchbar* (e.a.b.) *-operation, *, on the set of nonzero fractional ideals of R such that S coincides with

$$R^* = \{0\} \cup \{f/g : f, g \in R[X] \setminus \{0\} \text{ and } c(f)^* \subset c(g)^*\}.$$

(As usual, if $h \in R[X]$, then $c(h)$ denotes the ideal of R generated by the coefficients of h). Background on Kronecker function rings appears in [12] and [8, sections 32-34]; for ease of reference, we shall assume familiarity with the latter. Note, via [8, Corollary 32.8], that any R admitting an e.a.b. * as above must be integrally closed.

There are several reasons for interest in Kronecker function rings. First, if $*_1$ and $*_2$ are each e.a.b. *-operations on the nonzero fractional ideals of R, then $*_1$ and $*_2$ are equivalent (in the sense of agreeing at each nonzero finitely

(*) This work is partially supported by a NATO Collaborative Research Grant.
(1) Supported in part by Università di Roma «La Sapienza» and a Faculty Development Grant from the University of Tennessee.
(2) Supported in part by Università di Roma «La Sapienza» and University of Tennessee.
(3) Work done under the auspices of G.N.S.A.G.A. of the Consiglio Nazionale delle Ricerche.

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generated ideal of R) if and only if $R^{*_1} = R^{*_2}$ (cf. [8, Remark 32.9]). Secondly, if T is any domain, then $X(T)$, the abstract Riemann surface of T, is homeomorphic to Spec (R^*) with the Zariski topology for a suitable R and e.a.b. $*$: see [6, Theorem 2]. (The underlying set of $X(T)$ is the collection of all valuation overrings of T. $X(-)$ is a functor of considerable importance in classical algebraic geometry). Thirdly, each KFR is a Bézout domain [8, Theorem 32.7. (*b*)]; in particular, it is treed, in the sense that its prime spectrum, under partial order by inclusion, forms a tree. Finally (cf. [6, Lemma 6 (*c*)]), if T is any treed domain, then Spec (T) is order-isomorphic to Spec (R^*), for a suitable R and e.a.b. $*$.

By the above remarks, Spec fails to distinguish the KFR's in the (larger) class of treed domains. In fact, KFR's form a proper subclass of all Bézout domains having rational function quotient fields. For instance, no polynomial ring can be a KFR (see Proposition 2.3 (*a*)). Section 2 is devoted to such «rarity» results (cf. also Proposition 2.5) and actually addresses scarcity for the more instrinsic concept of a *kfr*, which is defined next. We shall say that S is a *kfr* (*with respect to K and X*) in case $S = R^*$ where R is a subring of $K(X)$ having quotient field F, $K(X) = F(Y)$ for some indeterminate Y over F, $*$ is an e.a.b. $*$-operation on the nonzero fractional ideals of R, and R^* is constructed with respect to the variable Y. It is evident (by choosing $F = K$ and $Y = X$) that each KFR is a *kfr*; however, as Example 2.2 shows, the converse is false.

In a more positive vein, Section 3 characterizes KFR's (and, by varying K and X, thus characterizes *kfr*'s). Specifically, Theorem 3.2 shows, i.a., that a subring S of $K(X)$ is a KFR if and only if S is integrally closed, K is the quotient field of $S \cap K$, and $(W \cap K)^* = W$ for each $W \in X(S)$. (As usual, if V is a valuation ring of K, then V^* denotes the *trivial extension* of V to $K(X)$, that is, the valuation ring of the inf-extension of any valuation on K having valuation ring V). In case S is itself a valuation domain, this criterion reduces to $(S \cap K)^* \subset S$: see Proposition 3.5 for this and other equivalents.

In Proposition 3.3, the question whether a given KFR $S = R^*$ is a suitable ring of fractions of $R[X]$ is related to whether $*$ is equivalent to the v-operation and whether $S \cap K$ is a Prüfer v-multiplication domain.

In the final section, we give some deeper «scarcity» results, and give two important classes of fields K for which all *kfr* subrings of $K(X)$ may be listed. In particular, we show that if either char $K = p$ and $t.d._{F_p}(K) \leq 1$ or K is an algebraic extension of $\boldsymbol{Q}$, then each *kfr* subring of $K(X)$ is a Nagata ring.

Several of the results in Sections 2 and 3 of this paper were announced in [2].

Any unreferenced material is standard, typically in [8].

2. - Rarity.

Before constructing a *kfr* which is not a *KFR*, we collect some useful facts.

