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ON TREED NAGATA RINGS

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Let R be an integral domain. It is proved that R', the integral closure of R, is a Prüfer domain if and only if the canonical map $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ is a homeomorphism. As a consequence, $R(X)$ is treed if and only if R is treed and R' is Prüfer. We also characterize when $R(X)$ is going-down, an i-domain, universally going-down. Examples show that these situations are distinct. Analogous characterizations are obtained for the ring $R\langle X\rangle$. Also, it is proved that if R is integrally closed, its inclusion map into the completion R^{b} is universally going-down if and only if R is a Prüfer domain.

1. Introduction

Let R be a (commutative integral) domain, with integral closure R'. If X is an indeterminate over R, the *Nagata ring* $R(X)$ is the ring of fractions $R[X]_S$, where S is the set of polynomials with unit content. (Useful references on Nagata rings are [28, p. 18] and [21, Section 33]. The latter is especially useful for the extension to several variables.) Nagata rings are rather well behaved: the maximal ideals of $R(X)$ are the ideals $MR(X)$, where M ranges over the maximal ideals of R [28, 6.17(4)]. Nagata rings have been helpful in studying Prüfer domains because a (integrally closed) domain R is a Prüfer domain if and only if $R(X)$ is a Prüfer domain [21, Theorem 33.4]. Our main purpose here is to show that Nagata rings are equally helpful in studying related properties.

The definition of one such property is recalled next. As in [12], R is said to be *treed* in case $\mathrm{Spec}(R)$, as a poset under inclusion, is a tree; that is, in case no maximal ideal of R contains incomparable prime ideals. Each Prüfer domain is treed (since it is locally a valuation domain), as is each domain of (Krull) dimension at most 1. Given the above information, it is reasonable to ask *this article's motivating*

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question: when is $R(X)$ treed? The answer is *not* "if and only if R is treed". As Theorem 2.10 establishes, the answer is "if and only if R is treed *and* R' is a Prüfer domain". Along the way, we find the following useful technical fact in Theorem 2.7: R' is a Prüfer domain if and only if the canonical surjection $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ is injective. Other characterizations of Prüfer domains appear in Corollaries 2.9, 2.13 and 2.14; Proposition 2.19; and Theorem 2.20.

Much of Section 2 concerns analogues of Theorem 2.10. Many classes of rings are naturally interposed between the Prüfer domains and the treed domains. For instance,

$$\text{Prüfer domain} \Rightarrow \text{universally going-down domain}$$
$$\Rightarrow i\text{-domain} \Rightarrow \text{going-down domain} \Rightarrow \text{treed domain.}$$

(Definitions and background are recalled in Section 2 as needed.) In general, none of these implications is reversible. Does this remain the case when attention is restricted to Nagata rings? For instance, is there a treed Nagata ring which is not a going-down domain? Section 2 details an affirmative answer to such questions in a comprehensive way, by characterizing when $R(X)$ is a universally going-down domain, when $R(X)$ is an i-domain, and when $R(X)$ is a going-down domain.

Another result in Section 2 is worthy of note. Assume now that R is integrally closed, with completion R^{b}. A result of Arnold and Brewer [4] asserts that R is a Prüfer domain if and only if R^{b} is flat over $R[X]$. This is recovered in Theorem 2.20, where we show that R is a Prüfer domain if and only if the inclusion map $R \to R^{\mathrm{b}}$ is a universally going-down homomorphism (in the sense of [18]).

Section 3 is briefer, and concerns related results for $R\langle X\rangle$. By definition, $R\langle X\rangle = R[X]_U$, where U is the set of monic polynomials in $R[X]$. The ring $R\langle X\rangle$ has received much attention because of its role in [30]; convenient references on $R\langle X\rangle$ are [2,26]. It is known [8] that $R\langle X\rangle$ is a Prüfer domain if and only if R is a Prüfer domain and $\dim(R) \leq 1$. We characterize when $R\langle X\rangle$ is a universally going-down domain, an i-domain, a going-down domain, or a treed domain. Many of the themes established in Section 2 carry over to $R\langle X\rangle$. One exception is given in Theorem 3.2: $R\langle X\rangle$ is treed if *and only if* $R\langle X\rangle$ is a going-down domain.

We shall let D' denote the integral closure of a domain D. It will be convenient to let D^+ and D^* be the seminormalization and the weak normalization of D, in the sense of [32] and [3], respectively. As in [25, p. 28], the properties of incomparability, going-up and going-down will be denoted by INC, GU and GD, respectively. If f is a polynomial, then $c(f)$ denotes the ideal generated by the coefficients of f. We assume familiarity with the material in [27, p. 707] on uppers and in [5, Chapters I–II] on flatness. Any unexplained material is standard, as in [21,25].

2. Results on $R(X)$

It is convenient to begin with six preliminary results. Only the first three of these, 2.1–2.3, are needed for the proof of our first main result, Theorem 2.7. The others, 2.4–2.6, are first needed in the proof of Corollary 2.9. In the interest of economy, we have included parallel treatment of $R\langle X\rangle$ in 2.5 and 2.6, but the reader may defer that discussion until beginning Section 3.

Lemma 2.1. *Let R be a domain. For each $r\in R$, let f_r be the R-algebra automorphism of $R[X]$ sending X to $X+r$. Then*:

(a) *Let $r\in R$ and $Q_1, Q_2 \in \mathrm{Spec}(R[X])$. Then $Q_1\subset Q_2$ if and only if $f_r(Q_1)\subset f_r(Q_2)$.*

(b) *If $r\in R$ and $P\in \mathrm{Spec}(R)$, then $f_r(P[X])=P[X]$.*

(c) *If $r\in R$ and $Q\in\mathrm{Spec}(R[X])$, then $f_r(Q)\cap R=Q\cap R$.*

(d) *If $P\in\mathrm{Spec}(R)$ and $P[X]$ contains an upper of 0, then $P[X]$ contains infinitely many uppers of 0.*

Proof. (a) Only the 'if' assertion needs attention. This follows since $f_{-r}(f_r(Q))=Q$ for each $Q\in\mathrm{Spec}(R[X])$.

(b) Since f_r preserves R elementwise, $P\subset f_r(P[X])$. Thus $P[X]\subset f_r(P[X])$ since $f_r(P[X])$ is an ideal of $R[X]$; similarly, $P[X]\subset f_{-r}(P[X])$. Applying f_r and using (a), we have $f_r(P[X])\subset f_r(f_{-r}(P[X]))=P[X]\subset f_r(P[X])$, yielding (b).

