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Catenarity of formal power series rings over a pullback

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Abstract

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Let (T, M, K) be a quasilocal domain with maximal ideal M and residue field K, $\varphi : T \rightarrow K$ the natural surjection, and R the pullback $\varphi^{-1}(D)$, where D is a subring of K. It is shown that $R[[X]]$ is catenarian if and only if $T[[X]]$ and $D[[X]]$ are each catenarian. We also construct a non-Noetherian domain R such that $\dim(R) > 1$ and $R[[X_1, \ldots, X_n]]$ is catenarian for each integer $n \geq 1$. This work leads to the question of determining the field extensions $k \subset K$ such that $\operatorname{Spec}(K[[X_1, \ldots, X_n]]) \rightarrow \operatorname{Spec}(k[[X_1, \ldots, X_n]])$ is a homeomorphism for each integer $n \geq 1$. It is shown that any such extension must be purely inseparable; the converse holds if K is a finitely generated extension of k.

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1. Introduction

The rings considered in this paper are (commutative integral) domains, typically of finite Krull dimension. We denote the rings of polynomials and formal power series in n indeterminates over a ring R by $R[X_1, \ldots, X_n]$ and $R[[X_1, \ldots, X_n]]$, respectively. Following [9], we say a (not necessarily Noetherian) ring R is *catenarian* if for each pair $P \subset Q$ of prime ideals of R, every saturated chain of prime ideals of R between P and Q has the same finite length. It is well known (cf. [21]) that each Cohen–Macaulay ring R is *universally catenarian*, in the sense that $R[X_1, \ldots, X_n]$ is catenarian for each integer $n \geq 1$. Moving beyond the Noetherian context, one showed in [24, 20, 10] that each locally finite-dimensional (LFD) Prüfer domain is universally catenarian. Hopes for an analogous theory for formal power series rings were somewhat stimulated by the following result of Lequain [19] (where '$[X]]$' denotes either '$[X]$' or '$[[X]]$'): if R is a Noetherian ring, then $R[X_1]] \cdots [X_n]]$ is catenarian for each integer $n \geq 1$ if and only if $R[X_1]] \cdots [X_n]]$ is catenarian for some integer $n \geq 1$. Nevertheless, it was shown in [7, 23] that if R is an LFD Prüfer SFT-ring, then $R[[X_1, \ldots, X_n]]$ is catenarian if and only if either $n = 1$ or $\dim(R) = 1$. (Another positive result on catenarity of formal power series rings in one variable, over a globalized pseudo-valuation SFT-ring, appears in [16].) It thus seems reasonable to ask if there exists a non-Noetherian ring R with $\dim(R) > 1$ such that $R[[X_1, \ldots, X_n]]$ is catenarian for each integer $n \geq 1$. One consequence of our work is an affirmative answer to this question; see Example 3.6 below.

In this paper, we are principally interested in when $R[[X_1, \ldots, X_n]]$ is catenarian for a pullback R. Recall from [3] the corresponding facts about polynomial rings. Namely, if $T = K + M$ is a quasilocal domain and $R = D + M$, where D is a subring of K with quotient field k, then, if $k \subset K$ is algebraic and $T[X_1, \ldots, X_n]$ and $D[X_1, \ldots, X_n]$ are each catenarian, $R[X_1, \ldots, X_n]$ is catenarian. Moreover, the converse holds when $k = K$.

Let (T, M, K) be a quasilocal domain with maximal ideal M and residue field K, $\varphi : T \to K$ the natural surjection, and $R = \varphi^{-1}(D)$, where D is a subring of K. In Theorem 2.6, we show that $R[[X]]$ is catenarian if and only if $T[[X]]$ and $D[[X]]$ are each catenarian. Section 3 gives two examples to show that this equivalence need not hold when we pass to two or more indeterminates. However, suppose that $D = k$ is a subfield of K with $\operatorname{char}(k) = p > 0$ and that $k \subset K$ is an extension of *finite exponent*, in the sense that $K^{p^e} \subset k$ for some integer $e \geq 1$; then Corollary 3.5 establishes that, for each integer $n \geq 1$, $R[[X_1, \ldots, X_n]]$ is catenarian if and only if $T[[X_1, \ldots, X_n]]$ is catenarian. This leads, as promised, in Example 3.6 to a non-Noetherian ring R of dimension greater than 1 such that $R[[X_1, \ldots, X_n]]$ is catenarian for each integer $n \geq 1$.