LEMMA 2.1. *Let $S = R^*$ be a kfr with respect to K and X, where R has quotient field F, $K(X) = F(Y)$, and R^* is constructed using the e.a.b. *-operation * with respect to the variable Y. Then:*

(a) S is a Bézout domain; $S \cap F = R$; the quotient field of $S \cap F$ is F; R is integrally closed; and $(W \cap F)^*$, the trivial extension of $W \cap F$ to $F(Y)$, coincides with W for each $W \in X(S)$.

(b) S is a field if and only if R is a field; that is, $S = K(X)$ if and only if $S \cap F = F$.

The assertions in Lemma 2.1 (*a*) are well known: cf. [8, Theorem 32.7 (1), (2); Corollary 32.8; and Theorem 32.10]. They will figure prominently in Theorem 3.2's characterization of *KFR*'s. As for the proof of (*b*), its «only if» half is clear: $S = K(X) \Rightarrow S \cap F = F(Y) \cap F = F$. However, to prove the «if» half of (*b*), we first need to recall some facts about completion and the Nagata ring.

Consider an e.a.b. *-operation * on the nonzero fractional ideals of a(n integrally closed) domain R. By [8, Theorem 32.12], * is equivalent to a suitable w-operation; that is, $R = \cap V_i$ for some set $\{V_i\}$ of valuation overrings of R such that $J^* = \cap JV_i$ for each nonzero finitely generated fractional ideal J of R. In case $\{V_i\} = X(R)$, the associated w-operation is called *completion* and the associated Kronecker function ring is denoted by R^b. Thus $R^b \subset R^*$ for each e.a.b. *. On the other hand, [8, Theorem 33.3] records that $R(X) \subset R^b$, where the Nagata ring $R(X)$ is defined as the ring of fractions of $R[X]$ with respect to the multiplicatively closed set $\{f \in R[X] : c(f) = R\}$. An oftcited result [8, Theorem 33.4], needed in Example 2.2 below, is that $R(X) = R^b$ if and only if R is a Prüfer domain.

We may now prove the «if» half of Lemma 2.1 (*b*). Let $S \cap F = F$; that is, $R = F$. Then, necessarily, * is equivalent to the (identity) w-operation induced by the singleton set, $\{F\}$. However, the Kronecker function ring constructed from the identity operation is clearly $F(Y)$, and so $S = F(Y)$, completing the proof of Lemma 2.1.

Lemma 2.1 easily implies that if a ring S is contained properly between K and $K(X)$, then S cannot be a *KFR*. However, such an S *can* be a *kfr*, as we see in

EXAMPLE 2.2. If $K = \mathbf{Q}(Y)$, then $S = \mathbf{Q}[X](Y)$ is a *kfr* but not a *KFR*. Indeed, if R denotes the (Prüfer) domain $\mathbf{Q}[X]$ and the ring R^b is constructed with respect to the variable Y, then $S = R^b$, a *kfr* (with respect to K and X). However, Lemma 2.1 (b) shows that S is not a *KFR* (with respect to K and X); the point is that $S \cap K = K$ although, since $X^{-1} \notin S$, $S \neq K(X)$.

Despite the preceding result, not every Bézout domain with quotient field $K(X)$ is a *kfr*. For instance, we have

PROPOSITION 2.3. (*a*) *No polynomial ring is a kfr.*

(b) *If T is an indeterminate over $\mathbf{Q}$, and $f \in \mathbf{Q}[T]$ is irreducible, then the DVR, $\mathbf{Q}[T]_{(f)}$, is not a kfr.*

PROOF. (*a*) Suppose that $S = A[\{Y_i\}]$ is a *kfr*, for some domain A and nonempty set $\{Y_i\}$ of variables. If P is a nonzero maximal ideal of A, then PS and $(\{Y_i\})$ are incomparable primes each contained in the maximal ideal $(P, \{Y_i\})$, contradicting the fact that S is treed. Thus A is a field, and it follows easily that $\{Y_i\}$ is a singleton set, say $\{Y\}$. By hypothesis, $A[Y] = R^*$ where F is the quotient field of R, $A(Y) = F(Z)$ for some indeterminate Z over F, and R^* is contructed with respect to the variable Z. Since S is not a field, Lemma 2.1 (*b*) assures that R is not a field; pick a nonzero nonunit $r \in R$. Set $u = (r-1) + Z$ and $v = 1 + (r-1)Z$. Clearly, both u and v are units of R^*; that is, u and v are in A. Thus $w = u + v = r + rZ$ is a nonzero element of A and hence a unit of R^*. However, $w^{-1} \notin R^*$ since $c(1) = R \not\subseteq Rr = c(w)$. This contradiction establishes (*a*).