(c) $Q\cap R\subset f_r(Q)\cap R$ since f_r fixes R elementwise. Similarly, $f_r(Q)\cap R\subset f_{-r}(f_r(Q))\cap R=Q\cap R$, yielding (c).

(d) $P[X]$ contains Q, an upper of 0. Thus $P\neq 0$, R is not a field, and R is infinite. Note, via (b) and (c), that $f_s(Q)$ is an upper of 0 for each $s\in R$; moreover, by (a) and (b), $f_s(Q)\subset P[X]$. Thus it suffices to show, for each $r\in R$, that $f_r(Q)=f_s(Q)$ for only finitely many $s\in R$. Choose nonzero $g\in Q$ of minimal degree. Suppose $f_r(Q)=f_s(Q)$. It suffices to show that $g(r)=g(s)$, since the (nonzero) polynomial $g-g(r)$ has only finitely many roots. Now, since f_r is degree-preserving, it is easy to see that $f_r(g)=g(X+r)$ is of minimal degree in $f_r(Q)$; similarly, $g(X+s)$ is of minimal degree in $f_s(Q)$. Since $g(X+r)$ and $g(X+s)$ have the same leading term and $f_r(Q)=f_s(Q)$, it follows from minimality that $g(X+r)-g(X+s)=0$. Applying the R-algebra homomorphism $R[X]\to R$ that sends X to 0, we find $g(r)-g(s)=0$; i.e., $g(r)=g(s)$, as desired. □

Proposition 2.2. *Let R be a domain such that $R(X)$ is treed. Then*:

(a) *R is treed.*

(b) *R is a strong S-domain.*

(c) *The canonical map $h:\mathrm{Spec}(R(X))\to\mathrm{Spec}(R)$ is a bijection.*

Proof. (a) If a maximal ideal M of R contained incomparable primes P_1 and P_2, then $MR(X)$ would contain incomparable primes $P_1R(X)$ and $P_2R(X)$. In other words, if R were not treed, then neither would $R(X)$ be.

(b) Deny. Then there exist adjacent primes $P_1\subset P_2$ of R and a prime Q of $R[X]$ strictly between $P_1[X]$ and $P_2[X]$. Note that Q is an upper of P_1. Thus, for polynomials over the domain R/P_1, we have that $\bar{Q}=Q/P_1[X]$ is an upper of 0 contained in $\bar{P}_2[X]$, where $\bar{P}_2=P_2/P_1$. By Lemma 2.1(d), $\bar{P}_2[X]$ contains infinitely many (incomparable) uppers of 0. Hence $P_2[X]$ in $R[X]$ contains infinitely many incomparable primes (in fact, uppers of P_1). Passing to $R(X)$, we find $P_2R(X)$ containing infinitely many incomparable primes, contradicting the hypothesis that $R(X)$ is treed.

(c) For each $P\in \mathrm{Spec}(R)$, $PR(X)\in \mathrm{Spec}(R(X))$ and $h(PR(X))=P$. If the assertion fails, there exist $P\in \mathrm{Spec}(R)$ and $W\in \mathrm{Spec}(R(X))$ such that $h(W)=P$ and $W\neq PR(X)$. Then $q=W\cap R[X]$ is an upper of P. Choose a maximal ideal $MR(X)$ of $R(X)$ containing W (for a suitable maximal ideal M of R). By analyzing $P[X]\subset q\subset M[X]$ as we did for $P_1[X]\subset Q\subset P_2[X]$ in the proof of (b), we find the desired contradiction. □

We next show that condition (c) in Proposition 2.2 descends under integrality.

Lemma 2.3. *Let $R\subset T$ be an integral extension of domains. If the canonical map $h_T:\mathrm{Spec}(T(X))\to \mathrm{Spec}(T)$ is a bijection, then $h_R:\mathrm{Spec}(R(X))\to \mathrm{Spec}(R)$ is also a bijection.*

Proof. Deny. As in the proof of Proposition 2.2(c), we have

$$P[X]\subset N_1\subset M[X]\subset N_2$$

where P is some prime of R, N_1 is an upper of P, M is a maximal ideal of R, and N_2 is an upper of M. By integrality, $R[X]\subset T[X]$ satisfies GU and so there is a chain $Q_1\subset Q_2\subset Q_3\subset Q_4$ of primes in $T[X]$ which lies over the displayed chain in $R[X]$. Thus $Q_1\cap R=P=Q_2\cap R$, and so $(Q_1\cap T)\cap R=Q_2\cap T\cap R$. Since $R\subset T$ satisfies INC, $Q_1\cap T=Q_2\cap T$. In other words, Q_2 is an upper. Similarly, Q_4 is an upper and $Q_3=N[X]$ for some maximal ideal N of T. Put $q=Q_2\cap T$. Then $h_T(qT(X))=q=Q_2\cap T=h_T(Q_2T(X))$. As h_T is an injection, $qT(X)=Q_2T(X)$. Thus, intersecting with $T[X]$, we have $q[X]=Q_2$, contradicting the fact that Q_2 is an upper. □

The next three results concern the behavior of $R(X)$ and $R\langle X\rangle$ under semi- (or weak) normalization. Note that the proof of Lemma 2.4 carries over for reduced rings (provided that the total quotient rings of R and T compare).

Lemma 2.4. (a) *If R is a domain, then R^+ is the intersection of the seminormal overrings of R.*

(b) *If $R\subset T$ is an extension of domains, then $R^+\subset T^+$.*

Proof. (a) Let S be the intersection of the seminormal overrings of R. Then $S\subset R^+$, since R^+ is a seminormal overring of R. (For the reduced case, cf. [31, Theorem 2.5, Lemma 2.4, Lemma 2.3].) For the reverse inclusion, it suffices to show that $R^+\subset D$ for each seminormal overring D of R. This, in turn, follows via [31, Corollary 4.2], since $R^+={}_D^+R\subset D$.