The path to Corollary 3.5 depends on the observation that if $k \subset K$ is a (necessarily purely inseparable) field extension of finite exponent, then the canonical map $\operatorname{Spec}(K[[X_1, \ldots, X_n]]) \to \operatorname{Spec}(k[[X_1, \ldots, X_n]])$ is a homeomorphism for each integer $n \geq 1$. Section 4 is devoted to studying just which field extensions $k \subset K$ induce such homeomorphisms. Theorem 4.6, the main result of Section 4, is that any such $k \subset K$ must be purely inseparable. (Thus, in Corollary 4.8, the question is answered completely in case K is finitely generated as a field extension of k.) Section 4 also includes Corollary 4.9, giving a related characterization of purely inseparable field extensions of finite exponent, and Proposition 4.1, which treats analogous questions for the easier case of polynomial rings.

2. Catenarity of a formal power series ring in one indeterminate over a pullback

It is convenient to recall from [5] the following definition. A ring R is an *SFT-ring* in case, for each ideal I of R, there is a finitely generated ideal $J \subset I$ and an integer $k \geq 1$ such that $r^k \in J$ for each $r \in I$. The proof of [5, Theorem 1] shows that if R is not an SFT-ring, then $R[[X]]$ has an infinite chain of prime ideals. Hence, if $R[[X]]$ is catenarian, then R is necessarily an SFT-ring.

We begin with several preliminary results.

Proposition 2.1 (Arnold [6, Proposition 2.1]). *Let R be an SFT-ring and let $X_1, \ldots, X_n$ be indeterminates over R. Let P be a prime ideal of $R[[X_1, \ldots, X_n]]$ and $p = R \cap P$. Then $P \supset p[[X_1, \ldots, X_n]]$.* □

Proposition 2.2. *Let R be a ring and p a prime ideal of R. Then the prime ideals $p[[X]] \subset p + XR[[X]]$ are adjacent in $R[[X]]$.*

Proof. Note that the prime ideal (X) has height 1 in $(R/p)[[X]]$, as a straightforward consequence of the fact that $\bigcap (X^n) = 0$. Since

$$R[[X]]/p[[X]] \approx (R/p)[[X]] \quad \text{and} \quad (p + XR[[X]])/p[[X]] \approx (X),$$

we have $\mathrm{ht}((p + XR[[X]])/p[[X]]) = 1$. □

In the rest of this section, we let (T, M, K) denote a quasilocal domain with maximal ideal M and residue field K, $\varphi : T \to K$ the natural surjection, and $R = \varphi^{-1}(D)$, where D is a subring of K. The domain R is then a pullback given by the following diagram:

$$\begin{tikzcd} R = \varphi^{-1}(D) \arrow[r] \arrow[d] & T \arrow[d, "\varphi"] \\ D \arrow[r] & K \end{tikzcd}$$

Proposition 2.3 (Khalis [18]). *With the same hypotheses as above, we have*:

(1) *R is an SFT-ring if and only if T and D are each SFT-rings.*

(2) *If R is an SFT-ring, then*

(a) *the prime ideal $M[[X]]$ has the same height in both $T[[X]]$ and $R[[X]]$, and*

(b) $\mathrm{ht}(M[[X]]) + \dim(D[[X]]) \leq \dim(R[[X]]) \leq \dim(T[[X]]) + \dim(D[[X]]) - 1$. *If ($R$ is an SFT-ring and) either D is a field or T is either a Noetherian domain or a valuation domain, then* $\dim(R[[X]]) = \dim(T[[X]]) + \dim(D[[X]]) - 1$. □

Lemma 2.4. *If T and D are SFT-rings, then the function $\xi : \mathrm{Spec}(T[[X]]) \to \mathrm{Spec}(R[[X]])$ given by $\xi(Q) = Q \cap R[[X]]$ induces an order-preserving bijection between $\mathrm{Spec}(T[[X]])$ and $\Omega_R = \{P \in \mathrm{Spec}(R[[X]]) \mid P \subset M + XR[[X]]\}$.*