(*b*) Suppose that $S = \mathbf{Q}[T]_{(f)}$ is a *kfr*. Then $S = R^*$ where F is the quotient field of R, $K(X) = \mathbf{Q}(T) = F(Y)$ for some indeterminate Y over F, and R^* is constructed using Y. It follows easily from Luroth's theorem that $F = = \mathbf{Q}(= K)$. As $S \cap F = F$ and (since $f^{-1} \notin S$) $S \neq K(X)$, this contradicts Lemma 2.1 (*b*). The proof of Proposition 2.3 is complete.

REMARK 2.4. (*a*) Another (quick) proof of Proposition 2.3 (*a*) is available using the ideas in [7]. First, reduce as above to the case $S = A[Y]$, A a field. Next, apply [7, Proposition 4.1 and Corollary 5.2] to conclude that 1 is in the stable range of the supposed *kfr* $S = R^*$. We then obtain the (desidered) contradiction, for $A[Y]$ fails to satisfy criterion (iii) of [7, Proposition 5.1]; the point is that no $d \in A[Y]$ satisfies $(Y, 1 - Y^2) = = (Y + d \cdot (1 - Y^2))$.

(*b*) As noted in (*a*), each *kfr* has 1 in the stable range. Thus, by [7, Theorem 5.3], a *kfr* S is a *PID* (if and) only if S is a Euclidean domain. Note, however, that $\mathbf{Q}[T]_{(T)}$ is Euclidean (hence, a *PID* and in fact, a *DVR*), has 1 in the stable range (by, for instance, criterion (iii) in [7, Proposition 5.1]), but is not a *kfr* (by Proposition 2.3 (*b*)).

(*c*) Besides *DVR*'s of the type considered in Proposition 2.3 (*b*), no formal power series ring $A[\![Y_1, \dots, Y_n]\!]$ can be a *kfr*. To see this, use treedness as in the proof of Proposition 2.3 (*a*) to reduce to the case A a field, $\{Y_i\} = \{Y\}$; then one need only note that $A((Y))$ cannot take the form $F(Z)$, Z transcendental over a field F. (We are indebted to S. Mulay for the preceding observation). To find which (not necessarily discrete) valuation domains *can* be *KFR*'s, see (*d*) and Proposition 3.6.

(*d*) Despite Proposition 2.3 (*b*), the *DVR*, $F[T]_{(f)}$, *can* be a Kronecker function ring for suitable F and f. To see this, let T and U be algebraically independent indeterminates over a field k, set $F = k(U)$, and set $S = F[T]_{(T)}$. One may verify essentially via Gauss' lemma, that S is an overring of $k[T](U)$. However, $k[T](U)$ is the Kronecker function ring of the Prüfer domain $k[T]$ arising from completion with respect to U (cf. [8, Theorem 33.4]). Thus, [8, Theorem 32.15] implies that S is a *kfr* with respect to $K = k(T)$ and $X = U$.

(*e*) In view of the comments in the introduction, it is interesting to note that $R_1^{*_1} = R_2^{*_2}$ does not imply $R_1 = R_2$. To see this, let Y_1, Y_2 be independent indeterminates over a field F, let R be a Prüfer domain with quotient field F, and use [8, Theorem 33.4] to conclude that $R_1 = R(Y_1)$ and $R_2 = R(Y_2)$ are Prüfer domains. Evidently $R_1 \neq R_2$ since $Y_1 \in R_1 \setminus R_2$. By another appeal to [8, Theorem 33.4], completion of R_1 (resp., R_2) with respect to Y_2 (resp., Y_1) leads to $R_1^{*_1} = R_1(Y_2)$ and $R_2^{*_2} = R_2(Y_1)$. It remains only to verify that $R_1^{*_1} = R_2^{*_2}$, and this follows directly from [1, Lemma].

Our final «rarity» result in this section is

PROPOSITION 2.5. *Let L be a field of positive characteristic which is algebraic over its prime subfield, and let T be an indeterminate over L. If a subring S of $L(T)$ is a kfr, then S is a field.*

PROOF. $S = R^*$, where F is the quotient field of R, $K(X) = F(Z) \subset L(T)$ for some indeterminate Z over F, and R^* is constructed using Z. If k denotes the prime subfield of L (and, hence, of F), then k is ring-generated by $\{1\}$, whence $k \subset R$. It follows from the construction of the Nagata ring that $k(Z) \subset R(Z)$; moreover, as recalled after the statement of Lemma 2.1, $R(Z) \subset R^* (= S \subset F(Z) \subset L(T))$. Now, to complete the proof, it suffices to show that $L(T)$ is algebraic over $k(Z)$: cf. [8, Lemma 11.1]. This, in turn, follows by considering transcendence degrees:

$$1 = 0 + 1 = t.d._k(L) + t.d._L(L(T)) = t.d._k(L(T)) =$$
$$= t.d._k(k(Z)) + t.d._{k(Z)}(L(T)) = 1 + t.d._{k(Z)}(L(T)),$$

whence $t.d._{k(Z)}(L(T)) = 0$, as desired.