(b) View R^+ and T^+ via (a). The assertion is now an easy consequence of the following observation. If D is a seminormal overring of T and K is the quotient field of R, then $D \cap K$ is a seminormal overring of R. □

Lemma 2.5. *Let $R \subset T$ be an integral extension of domains. Let U be the set of monic polynomials in $R[X]$. Let $U' = \{f \in R[X]: c(f) = R\}$ and $V' = \{f \in T[X]: c(f) = T\}$. Then:*

(a) *V' is the saturation of U' in $T[X]$. Hence $T[X]_{U'} = T(X)$.*

(b) *$T[X]_U = T\langle X\rangle$.*

Proof. (a) The first assertion was established in the proof of [23, Theorem 3(a)]. Hence $T[X]_U = T[X]_{V'}$ which, by definition, is $T(X)$.

(b) Let V be the set of all monic polynomials in $T[X]$. It was shown in the proof of [10, Lemma 1] (even in the presence of zero-divisors) that $T[X]_U = T[X]_V$ which, by definition, is $T\langle X\rangle$. □

Proposition 2.6. *Let R be a domain. Then:*

(a) *$R(X)' = R'(X)$ and $R\langle X\rangle' = R'\langle X\rangle$.*

(b) *$R(X)^+ = R^+(X)$ and $R\langle X\rangle^+ = R^+\langle X\rangle$.*

(c) *$R(X)^* = R^*(X)$ and $R\langle X\rangle^* = R^*\langle X\rangle$.*

Proof. (a) With $T = R'$, let U' and V' be as in Lemma 2.5. Then

$$R(X)' = (R[X]_{U'})' = (R[X]')_{U'} = R'[X]_{U'} = R'(X).$$

Indeed, the first equation holds by definition of the Nagata ring; the second, because integral closure commutes with localization; the third, because $R[X]' = R'[X]$; and the fourth, by Lemma 2.5(a). The second assertion admits a parallel proof, with U replacing U' and Lemma 2.5(b) replacing Lemma 2.5(a).

(b) This is proved as in (a), with T now taken as R^+. Use the following documentation. Seminormalization commutes with localization (cf. [31, Proposition 2.9]); and $R[X]^+ = R^+[X]$ (a consequence of [22, Theorem 1.6] and Lemma 2.4(b)).

(c) Argue as above, with T now taken as R^*. The documentation for (c) follows from the universality of weak normalization (cf. [3, Teorema 1]). Alternatively, note that weak normalization commutes with localization (cf. [33, Corollary of Proposition 2]); and a polynomial ring over a weakly normal domain is weakly normal (verify the criterion in [33, Theorem 1] by reasoning as in the proof of [9, Theorem 1]). □

We may now present our first main result, giving characterizations of condition (c) in Proposition 2.2.

Theorem 2.7. *Let R be a domain and $h = h_R: \mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ the canonical map. Then the following conditions are equivalent:*

(1) *h is a bijection*;
(2) *h is an injection*;
(3) *h is an isomorphism of posets* (*with respect to inclusion*);
(4) *h is a homeomorphism* (*with respect to the Zariski topology*);
(5) *If* $P \in \mathrm{Spec}(R)$ *and M is a maximal ideal of R, then no upper of P is contained in* $M[X]$;
(6) *If M is a maximal ideal of R, then no upper of* 0 *is contained in* $M[X]$;
(7) R' *is a Prüfer domain.*

Proof. As we recalled in the proof of Proposition 2.2(c), h is surjective, with $h(PR(X)) = P$ for each $P \in \mathrm{Spec}(R)$. Hence, (1) ⇔ (2). Since h is inclusion-preserving, we also see that (1) ⇔ (3). Moreover, (4) ⇒ (3) trivially; and, since h is a spectral map, [24, Proposition 15] assures that (3) ⇒ (4).

Next, one may establish (the contrapositive of) (5) ⇒ (2) as in the proofs of Proposition 2.2(c) and Lemma 2.3. Conversely, to establish (the contrapositive of) (2) ⇒ (5), note that if $Q_1 = P[X] \subset Q_2 \subset M[X]$ for some upper Q_2 of P, then $h(Q_1R(X)) = P = h(Q_2R(X))$, so that h is not injective.

It now suffices to show that (5), (6), and (7) are equivalent. Given (7), we see easily that $h_{R'} : \mathrm{Spec}(R'(X)) \to \mathrm{Spec}(R')$ is a bijection (cf. [21, Theorems 32.10, 32.15, and 33.4]), and so an appeal to Lemma 2.3 yields (5). Thus, (7) ⇒ (5). Moreover, (5) ⇒ (6) trivially.

Finally, we shall establish (the contrapositive of) (6) ⇒ (7). Assuming (7) fails, we have (cf. [29, Proposition 2.26]) that $R \subset R[u]$ does not satisfy INC for some u in K, the quotient field of R. Hence, by [14, Theorem 2.3], there exist distinct primes $q_1 \subset q_2$ of $R[u]$ and a *maximal* ideal M of R such that $q_i \cap R = M$. Let $e : R[X] \to K$ be the R-algebra homomorphism sending X to u. Put $Q_i = e^{-1}(q_i)$ and $Q = \ker(e)$. As $Q_1 \subset Q_2$ are distinct primes of $R[X]$ which each lie over M, it follows that $Q_1 = M[X]$. Since Q_1 contains Q, which is an upper of 0, (6) fails, as desired. □

Since each Prüfer domain is integrally closed, Theorem 2.7 immediately yields the following result.

Corollary 2.8. *A domain R is a Prüfer domain if and only if R is integrally closed and the canonical map* $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ *is a bijection.*

Corollary 2.9. *Let R be a domain. For each nonnegative integer n, let* $h_n : \mathrm{Spec}(R(X_1, \ldots, X_{n+1})) \to \mathrm{Spec}(R(X_1, \ldots, X_n))$ *and* $g_n : \mathrm{Spec}(R(X_1, \ldots, X_{n+1})) \to \mathrm{Spec}(R)$ *denote the canonical maps. Then the following conditions are equivalent*:
(1) h_n *is a bijection for each* $n \geq 0$;
(2) h_n *is an injection for some* $n \geq 0$;
(3) g_n *is a bijection for each* $n \geq 0$;
(4) g_n *is an injection for some* $n \geq 0$;
(5) R' *is a Prüfer domain.*

Proof. $h_0 = g_0 = h_R : \mathrm{Spec}(R(X_1)) \to \mathrm{Spec}(R)$ is surjective. Moreover, $g_n = h_0 \circ \cdots \circ h_{n-1} \circ h_n$ for each $n \geq 1$. Now, $R(X_1, \ldots, X_{d+1}) = R(X_1, \ldots, X_d)(X_{d+1})$ for each d [1, Lemma], and so each h_i is surjective. Thus, each g_i is surjective. Both (1) ⇒ (3) and (4) ⇒ (2) now follow easily. Also, (3) ⇒ (4) trivially. Thus, it suffices to show that (2) ⇒ (5) ⇒ (1).