Proof. Let $\mathcal{S}_T = \{Q \in \mathrm{Spec}(T[[X]]) \mid Q \cap T \subsetneq M\}$ and $\mathcal{S}_R = \{Q \in \mathrm{Spec}(R[[X]]) \mid Q \cap R \subsetneq M\}$. As in [18, Proposition 3.3], the contraction map $\tau : \mathcal{S}_T \to \mathcal{S}_R$ given by $\tau(Q) = Q \cap R[[X]]$ is an order-preserving bijection. Since T is a quasilocal SFT-ring with maximal ideal M and $\dim(T[[X]]/M[[X]]) = 1$, if Q is a prime ideal of $T[[X]]$ with $Q \cap T = M$, then either $Q = M[[X]]$ or $Q = M + XT[[X]]$. Similarly, if $Q \in \Omega_R$ and $Q \cap R = M$, then either $Q = M[[X]]$ or $Q = M + XR[[X]]$. Thus the function $\xi : \mathrm{Spec}(T[[X]]) \to \Omega_R$ is given by $\xi|_{\mathcal{S}_T} = \tau$, $\xi(M[[X]]) = M[[X]]$, and $\xi(M + XT[[X]]) = M + XR[[X]]$, and is an order-preserving bijection. □

In particular, with the hypotheses of Lemma 2.4, the two prime ideals $M + XR[[X]]$ and $M + XT[[X]]$ have the same height in both $R[[X]]$ and $T[[X]]$, respectively. Moreover, $\mathrm{ht}(M + XR[[X]]) = \mathrm{ht}(M[[X]]) + 1$ when $T[[X]]$ is catenarian.

Lemma 2.5. *Suppose that T and D are each SFT-rings. If $P_1 \subset P_2$ are adjacent prime ideals in $R[[X]]$ such that $P_1 \cap R \subsetneq M \subset P_2 \cap R$, then either $P_2 = M[[X]]$ or $P_2 = M + XR[[X]]$.*

Proof. Let $p_1 = P_1 \cap R \subsetneq M$. Then there exists $Q_1 \in \mathrm{Spec}(T[[X]])$ such that $Q_1 \cap R[[X]] = P_1$ and $Q_1 \cap T = p_1$ [18, Corollaire 3.2]. Since R is an SFT-ring and $M \subset P_2 \cap R$, we have $M[[X]] \subset P_2$ by Proposition 2.1. Suppose that $M[[X]] \subsetneq P_2$. As $P_1 \subsetneq P_2$ are adjacent and $p_1 \subsetneq M$, $P_1 \not\subset M[[X]]$ and therefore $Q_1 \not\subset M[[X]]$. Let $f = \sum a_i X^i \in Q_1 - M[[X]]$. Then each $a_i \in T$; let i_0 be the first index $i \geq 1$ with $a_i \notin M$. As a_{i_0} is a unit in T, we may assume that $a_{i_0} = 1$. Thus $f = a_0 + \cdots + a_{i_0-1}X^{i_0-1} + X^{i_0}g$, where g is a unit in $T[[X]]$. Hence $fg^{-1} = (a_0 + \cdots + a_{i_0-1}X^{i_0-1})g^{-1} + X^{i_0} \in Q_1$, and as $(a_0 + \cdots + a_{i_0-1}X^{i_0-1})g^{-1} \in M[[X]]$, we have $fg^{-1} \in R[[X]]$. Thus $fg^{-1} \in Q_1 \cap R[[X]] = P_1$, and therefore $fg^{-1} \in P_2$. Since $(a_0 + \cdots + a_{i_0-1}X^{i_0-1})g^{-1} \in M[[X]] \subset P_2$, we have $X^{i_0} = fg^{-1} - (a_0 + \cdots + a_{i_0-1}X^{i_0-1})g^{-1} \in P_2$, and so $X \in P_2$. Therefore, $M + XR[[X]] \subset P_2$. On the other hand, $Q_1 \subset M + XT[[X]]$ (because $T[[X]]$ is quasilocal with maximal ideal

$M + XT[[X]]$), and thus $P_1 \subset M + XR[[X]]$. Hence $P_1 \subsetneq M + XR[[X]] \subseteq P_2$, whence $P_2 = M + XR[[X]]$, since P_1 and P_2 are adjacent in $R[[X]]$. □

We can now state the main result of this section.