3. – Characterizations.

Before presenting a characterization of *KFR*'s, we shall introduce some notation and remark upon it. If $K(X)$ obtains a(n integrally closed) domain S expressed as $S = \cap W_i$ for some set $\mathcal{W} = \{W_i\}$ of valuation overrings W_i of S, set $\mathcal{W}_K = \{W_i \cap K\}$. Evidently each $W_i \cap K$ is a valuation ring of K. Moreover, if S were a *KFR*, then (cf. Lemma 2.1 (*a*)) K would be the quotient field of $S \cap K$, that is, each $W_i \cap K$ would then be an overring of $S \cap K$. However,

REMARK 3.1. In the above setting, the elements of $\mathcal{W}_K$ need not be overrings of $S \cap K$. To see this in case $K = \boldsymbol{Q}(Y)$, consider the integrally closed domain $S = \boldsymbol{Q}[X, XY^{-1}]$ and a particular (discrete) valuation overring, $W = \boldsymbol{Q}[X, Y]_{(X)}$. One verifies readily that $W \cap K = K$ is not an overring of $S \cap K = \boldsymbol{Q}$ (cf. [11, Exercise 5, page 73]).

Let S and $\mathcal{W}$ be as above and now *suppose* that $R = S \cap K$ has quotient field K. Then each $W_i \cap K$ is a valuation overring of R, and so it makes sense to let w_K denote the w-operation on the nonzero fractional ideals of R induced by $\mathcal{W}_K$. If $\mathcal{W} = X(S)$, then w_K will be denoted by b_K.

THEOREM 3.2. *Let S be an integrally closed subring of $K(X)$; set $R =$ $= S \cap K$. Then the following conditions are equivalent:*

(1) *S is a KFR (with respect to K and X);*

(2) *K is the quotient field of R, and $S = R^{b_K}$;*

(3) *Each overring of S is a KFR (with respect to K and X);*

(4) *K is the quotient field of R, and $R^{b_K} \subset S$;*

(5) *K is the quotient field of R, and there exists $\mathcal{W} = \{W_i\} \subset X(S)$ such that $S = \cap W_i$ and $R^{w_K} \subset S$;*

(6) *K is the quotient field of R, and there exists $\mathcal{W} = \{W_i\} \subset X(S)$ such that $S = \cap W_i$ and $(W_i \cap K)^* = W_i$ for each i;*

(7) *K is the quotient field of R, and $(W \cap K)^* = W$ for each $W \in X(S)$.*

PROOF. (4) $\Rightarrow$ (7): Since S is integrally closed, so is R. Given (4), [8, Theorem 32.15] then assures that S is a *KFR*, whence [8, Theorem 32.10] yields (7).

(7) $\Rightarrow$ (6): Take $\mathcal{W} = X(S)$.

(6) $\Rightarrow$ (5): Given (6), [8, Theorem 32.11] assures that $R^{w_K} = \cap(W_i \cap K)^*$ $(= \cap W_i = S)$.

(5) $\Rightarrow$ (4): It suffices to note, via [8, Theorem 32.11], that $R^{b_K} \subset R^{w_K}$.

(2) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (3): Apply [8, Theorem 32.15].

(3) $\Rightarrow$ (1): Trivial.

(1) $\Rightarrow$ (2): Assume (1). Then, using $\{W_i\} = X(S)$ and appealing to [8, Theorems 32.11 and 32.10], we have $R^{b_K} = \cap (W_i \cap K)^* = \cap W_i = S$. The proof is complete.

As usual, we shall let v denote the *-operation, $A \to (A^{-1})^{-1}$. The next result is an analogue of [8, Theorem 33.4] for Prüfer v-multiplication domains, *PVMD*'s. (By [10], a domain R is a *PVMD* if and only if R_P is a valuation domain for each maximal t-ideal P, and, for such R and P's, $R = \cap R_P$. In particular, each *PVMD* is an essential domain in the sense of [14]. For additional characterizations of *PVMD*'s, see, e.g., [13]).