Let $n \geq 1$ and put $S = R(X_1, \ldots, X_{n-1})$. We have the following chain of equivalences:

h_n is an injection

⇔ h_n is a bijection (by Theorem 2.7)

⇔ $R(X_1, \ldots, X_n)'$ is a Prüfer domain (by Theorem 2.7)

⇔ $S(X_n)'$ is a Prüfer domain (by the result recalled from [1] above)

⇔ $S'(X_n)$ is a Prüfer domain (by Proposition 2.6(a))

⇔ S' is a Prüfer domain (by [21, Theorem 33.4])

⇔ h_{n-1} is an injection (as above)

⇔ h_{n-1} is a surjection (as above).

By iterating the above argument and appealing to Theorem 2.7, we find that (2) ⇔ (5) ⇔ (1). □

The next main result answers the introduction's motivating question.

Theorem 2.10. *For a domain R, the following conditions are equivalent*:

(1) *$R(X)$ is treed*;

(2) *R is treed and the canonical map* $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ *is a bijection*;

(3) *R is treed and R' is a Prüfer domain*;

(4) *$R(X_1, \ldots, X_n)$ is treed for some $n \geq 1$*;

(5) *$R(X_1, \ldots, X_n)$ is treed for each $n \geq 1$.*

Proof. (1) ⇒ (2) by Proposition 2.2(a), (c).

(2) ⇒ (1). Deny. Then some maximal ideal M of R is such that $MR(X)$ contains incomparable primes Q_1, Q_2 of $R(X)$. Put $q_i = Q_i \cap R[X]$. As $q_i \subset M[X]$, it follows from condition (5) in Theorem 2.7 that $q_i = P_i[X]$ for some $P_i \in \mathrm{Spec}(R)$. Since $Q_i = q_i R(X) = P_i R(X)$, it follows that P_1 and P_2 are incomparable. As $P_i = q_i \cap R \subset MR(X) \cap R = M$, this contradicts the hypothesis that R is treed.

(2) ⇔ (3). Invoke Theorem 2.7.

Thus, (1) ⇔ (2) ⇔ (3).

Before completing the proof, we note one upshot of the proof of Corollary 2.9. To wit: R' is a Prüfer domain ⇔ $R(X_1, \ldots, X_d)'$ is a Prüfer domain for some $d \geq 1$ ⇔ $R(X_1, \ldots, X_d)'$ is a Prüfer domain for each $d \geq 1$.

In order to complete the proof, it now suffices to establish the following assertion. If $n \geq 2$, then $R(X_1, \ldots, X_n)$ is treed if and only if $S = R(X_1, \ldots, X_{n-1})$ is treed and R' is a Prüfer domain. Now, by the one-variable case established above, $R(X_1, \ldots, X_n) = S(X_n)$ is treed if and only if S is treed and S' is a Prüfer domain. But, by the above 'upshot', S' is a Prüfer domain if and only if R' is a Prüfer domain. □

Remark 2.11. The conditions mentioned in Theorem 2.10(3) are independent. First, to see that "R is treed" does not imply "R' is a Prüfer domain", consider an (one-dimensional, integrally closed) example such as $\mathbb{Q} + X\mathbb{Q}(Y)[[X]]$. Secondly, [29, Example 2.28] presents a two-dimensional example showing that "R' is a Prüfer domain" does not imply "R is treed".

Following [12], we say that a domain R is a *going-down domain* in case $R \subset T$ satisfies GD for each domain T containing R; according to [20, Theorem 1], attention may be restricted to valuation overrings T of R. Each Prüfer domain is a going-down domain, as is each one-dimensional domain. Each going-down domain is treed [12, Theorem 2.2]; according to an example of W.J. Lewis (cf. [16, Remark 2.1(a)]), the converse is false. We next present a 'going-down' analogue of Theorem 2.10.

Corollary 2.12. *For a domain R, the following conditions are equivalent*:

(1) *$R(X)$ is a going-down domain*;

(2) *R is a going-down domain and the canonical map* $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ *is a bijection*;

(3) *R is a going-down domain and R' is a Prüfer domain*;

(4) *$R(X_1, \ldots, X_n)$ is a going-down domain for some $n \geq 1$*;

(5) *$R(X_1, \ldots, X_n)$ is a going-down domain for each $n \geq 1$.*

Proof. (2) ⇔ (3) follows directly from Theorem 2.7. Next, recall that each going-down domain is treed. To show that (1) ⇒ (2), it suffices (by Theorem 2.10) to show that if $R(X)$ is a going-down domain, then so is R. Consider a valuation overring V of R, prime ideals $P \subset M$ of R, and a prime N of V such that $N \cap R = M$. Let V^* be the trivial extension of V (cf. [21, p. 401]), viewed as usual as a valuation overring of $R(X)$. Let I be *the* prime of V^* such that $I \cap V = N$. Note that $I \cap R(X) = MR(X)$ since $(I \cap R(X)) \cap R = M$ and $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ is injective. As $R(X) \subset V^*$ satisfies GD, I contains a prime J of V^* such that $J \cap R(X) = PR(X)$. Hence $Q = J \cap V$ is a prime of V such that $Q \subset N$ and $Q \cap R = P$. It follows that $R \subset V$ satisfies GD, and so R is a going-down domain. Hence, (1) ⇒ (2).

Next, we assume (2) and shall derive (1). By Theorem 2.7, 'test data' consist of primes $PR(X) \subset MR(X)$ of $R(X)$, arising from primes $P \subset M$ of R, and a valuation overring (W, I) of $R(X)$ such that $I \cap R(X) = MR(X)$. Consider $V = W \cap K$, where K is the quotient field of R. Then V is a valuation (hence Prüfer) overring of R, with maximal ideal $N = I \cap V$. Note that $N \cap R = M$. As $R \subset V$ satisfied GD, N contains a prime Q of V such that $Q \cap R = P$. Now, since V is a going-down domain, $V \subset W$ satisfies GD, producing a prime J of W such that $J \subset I$ and $J \cap V = Q$. Observe that $J \cap R(X) = PR(X)$ since $(J \cap R(X)) \cap R = P$ and $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ is injective. Hence $R(X) \subset W$ satisfies GD, and so $R(X)$ is a going-down domain. It follows that (2) $\Rightarrow$ (1). Thus, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

The equivalences involving (4) and (5) may now be established by repeating the last paragraph of the proof of Theorem 2.10, with 'treed' being replaced by 'a going-down domain'. □

Examples going back to Krull show that an integrally closed treed domain need not be a Prüfer domain. (In fact, the first ring noted in Remark 2.11 illustrates this too.) The next result shows that no such example is a Nagata ring.