Theorem 2.6. *Let (T, M, K) be a quasilocal domain with maximal ideal M and residue field K, $\varphi : T \to K$ the natural surjection, and $R = \varphi^{-1}(D)$, where D is a subring of K. Then $R[[X]]$ is catenarian if and only if $T[[X]]$ and $D[[X]]$ are each catenarian.*

Proof. Suppose that $R[[X]]$ is catenarian. Then $D[[X]] \approx R[[X]]/M[[X]]$ is catenarian. Moreover, R is an SFT-ring since $R[[X]]$ is catenarian. Thus T and D are each SFT-rings by Proposition 2.3(1), and hence $T[[X]]$ is catenarian by Lemma 2.4.

Conversely, suppose that $T[[X]]$ and $D[[X]]$ are each catenarian. Then T and D are each SFT-rings, and hence R is an SFT-ring. Also, $T[[X]]$ and $D[[X]]$ are each LFD, and hence $R[[X]]$ is also LFD (cf. [18, Proposition 3.5(2)] and [14, Proposition 2.2 and Theorem 2.4]).

Let $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_s = P$ be a saturated chain of $s+1$ prime ideals of $R[[X]]$. It suffices to show that $\mathrm{ht}(P/P_0) = s$.

(a) If $P_0 \cap R = p_0 \supset M$, then $M[[X]] \subset P_0$ by Proposition 2.1. The chain $P_0/M[[X] \subset P_1/M[[X]] \subset \cdots \subset P_s/M[[X]]$ is a saturated chain of prime ideals of $R[[X]]/M[[X]] \approx D[[X]]$, which is catenarian. As $\mathrm{ht}(P/M[[X]]) - \mathrm{ht}(P_0/M[[X]]) = \mathrm{ht}(P/P_0)$ and $\mathrm{ht}(P/M[[X]]) = \mathrm{ht}(P_0/M[[X]]) + s$, we have $\mathrm{ht}(P/P_0) = s$.

(b) Suppose that $P_0 \cap R = p_0 \not\supset M$, and thus $p_0 \subsetneq M$ (cf. [14, Theorem 2.4]). If $P \cap R \subsetneq M$, then by [18, Proposition 3.3] there is a saturated chain $P_0' \subset P_1' \subset \cdots \subset P_s'$ of prime ideals of $T[[X]]$ such that $P_i' \cap R[[X]] = P_i$ for each integer $0 \leq i \leq s$ and $\mathrm{ht}(P/P_0) = \mathrm{ht}(P_s'/P_0')$. As $T[[X]]$ is catenarian, $\mathrm{ht}(P_s'/P_0') = s$ and thus $\mathrm{ht}(P/P_0) = s$.

Without loss of generality, either $P \cap R \not\subset M$ or $P \cap R = M$. Then $M \subset P \cap R$. Choose i_0 with $0 \leq i_0 \leq s-1$ such that $P_{i_0} \cap R \subsetneq M$ and $M \subset P_{i_0+1} \cap R$. Since the primes $P_{i_0} \subset P_{i_0+1}$ are adjacent, Lemma 2.5 gives that P_{i_0+1} is either $M[[X]]$ or $M + XR[[X]]$. In particular, $M[[X]] \subset P_{i_0+1}$. Thus in the catenarian ring $D[[X]] \approx R[[X]]/M[[X]]$, we have $\mathrm{ht}(P/M[[X]]) - \mathrm{ht}(P_{i_0+1}/M[[X]]) = \mathrm{ht}(P/P_{i_0+1}) = s - i_0 - 1$. On the other hand, every chain which realizes the height of the prime P passes through either $M[[X]]$ or $M + XR[[X]]$ by Lemma 2.5. As $\mathrm{ht}(M + XR[[X]]) = \mathrm{ht}(M[[X]]) + 1$ by the remark after Lemma 2.4, we have $\mathrm{ht}(P) - \mathrm{ht}(P_{i_0+1}) = s - i_0 - 1$. The ring $T[[X]]$ is catenarian and since P_{i_0+1} is either $M[[X]]$ or $M + XR[[X]]$, we have $\mathrm{ht}(P_{i_0+1}/P_0) = \mathrm{ht}(P_{i_0+1}) - \mathrm{ht}(P_0) = i_0 + 1$ by Lemma 2.4. Thus $\mathrm{ht}(P) - \mathrm{ht}(P_0) = \mathrm{ht}(P) - \mathrm{ht}(P_{i_0+1}) + \mathrm{ht}(P_{i_0+1}) - \mathrm{ht}(P_0) = (s - i_0 - 1) + (i_0 + 1) = s$. □

3. Catenarity of $R[[X_1, \ldots, X_n]]$

When the number of indeterminates over R is greater than one, the assertion in Theorem 2.6 is no longer true, as the next two examples show. The first is quasilocal; the second, not semi-quasilocal.