PROPOSITION 3.3. (cf. [3, Lemma 1]). *Let S be a KFR and set $R = S \cap K$. Let Σ denote the (multiplicatively closed) set $\{f \in R[X]: c(f)_v = R\}$. Then the following conditions are equivalent:*

(1) *$S = R^v$ and R is a PVMD;*

(2) *$S = R[X]_\Sigma$.*

PROOF. If R is a *PVMD*, the above remarks on essentiality allow us to infer from [14, Proposition 12] that $R^v = R[X]_\Sigma$. It therefore remains only to show that (2) implies that R is a *PVMD*; and this follows from [14, Corollary 13] since each *KFR* is a Bézout domain.

REMARK 3.4. It is straightforward to convert the criteria in Theorem 3.2 into characterizations of *kfr*'s. The first assertion in the abstract makes this explicit for criterion (7), and we leave the others to the reader.

Next, we specialize Theorem 3.2 to the case of valuation domains. For motivation, see Proposition 2.3 (*b*) and Remark 2.4 (*c*), (*d*).

PROPOSITION 3.5. *Let (W, M) be a valuation ring of $K(X)$; set $R = W \cap K$. Then the following conditions are equivalent:*

(1) *W is a KFR;*

(2) *$W = R^b$;*

(3) *$W = R^*$, that is, W is the trivial extension of R to $K(X)$;*

(4) *$W_P = (W_P \cap K)^*$, for each $P \in \mathrm{Spec}(W)$;*

(5) *$R^* \subset W$;*

(6) *$R(X) \subset W$;*

(7) *$R(X) = W$;*

(8) *The canonical map $X(W) \to X(R)$ is bijective, with inverse map $X(R) \to X(W)$ given by $V \to V^*$.*

PROOF. Note first that $(R, M \cap K)$ is a valuation ring of K; in particular, K is the quotient field of R. Then, as a direct application of Theorem 3.2, we have (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5). (For (5) $\Rightarrow$ (1), one also needs to note via [8, Theorem 32.11] that if $\mathcal{W} = \{W\}$, when $R^{w_K} = R^*$). As $\{W_P : P \in \mathrm{Spec}(W)\}$ is the set of overrings of W, Theorem 3.2 now shows that (4) is equivalent to each overring of W being a *KFR*; hence, (1) $\Leftrightarrow$ (4). Next, as in the proof of Proposition 3.5, [8, Proposition 32.18 and Theorem 33.4] combines with the fact that R is a Prüfer domain to yield (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (7).

(7) $\Rightarrow$ (6): Trivial.

(6) $\Rightarrow$ (3): Given (6), W is a valuation overring of the *KFR*, $R(X)$, and so Theorem 3.2 [(1) $\Rightarrow$ (7)] yields (3).

(2) $\Rightarrow$ (8): Well known [8, Theorems 32.10 and 32.15].

(8) $\Rightarrow$ (1): Given (8), [8, Theorem 32.11] leads to

$$R^{b_K} = \cap \{(V \cap K)^* : V \in X(W)\} = \cap V = W,$$

and so (1) holds, completing the proof.

4. - Catalogues: more rarity.

In this final section, we investigate several classes of fields K for which all *kfr* subrings of $K(X)$ may be listed. When one tries to determine the *kfr* subrings of a field $K(X)$, the following problems naturally arise:

(1) *Let F and K be fields, with Y and X indeterminates over F and K, respectively. When does $F(Y) \subset K(X)$ imply $F \subset K$?*

(2) *In particular, when does $F(Y) = K(X)$ imply $F = K$?*

Of course, the answer to the above questions is not «Always». To see this, let X and Y be independent indeterminates over $\boldsymbol{Q}$, and set $F = \boldsymbol{Q}(X)$ and $K = \boldsymbol{Q}(Y)$; then $F(Y) = K(X)$, although F and K are not comparable.

We next give several cases in which $F(Y) \subset K(X)$ *does* imply $F \subset K$.

PROPOSITION 4.1. *Let L and K be fields, with Y and X indeterminates over F and K, respectively.*

(*a*) *If $F(Y) \subset K(X)$, then $F \subset K$ if either*

(i) *Each $\alpha \in F$ is algebraic over K, or*

(ii) *For each $\alpha \in F$, $\{n \in N \mid \alpha = \beta^n$ for some $\beta \in F\}$ is an infinite set.*

(*b*) *If $F(Y) = K(X)$, then $F \subset K$ implies $F = K$.*

PROOF. (*a*) For part (i), just notice that if an element $\alpha \in F$ also satisfies $\alpha \in K(X) - K$, then α is not algebraic over K.