Corollary 2.13. *For a domain R, the following conditions are equivalent*:

(1) *$R(X)$ is integrally closed and treed*;
(2) *$R(X)$ is an integrally closed going-down domain*;
(3) *R is integrally closed and $R(X)$ is treed*;
(4) *R is integrally closed and $R(X)$ is a going-down domain*;
(5) *R is a Prüfer domain.*

Proof. It is well known (and easy to see) that $R(X) \cap K = R$, where K is the quotient field of R. Hence, if $R(X)$ is integrally closed, so is R. The converse also holds: just apply Proposition 2.6(a). In other words, $R(X)$ is integrally closed if and only if R is integrally closed. Thus, (1) $\Leftrightarrow$ (3); and (2) $\Leftrightarrow$ (4). By Corollary 2.12, (5) $\Rightarrow$ (4). Moreover, since going-down domains are treed, (4) $\Rightarrow$ (3). It suffices now to establish that (3) $\Rightarrow$ (5), and an appeal to Theorem 2.10 accomplishes this. □

The next result gives another characterization of treed Nagata rings in the context of going-down domains. Following [6], we say that a domain R is *catenarian* if, for each pair $P \subset Q$ of prime ideals of R, all saturated chains of prime ideals from P to Q have a common finite length. If R is a Cohen–Macaulay domain, then $R[X_1, \ldots, X_n]$ is catenarian for each $n \geq 0$. If $R[X]$ is catenarian, then R is a strong S-domain [6, Lemma 2.3]. Note that if R is catenarian, then R is locally finite-dimensional (or LFD, for short), in the sense that each maximal ideal of R has finite height.

Corollary 2.14. *Let R be an LFD going-down domain. Then the following conditions are equivalent*:

(1) *$R[X]$ is catenarian*;
(2) *$R[X_1, \ldots, X_n]$ is catenarian for each $n \geq 1$*;
(3) *R is a strong S-domain*;
(4) *R' is a Prüfer domain*;
(5) *$R(X)$ is a going-down domain*;
(6) *$R(X)$ is treed.*

Proof. Using INC, one sees easily that R' is LFD. Hence, (2) $\Leftrightarrow$ (4) follows from [6, Theorem 6.2]. Trivially, (2) $\Rightarrow$ (1); and we recalled above that (1) $\Rightarrow$ (3). Moreover, (3) ⇔ (2) by [7, Theorem 1]. Also, (4) ⇔ (5) by Corollary 1.12. Finally, since going-down domains are treed, Theorem 2.10 yields (4) ⇔ (6). □

Remark 2.15. (a) In the absence of a 'going-down' hypothesis, the conditions in Corollary 2.14 are not equivalent. Indeed, consider a local Cohen–Macaulay domain R of dimension at least 2. Then R is LFD and $R[X_1,\dots,X_n]$ is catenarian for each $n\geq 1$. However, $R(X)$ is not treed, by Proposition 2.2(a), since a treed Noetherian domain has dimension ≤ 1 (cf. [25, Theorem 144]).

(b) Following [29], we say that a domain R is an *open domain* in case the canonical continuous map $\mathrm{Spec}(T)\rightarrow\mathrm{Spec}(R)$ is open for each domain T containing R. Each open domain is a going-down domain, but the converse is false (cf. [29, Theorem 3.16]). We can now state another result in the spirit of Theorem 2.10 and Corollary 2.12. For a domain R, the following five conditions are equivalent:

(1) $R(X)$ is an open domain;

(2) R is an open domain and the canonical map $\mathrm{Spec}(R(X))\rightarrow\mathrm{Spec}(R)$ is a bijection;

(3) R is an open domain and R' is a Prüfer domain;

(4) $R(X_1,\dots,X_n)$ is an open domain for some $n\geq 1$;

(5) $R(X_1,\dots,X_n)$ is an open domain for each $n\geq 1$.

To obtain a proof, one can show (1) ⇔ (2) ⇔ (3) by combining the characterization of open domains in [29, Theorem 3.16] with Corollary 2.12 and Theorem 2.7; and then establish the equivalences involving (4) and (5) as in Theorem 2.10, mutatis mutandis.

Before we develop additional results in the spirit of Theorem 2.10, it is convenient to recall the following material. As in [19], a domain R is said to be a *universally going-down domain* in case the inclusion map $R\rightarrow T$ is a universally going-down homomorphism for each domain T containing R. Each Prüfer domain is a universally going-down domain, but not conversely [19, Remark 2.5(a)]. In fact, a domain R is a universally going-down domain if and only if R^* is a Prüfer domain (cf. [19, Theorem 2.4]).

Following [29], we say that an extension $R\subset T$ of domains is an *i-extension* in case the canonical map $\mathrm{Spec}(T)\rightarrow\mathrm{Spec}(R)$ is an injection. A domain R is called an *i-domain* if $R\subset T$ is an *i*-extension for each overring T of R. Each universally going-down domain is an *i*-domain [19, Proposition 2.2(b)], but not conversely [19, Remark 2.5(c)]. Each *i*-domain is a going-down domain (cf. [13, Corollary 2.3; 29, Corollary 2.13]), but not conversely (cf. [29, Proposition 2.23]). A domain R is an *i*-domain if and only if R' is a Prüfer domain and $R\subset R'$ is an *i*-extension [29, Proposition 2.14]; equivalently, if and only if R'_M is a valuation domain for each maximal ideal M of R (cf. [29, Corollary 2.15]).