For part (ii), deny. Hence, we may choose $\alpha \in F$ such that $\alpha \in K(X) - K$. Then $\alpha = fg^{-1}$ with $f, g \in K[X]$, f and g relatively prime, g monic, and either f or g nonconstant. Consider $\alpha = \beta^n$ for some $\beta \in F$. Then also $\beta = pq^{-1}$ with $p, q \in K[X]$, p and q relatively prime, q monic, and either p or q nonconstant. By unique factorization in $K[X]$, $fg^{-1} = p^n q^{-n}$ implies that $g = q^n$ and $f = p^n$. However, this is clearly impossible for infinitely many n, by degree considerations, since either f or g (and hence p or q, respectively) is nonconstant. This (desired) contradiction establishes (ii).

(*b*) Deny, and pick $t \in K - F$. Then $t \in F(Y) - F$, and so t is transcendental over F. It follows that $t.d._F(F(t, X)) = 2$, since $F(t) \subset K$ and X is transcendental over K. This contradicts

$$t.d._F(F(t, X)) \leq t.d._F(K(X)) = t.d._F(F(Y)) = 1.$$

REMARK 4.2. (*a*) Three important cases in which Proposition 4.1 is applicable are

(i) F is algebraic over its prime subfield,

(ii) F is algebraically closed, or

(iii) $F = \boldsymbol{R}$.

(*b*) Note that a field F which satisfies either hypothesis in part (*a*) of Proposition 4.1 cannot be a purely transcendental extension of one of its subfields (cf. the reasoning preceding Proposition 4.1).

We next give an explicit and satisfying characterization of the *kfr* subrings of $K(X)$, when K is any algebraic extension field of $\boldsymbol{Q}$.

THEOREM 4.3. *Let K be an algebraic extension field of $\boldsymbol{Q}$ and X an indeterminate over K. Then:*

(*a*) *Each kfr subring A of $K(X)$ has the form $R_S(Y)$, where R is the integral closure of $\boldsymbol{Z}$ in a subfield F of K, S is a multiplicatively closed subset of R, and $Y \in K(X) - K$. Moreover, A has quotient field $K(X)$ if and only if R is the integral closure of $\boldsymbol{Z}$ in K and $Y = (aX + b)\cdot(cX + d)^{-1}$ for some $a, b, c, d \in K$ with $ad - bc \neq 0$.*

(*b*) *Each $R_S(Y)$ for R, S, and Y as in* (*a*) *is a kfr subring of $K(X)$.*

PROOF. (*b*) This follows easily form the fact that R, and hence R_S, is a Prüfer domain [8, Theorem 22.3 and Proposition 22.5], whence $R_S(Y) = R_S^b$ [8, Theorem 33.4].

(*a*) Let A be a *kfr* subring of $K(X)$. Then $A = B^*$, where $*$ is an e.a.b. $*$-operation on the nonzero fractional ideals of an integrally closed domain B with quotient field F and B^* is constructed with respect to the field F and an indeterminate $Y \in K(X)$, and $F(Y)$ is the quotient field of A. Then, necessarily, $t.d._{\boldsymbol{Q}}(F) = 0$, i.e., F is algebraic over $\boldsymbol{Q}$. Hence $F \subset K$, by Proposition 4.1 (*a*), (i). Certainly the integrally closed domain B contains R, the integral closure of $\boldsymbol{Z}$ in F. By [5, Corollary 1, page 202], $B = R_S$ for some multiplicatively closed subset S of R. As in part (*b*), $B^* = B^b = B(Y) = R_S(Y)$ since B is a Prüfer domain. Finally, $Y \in K(X) - K$ since Y is not algebraic over $\boldsymbol{Q}$.

The «only if» part of the «moreover» statement follows because $F(Y) = K(X)$ implies $F = K$ by Proposition 4.1, and Lüroth's theorem yields that Y has the desired from. The «if» part follows similarly via Lüroth.

REMARKS 4.4. (*a*) As a special case of Theorem 4.3, *the kfr subrings A of $\boldsymbol{Q}(X)$ are the rings of the form $\boldsymbol{Z}_S(Y)$ for some multiplicatively closed subset S of $\boldsymbol{Z}$ and $Y \in \boldsymbol{Q}(X) - \boldsymbol{Q}$. Moreover, such an A has quotient field $\boldsymbol{Q}(X)$ if and only if $Y = (aX + b)\cdot(cX + d)^{-1}$ for some $a, b, c, d \in \boldsymbol{Z}$ with $ad - bc \neq 0$.*

(*b*) In Proposition 2.3 (*b*), we have seen that for each irreducible $f \in \boldsymbol{Q}[X]$, $\boldsymbol{Q}[X]_{(f)}$ is a Bézout subring of $\boldsymbol{Q}(X)$ which is not a *kfr*. Another large class of Bézout subrings of $\boldsymbol{Q}(X)$ with quotient field $\boldsymbol{Q}(X)$ which are not *kfr*'s are the domains of the form $\boldsymbol{Z}_S + X\boldsymbol{Q}[X]$ for S a multiplicatively closed subset of $\boldsymbol{Z}$. Indeed, by [4, Theorem 7], such domains are Bézout domains; but they are evidently not Nagata rings, and hence by Theorem 4.3, not *kfr*'s.

We next prove the analogue of Theorem 4.3 for fields of positive characteristic. Since the algebraic case has been handled in Proposition 2.5, we restrict ourselves here to the case in which K and F each have transcendence degree one over $\boldsymbol{F}_p$.

THEOREM 4.5. *Let K be a field of positive characteristic p, such that $t.d._{\boldsymbol{F}_p}(K) = 1$ and let X be an indeterminate over K.*

(*a*) *Each kfr subring A of $K(X)$ has the form $R_S(Y)$, where R is either a subfield of $K(X)$ algebraic over $\boldsymbol{F}_p$ or the integral closure of $\boldsymbol{F}_p[t]$ in a subfield F of $K(X)$ with $t.d._{\boldsymbol{F}_p}(F) = 1$ and $t \in F$ transcendental over $\boldsymbol{F}_p$, S is a multiplicatively closed subset of R, and $Y \in K(X)$ is transcendental over F.*

(*b*) *Each $R_S(Y)$, for R, S, and Y as in* (*a*), *is a kfr subring of $K(X)$.*

PROOF. (*b*) As in the proof of Theorem 4.3 (*b*), it may be shown that each such $R_S(Y)$ is a *kfr* subring of $K(X)$.

(*a*) Conversely, let A be a *kfr* subring of $K(X)$. Then $A = B^*$ for an integrally closed domain B with quotient field F, * an e.a.b. *-operation on the nonzero fractional ideals of B, and B^* is constructed with respect to the field F and some $Y \in K(X)$ transcendental over F. By Proposition 2.5, we may assume that $t.d._{\boldsymbol{F}_p}(F) = 1$. Let t be any element of B that is not algebraic over $\boldsymbol{F}_p$. Then the integrally closed domain B contains R, the integral closure of $\boldsymbol{F}_p[t]$ in F, and by [5, Corollary 2, page 202], $B = R_S$ for some multiplicatively closed subset S of R. Since B is a Prüfer domain, $B^* = B^b = R_S(Y)$. This completes the proof.

It is interesting to note that when either K is algebraic over $\boldsymbol{Q}$ or $t.d._{\boldsymbol{F}_p}(K) \leq 1$, all the *kfr* subrings of $K(X)$ are actually Nagata rings. (Apply Theorems 4.3 and 4.5). This observation also follows from our next proposition (which is a slight refinement of [9, Theorem 3.1]) since its hypothesis descends from K to F, when $F(Y)$ is any subfield of $K(X)$.

PROPOSITION 4.6. *For any field K, the following conditions are equivalent:*

(1) *Each integrally closed subring of K is a Prüfer domain.*

(2) *Each integrally closed ring with quotient field K is a Prüfer domain.*

(3) *Either K is a algebraic over $\boldsymbol{Q}$ or $t.d._{\boldsymbol{F}_p}(K) \leq 1$.*

PROOF. (3) $\Rightarrow$ (1) follows from [9, Theorem 3.1]. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): Assume (2), let char $K = 0$, and suppose that K is not alge-

braic over $\boldsymbol{Q}$. Let $\{X_\alpha\}$ be a transcendence basis for $K/\boldsymbol{Q}$ and put $R = \boldsymbol{Z}[\{X_\alpha\}]$. Next, let A be the integral closure of R in K. Then A is not a Prüfer domain since R is not a Prüfer domain. (Apply [8, Theorem 22.4]). Since K is the quotient field of A, we have the desired contradiction. The proof for the characteristic p case is similar and hence will be omitted. The proof of Proposition 4.6 is complete.

We close with some observations on the *kfr* subrings A of $\boldsymbol{R}(X)$. (These observations apply, *mutatis mutandis*, to the *kfr* subrings of any field $F(X)$ where F and X satisfy the hypotheses of Proposition 4.1 (*a*)).