Corollary 2.16. *Let R be a domain. Then*:

(a) *$R(X)$ is a universally going-down domain if and only if R is a universally going-down domain.*

(b) *$R(X)$ is an i-domain if and only if R is an i-domain.*

Proof. (a) We have the following equivalences:

$R(X)$ is a universally going-down domain
$\Leftrightarrow$ $R(X)^*$ is a Prüfer domain (by [19, Theorem 2.4])
$\Leftrightarrow$ $R^*(X)$ is a Prüfer domain (by Proposition 2.6(c))
$\Leftrightarrow$ R^* is a Prüfer domain
$\Leftrightarrow$ R is a universally going-down domain.

(b) We noted in the proof of Theorem 2.10 that $R(X)'$ is a Prüfer domain if and only if R' is a Prüfer domain. Moreover, if R' is a Prüfer domain, then Theorem 2.7 shows that $R \subset R(X)$ is an i-extension. This in turn implies that if $Q \in \mathrm{Spec}(R')$, then $QR'(X) \cap R(X) = (Q \cap R)R(X)$. Thus, if R' is a Prüfer domain, then $R(X) \subset R(X)' = R'(X)$ is an i-extension if and only if $R \subset R'$ is an i-extension. We therefore have the following equivalences: $R(X)$ is an i-domain $\Leftrightarrow$ $R(X)'$ is a Prüfer domain and $R(X) \subset R(X)'$ is an i-extension $\Leftrightarrow$ R' is a Prüfer domain and $R \subset R'$ is an i-extension $\Leftrightarrow$ R is an i-domain. □

Remark 2.17. The interested reader can use the criteria recalled above to develop several other proofs of Corollary 2.16(b). We next sketch one such proof. It is known that $R(X)_{MR(X)} = R_M(X)$ for each maximal ideal M of R (cf. [11, Proposition 1(3)]). But $R(X)'_{MR(X)} = R'_M(X)$ is a valuation domain if and only if R'_M is a valuation domain (cf. [21, Theorem 33.4 and Proposition 33.1(3)]). Universally quantifying over M completes the proof.

The material recalled above includes the implications

$$\text{Prüfer domain} \Rightarrow \text{universally going-down domain}$$
$$\Rightarrow i\text{-domain} \Rightarrow \text{going-down domain} \Rightarrow \text{treed domain};$$

and the fact that none of these implications is reversible in general. We next show that these implications' converses remain false even when attention is restricted to Nagata rings.

Examples 2.18. (a) A universally going-down domain of the form $R(X)$ need not be a Prüfer domain: by Corollary 2.6(a) and [21, Theorem 33.4], it suffices to take R to be a universally going-down non-Prüfer domain.

(b) Similarly, Corollary 2.16 leads to an i-domain of the form $R(X)$ which is not a universally going-down domain.

(c) Next, recall that [29, Example 2.17] constructs a one-dimensional Noetherian domain R such that $R \subset R'$ is not an i-extension. By Corollary 2.16(b), $R(X)$ is not an i-domain. However, according to Corollary 2.12, $R(X)$ is a going-down domain, since R is a going-down domain and R' (being a Dedekind domain) is a Prüfer domain.

(d) Finally, we produce a domain R such that $R(X)$ is a treed domain which is not a going-down domain. According to Theorem 2.10 and Corollary 2.12, this means finding a treed domain R such that R' is a Prüfer domain and R is not a going-down domain. A two-dimensional domain R with these properties is developed in [16, Example 2.3].

We close this section with some material on completions. As in [21, section 32], if R is an integrally closed domain, then the completion R^{b} is the minimum Kronecker function ring of R. Part of the next (easy) result was implicit in the proof that (7) ⇒ (5) in Theorem 2.7.

Proposition 2.19. *For a domain R, the following conditions are equivalent:*

(1) *R is integrally closed and the canonical map $g:\mathrm{Spec}(R^{\mathrm{b}})\to\mathrm{Spec}(R)$ is a bijection;*

(2) *R is a Prüfer domain.*

Proof. As in [17], if D is a domain, we let $X(D)$ denote the set (actually, spectral topological space) of all valuation overrings of D. According to the natural transformation established in [17, Corollary 4.5(b)], there is a commutative diagram

$$\begin{array}{ccc} X(R^{\mathrm{b}}) & \xrightarrow{F} & \mathrm{Spec}(R^{\mathrm{b}}) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ X(R) & \xrightarrow{f} & \mathrm{Spec}(R) \end{array}$$

(We have assumed, without loss of generality, that R is integrally closed. F (resp., f) sends a valuation domain to its center on R^{b} (resp., R).) Now, G is an isomorphism (cf. [21, p. 404]); and F is an isomorphism since R^{b} is a Prüfer domain. Hence, g is an isomorphism if and only if f is an isomorphism. Thus, by [17, Proposition 2.2], (1) holds if and only if R is a Prüfer domain. □

A result of Arnold and Brewer [4] states that an integrally closed domain R is a Prüfer domain if and only if R^{b} is a flat $R[X]$-module. Our next result recovers, and sharpens, this assertion.

Theorem 2.20. *Let R be an integrally closed domain. Then the following conditions are equivalent:*

(1) *The inclusion map $R\to R^{\mathrm{b}}$ is a universally going-down homomorphism;*

(2) *R^{b} is a flat R-module;*

(3) *R is a universally going-down domain;*

(4) *The inclusion map $R[X]\to R^{\mathrm{b}}$ is a universally going-down homomorphism;*

(5) *R^{b} is a flat $R[X]$-module;*
(6) *R is a Prüfer domain.*

Proof. (6) ⇔ (3) by [19, Theorem 2.4] since $R=R'$ forces $R^*=R$.

(3) ⇒ (1). Trivial.

(6) ⇒ (5). If R is a Prüfer domain, then $R^{\mathrm{b}}=R(X)$ [21, Theorem 33.4], which is a ring of fractions of, and hence flat over, $R[X]$.

(5) ⇒ (4). Any flat homomorphism is universally going-down (cf. [18, p. 418]).

(4) ⇒ (5). Any universally going-down overring extension of an integrally closed domain must be flat [15, Theorem 3].

(5) ⇒ (2). $R[X]$ is R-free (hence R-flat). Thus, if R^{b} is $R[X]$-flat, then R^{b} is R-flat.

(2) ⇒ (1). Flat implies universally going-down.

(1) ⇒ (3). Assume (1). According to [18, Corollary 2.3] and [19, Theorem 2.6], it suffices to show that $A=R[X_1,\ldots,X_n]\subset B=V[X_1,\ldots,X_n]$ satisfies GD for each valuation overring V of R and each positive integer n. Consider prime ideals $P\subset M$ of A and a prime N of B such that $N\cap A=M$. Let $D=V^*[X_1,\ldots,X_n]$, where V^* is the trivial extension of V. Since V^* is a faithfully-flat V-module, $D\cong V^*\otimes_V B$ is a faithfully flat B-module. In particular, there exists $I\in\mathrm{Spec}(D)$ such that $I\cap B=N$; hence, $I\cap A=M$.

Put $E=R^{\mathrm{b}}[X_1,\ldots,X_n]$. Since R^{b} is a Prüfer (hence, universally going-down) domain, $E\subset D$ satisfies GD. Moreover, $A\subset E\cong A\otimes_R R^{\mathrm{b}}$ satisfies GD since, by hypothesis, $R\to R^{\mathrm{b}}$ is universally going-down. Thus, $A\subset D$ satisfies GD. This produces $J\in\mathrm{Spec}(D)$ such that $J\subset I$ and $J\cap A=P$. Then $Q=J\cap B\in\mathrm{Spec}(B)$ satisfies $Q\subset N$ and $Q\cap A=P$. Hence, $A\subset B$ satisfies GD. □

By formally taking $n=0$ and deleting "universally" in the preceding proof that (1) ⇒ (3), we obtain a proof of the following assertion. An integrally closed domain R is a going-down domain if (and only if) $R\subset R^{\mathrm{b}}$ satisfies GD.

3. Results on $R\langle X\rangle$

This section discusses topics for $R\langle X\rangle$ (where R is a domain) analogous to those considered for $R(X)$ in Section 2. One striking difference in behavior is this: the canonical map $\mathrm{Spec}(R\langle X\rangle)\to\mathrm{Spec}(R)$ is injective (if and) only if R is a field. Indeed, if $r\in R$ is a nonzero nonunit, then $(rX-1)R\langle X\rangle$ is a prime of $R\langle X\rangle$ that meets R in 0 (cf. [5, Exercise 15(a), p. 549]). Another way in which $R\langle X\rangle$ behaves differently from $R(X)$ is given in Theorem 3.2: $R\langle X\rangle$ is treed if *and only if* $R\langle X\rangle$ is a going-down domain. (Contrast this with Examples 2.18(d).) First, we record some useful material from [8]. Note that Lemma 3.1(a) is due independently to LeRiche [26, Theorem 2.1].

Lemma 3.1. (Brewer–Costa [8, Lemma 1, Theorem 1(i)]). *Let R be a domain. Then:*

(a) $\dim(R\langle X\rangle)=\dim(R[X])-1$.

(b) *$R\langle X\rangle$ is a Prüfer domain if and only if R is a Prüfer domain and* $\dim(R)\leq 1$.

Theorem 3.2. *For a domain R, the following conditions are equivalent:*

(1) *$R\langle X\rangle$ is treed;*

(2) *$R\langle X\rangle$ is a going-down domain;*

(3) $\dim(R\langle X\rangle)\leq 1$;

(4) $\dim(R[X])\leq 2$;

(5) *R' is a Prüfer domain and* $\dim(R)\leq 1$.

Proof. (4) $\Leftrightarrow$ (3). Invoke Lemma 3.1(a). Moreover, (3) $\Rightarrow$ (2) $\Rightarrow$ (1) on general principles.

(1) $\Rightarrow$ (5). Suppose that $R\langle X\rangle$ is treed. Since $R\subset R\langle X\rangle\subset R(X)$, we see that $R(X)$ is a ring of fractions of $R\langle X\rangle$, and so $R(X)$ is treed. Then, by Theorem 2.10, R' is a Prüfer domain. It remains to show that $\dim(R)\leq 1$. Deny, and choose nonzero prime ideals $P\subset Q$ of R. Pick $r\in Q\setminus P$. By [5, Exercise 15(a), p. 549], $P_1=(rX-1)\in \operatorname{Spec}(R[X])$. Similarly, $N=P[X]+P_1\in\operatorname{Spec}(R[X])$, since $R[X]/N\cong (R/P)[X]/(\bar{r}X-1)$. Observe that P_1 and $P[X]$ are incomparable primes of $R[X]$ which are both contained in N. This will contradict the fact that $R\langle X\rangle$ is treed, provided we show that N survives in $R\langle X\rangle$. But if survival fails, some monic polynomial is in N, whence $1\in P+Rr\subset Q$, a contradiction.

(5) $\Rightarrow$ (4). Assuming (5) and appealing to [21, Proposition 30.14], we have $\dim(R[X])=\dim(R'[X])=\dim(R')+1=\dim(R)+1\leq 2$. □

We next state an analogue of Corollary 2.13. Its straightforward proof may be fashioned from Lemma 3.1(b), Theorem 3.2, and the fact that $R\langle X\rangle$ integrally closed implies R integrally closed.

Corollary 3.3. *For a domain R, the following conditions are equivalent:*

(1) *$R\langle X\rangle$ is integrally closed and treed;*

(2) *$R\langle X\rangle$ is an integrally closed going-down domain;*

(3) *$R\langle X\rangle$ is a Prüfer domain;*

(4) *R is a Prüfer domain and* $\dim(R)\leq 1$.

The next result is an analogue of Corollary 2.16.

Corollary 3.4. *Let R be a domain. Then:*

(a) *$R\langle X\rangle$ is a universally going-down domain if and only if R is a universally going-down domain and* $\dim(R)\leq 1$;

(b) *$R\langle X\rangle$ is an i-domain if and only if R is an i-domain and* $\dim(R)\leq 1$.

Proof (Sketch). The proof is analogous to that of Corollary 2.16. In the proof of (b), one must note if $\dim(R)\leq 1$ and R' is a Prüfer domain, then $R\langle X\rangle\subset R'\langle X\rangle$ is an i-extension. This follows essentially because each upper of 0 in $R[X]$ is lain over by only one upper of 0 in $R'[X]$ (cf. [27, Theorem 2(i)]). □

In the spirit of Remark 2.17, we note that another proof of Corollary 3.4(b) may be built by observing that $R\langle X\rangle_{MR\langle X\rangle}=R_M(X)$ for each maximal ideal M of R. Details are left to the interested reader.

We saw in Theorem 3.2 that $R\langle X\rangle$ does not behave as Examples 2.18(d) might lead us to expect. We show next that the rest of Examples 2.18 *does* carry over to $R\langle X\rangle$.