If the *kfr* A has quotient field $\boldsymbol{R}(X)$ and A is constructed with respect to F and Y, then $\boldsymbol{R}(X) = F(Y)$. By Proposition 4.1 (*a*) (ii), $F = \boldsymbol{R}$, and, by Lüroth's theorem, $Y = (aX + b)\cdot(cX + d)^{-1}$ for some $a, b, c, d \in \boldsymbol{R}$ with $ad - bc \neq 0$. (If we merely assume that $F(Y) \subset \boldsymbol{R}(X)$ then $F \subset \boldsymbol{R}$ and $Y \in \boldsymbol{R}(X)$ is transcendental over F). Then $A = R^*$ where R is an integrally closed subring of $\boldsymbol{R}$ with quotient field $\boldsymbol{R}$,* is an e.a.b. *-operation on the nonzero fractional ideals of R, and A is constructed with respect to $\boldsymbol{R}$ and Y. The integrally closed rings R with quotient field $\boldsymbol{R}$ are classified in the following manner. Let $\{X_\alpha\}$ be a transcendence basis for $\boldsymbol{R}/\boldsymbol{Q}$; then R is an integrally closed overring of the integral closure of $\boldsymbol{Z}[\{X_\alpha\}]$ in $\boldsymbol{R}$. Unlike the domains in Theorems 4.3 and 4.5 and Proposition 4.6, such domains R need not be Prüfer domains.

Pervenuto in Redazione il 22 dicembre 1986.

RIASSUNTO

Si dimostra che un dominio S è un anello di funzioni di Kronecker (in una variabile) se e soltanto se S è integralmente chiuso, il suo campo dei quozienti è della forma $F(Y)$, dove Y è trascendente su F, il campo dei quozienti di $S \cap F$ è F, e ogni sopraanello di valutazione W di S coincide con l'estensione banale di $W \cap F$. Inoltre, vengono classificati tutti i sottoanelli di funzioni di Kronecker di $K(X)$, per vari importanti casi del campo K.

SUMMARY

We show that an integral domain S is a Kronecker function ring if and only if S is integrally closed, its field of quotients is $F(Y)$, where F is a field such that the quotient field of $S \cap F$ is exactly F, Y is transcendental over F and every valuation overring W of S coincides with the trivial extension of $W \cap F$. Furthermore, we classify all the Kronecker function rings, subrings of $K(X)$, for several important types of fields K.

REFERENCES

[1] D. D. ANDERSON, *Some remarks on the ring $R(X)$*, Comm. Math. Univ. St. Pauli, **26** (1977), pp. 137-140.

[2] D. F. ANDERSON - D. E. DOBBS - M. FONTANA, *When is a Bézout domain a Kronecker function ring*, C.R. Math. Rep. Acad. Sci. Canada, 9 (1987), 25-30.

[3] J. T. ARNOLD - R. MATSUDA, *An almost Krull domain with divisorial height one primes*, Canad. Math. Bull., **29** (1986), pp. 50-53.

[4] J. W. BREWER - E. A. RUTTER, *$D+M$ constructions with general overrings*, Michigan Math. J., **23** (1976), pp. 33-42.

[5] E. D. DAVIS, *Overrings of commutative rings. II. Integrally closed overrings*, Trans. Amer. Math. Soc., **110** (1964), pp. 196-212.

[6] D. E. DOBBS - M. FONTANA, *Kronecker function rings and abstract Riemann surfaces*, J. Algebra, **99** (1986), pp. 263-274.

[7] D. ESTES - J. OHM, *Stable range in commutative rings*, J. Algebra, 7 (1967), pp. 343-362.

[8] R. GILMER, *Multiplicative Ideal Theory*, Dekker, New York (1972).

[9] R. GILMER - J. MOTT, *Integrally closed subrings of an integral domain*, Trans. Amer. Math. Soc., **154** (1971), pp. 239-250.

[10] M. GRIFFIN, *Some results on v-multiplication rings*, Canad. J. Math., **19** (1967), pp. 710-722.

[11] I. KAPLANSKY, *Commutative Rings*, rev. ed., Univ. of Chicago Press, Chicago (1974).

[12] W. KRULL, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*, Math. Z., **41** (1936), pp. 545-577.

[13] J. L. MOTT - M. ZAFRULLAH, *On Prüfer v-multiplication domains*, Manuscripta Math., **35** (1981), pp. 1-26.

[14] M. ZAFRULLAH, *Some polynomial characterizations of Prüfer v-multiplication domains*, J. Pure Appl. Algebra, **32** (1984), pp. 231-237.