Examples 3.5. (a) A universally going-down domain of the form $R\langle X\rangle$ need not be a Prüfer domain: by Corollary 3.4(a) and Lemma 3.1(b), it suffices to take R to be a one-dimensional universally going-down domain, as in [19, Remark 2.5(a)].

(b) Similarly, Corollary 3.4 shows that the domain $R=\mathbb{R}+X\mathbb{C}[[X]]$ in [19, Remark 2.5(c)] is such that $R\langle X\rangle$ is an i-domain but not a universally going-down domain.

(c) Finally, a going-down domain of the form $R\langle X\rangle$ need not be an i-domain. Indeed, by Theorem 3.2 and Corollary 3.4(b), the ring in Examples 2.18(c) is such an R. □

We close by pursuing additional analogies with Section 2.

Remark 3.6. (a) Let R be a domain. If $R\langle X\rangle$ is treed, then $R[X]$ is catenarian. (This may be regarded as an analogue of the implication (6) ⇒ (1) in Corollary 2.14.) Indeed, under the given hypothesis, it follows from Theorem 3.2 that $R[X]$ has dimension at most 2, and so is trivially catenarian.

The converse, however, fails. To see this, let R be any valuation domain of finite dimension at least 2. Then $R[X]$ is catenarian by Corollary 2.14, but Theorem 3.2 shows that $R\langle X\rangle$ is not treed.

(b) The analogue of Remark 2.15(b) asserts the following. If R is a domain, then $R\langle X\rangle$ is an open domain (if and) only if R is a field. The proof uses Theorem 3.2; the result that all open domains are semiquasilocal [29, Theorem 3.16]; our earlier observation that $(rX-1)\in \operatorname{Spec}(R\langle X\rangle)$ for each nonzero nonunit $r\in R$; and the fact that any domain with only finitely many nonunits is a field.

References

[1] D.D. Anderson, Some remarks on the ring $R(X)$, Comment. Math. Univ. St. Pauli 26 (1977) 137–140.

[2] D.D. Anderson, D.F. Anderson and R. Markanda, The rings $R(X)$ and $R\langle X\rangle$, J. Algebra 95 (1985) 96–115.

[3] A. Andreotti and E. Bombieri, Sugi omeomorfismi delle varietà algebriche, Ann. Scuola Norm. Sup. Pisa 23 (1969) 431–450.

[4] J.T. Arnold and J.W. Brewer, Kronecker function rings and flat $D[X]$-modules, Proc. Amer. Math. Soc. 27 (1971) 483–485.
[5] N. Bourbaki, Commutative Algebra (Addison-Wesley, Reading, MA, 1972).
[6] A Bouvier, D.E. Dobbs and M. Fontana, Universally catenarian integral domains, Adv. in Math., 72 (1988) 211–238.
[7] A Bouvier, D.E. Dobbs and M. Fontana, Two sufficient conditions for universal catenarity, Comm. Algebra 15 (1987) 861–872.
[8] J.W. Brewer and D.L. Costa, Projective modules over some non-Noetherian polynomial rings, J. Pure Appl. Algebra 13 (1978) 157–163.
[9] J.W. Brewer, D.L. Costa and K. McCrimmon, Seminormality and root closure in polynomial rings and algebraic curves, J. Algebra 58 (1979) 217–226.
[10] J.W. Brewer and W.J. Heinzer, R Noetherian implies $R\langle X\rangle$ is a Hilbert ring, J. Algebra 67 (1980) 204–209.
[11] J.W. Brewer, P.R. Montgomery, E.A. Rutter and W.J. Heinzer, Krull dimension of polynomial rings, Lecture Notes in Mathematics 311 (Springer, Berlin, 1972) 26–45.
[12] D.E. Dobbs, On going down for simple overrings II, Comm. Algebra 1 (1974) 439–458.
[13] D.E. Dobbs, Ascent and descent of going-down rings for integral extensions, Bull. Austral. Math. Soc. 15 (1976) 253–264.
[14] D.E. Dobbs, Lying-over pairs of commutative rings, Canad. J. Math. 33 (1981) 454–475.
[15] D.E. Dobbs, A note on universally going-down, Math. J. Okayama U. 30 (1988) 1–4.
[16] D.E. Dobbs, On treed overrings and going-down domains, Rend. Math. 7 (1987) 317–322.
[17] D.E. Dobbs, R. Fedder and M. Fontana, Abstract Riemann surfaces of integral domains and spectral spaces, Ann. Mat. Pura Appl. 148 (1987) 101–115.
[18] D.E. Dobbs and M. Fontana, Universally going-down homomorphisms of commutative rings, J. Algebra 90 (1984) 410–429.
[19] D.E. Dobbs and M. Fontana, Universally going-down integral domains, Arch. Math. 42 (1984) 426–429.
[20] D.E. Dobbs and I.J. Papick, On going-down for simple overrings III, Proc. Amer. Math. Soc. 54 (1976) 35–38.
[21] R. Gilmer, Multiplicative Ideal Theory (Dekker, New York, 1972).
[22] R. Gilmer and R.C. Heitmann, On Pic($R[X]$) for R seminormal, J. Pure Appl. Algebra 16 (1980) 251–257.
[23] R. Gilmer and J.F. Hoffmann, A characterization of Prüfer domains in terms of polynomials, Pacific J. Math. 60 (1975) 81–85.
[24] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969) 43–60.
[25] I. Kaplansky, Commutative Rings, rev. ed. (Univ. of Chicago Press, Chicago, 1974).
[26] L.R. LeRiche, The ring $R\langle X\rangle$, J. Algebra 67 (1980) 327–341.
[27] S. McAdam, Going down in polynomial rings, Canad. J. Math. 23 (1971) 704–711.
[28] M. Nagata, Local Rings (Interscience, New York, 1962).
[29] I.J. Papick, Topologically defined classes of going-down domains, Trans. Amer. Math. Soc. 219 (1976) 1–37.
[30] D. Quillen, Projective modules over polynomial rings, Invent. Math. 36 (1976) 167–171.
[31] R.G. Swan, On seminormality, J. Algebra 67 (1980) 219–229.
[32] C. Traverso, Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa 24 (1970) 585–595.
[33] H. Yanagihara, Some results on weakly normal ring extensions, J. Math. Soc. Japan 35 (1983) 649–661.