Example 3.1. Let $V = k(Y) + Zk(Y)[Z]_{(Z)} = k(Y) + M$, where k is a field, and Y and Z are indeterminates over k. Let $R = k[Y]_{(Y)} + M = D + M$. The rings V and $D = k[Y]_{(Y)}$ are Noetherian valuation rings. Thus, for each integer $n \geq 1$, the rings $V[[X_1, \ldots, X_n]]$ and $D[[X_1, \ldots, X_n]]$ are Cohen–Macaulay and, hence, catenarian. Note that R is a 'discrete' valuation ring of dimension 2 since $k(Y)$ is the quotient field of D. Since R is an SFT-ring by Proposition 2.3(1), $R[[X_1, \ldots, X_n]]$ is catenarian if and only if $n = 1$ [7, 23]. Thus, for each integer $n \geq 2$, the rings $V[[X_1, \ldots, X_n]]$ and $D[[X_1, \ldots, X_n]]$ are each catenarian, but $R[[X_1, \ldots, X_n]]$ is not catenarian.

Example 3.2. Let $V = \mathbb{Q} + Y\mathbb{Q}[[Y]] = \mathbb{Q}[[Y]]$ and $R = \mathbb{Z} + Y\mathbb{Q}[[Y]]$. Of course, the rings $V[[X_1, \ldots, X_n]]$ and $\mathbb{Z}[[X_1, \ldots, X_n]]$ are Cohen–Macaulay and, hence, catenarian for each integer $n \geq 1$. Note that R is a two-dimensional Bézout domain since $\mathbb{Z}$ is a PID with quotient field $\mathbb{Q}$. Since R is an SFT-ring by Proposition 2.3(1), $R[[X_1, \ldots, X_n]]$ is catenarian if and only if $n = 1$ [7, 23]. Thus, for each integer $n \geq 2$, the rings $V[[X_1, \ldots, X_n]]$ and $\mathbb{Z}[[X_1, \ldots, X_n]]$ are each catenarian, but $R[[X_1, \ldots, X_n]]$ is not catenarian.

Despite the above examples, Theorem 2.6 can, in some cases, be extended to more indeterminates by imposing additional conditions on the rings T and D. For example, see Theorem 3.4 below.

Lemma 3.3 (Girolami [16, Lemma 2.3]). *If*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi'"'] & C \arrow[d, "\varphi"] \\ B \arrow[r, "\psi'"'] & D \end{tikzcd}$$

is a pullback of commutative rings, then so is

$$\begin{tikzcd} A[[X]] \arrow[r, "\bar{\psi}"] \arrow[d, "\bar{\varphi}'"'] & C[[X]] \arrow[d, "\bar{\varphi}"] \\ B[[X]] \arrow[r, "\bar{\psi}'"'] & D[[X]]. \end{tikzcd} \qquad \square$$

We remark that the above lemma is also true for power series rings in any finite number of indeterminates.

Theorem 3.4. *Let (T, M, K) be a quasilocal domain with maximal ideal M and residue field K, $\varphi : T \to K$ the natural surjection, and $R = \varphi^{-1}(k)$, where k is a subfield of K. Suppose the contraction map $\mathrm{Spec}(K[[X_1, \ldots, X_n]]) \to \mathrm{Spec}(k[[X_1, \ldots, X_n]])$ is a homeomorphism for each integer $n \geq 1$. (This holds if $\mathrm{char}(k) = p > 0$ and $K^{p^e} \subset k$ for some integer $e \geq 1$.) Then the contraction map $\mathrm{Spec}(T[[X_1, \ldots, X_n]]) \to \mathrm{Spec}(R[[X_1, \ldots, X_n]])$ is a homeomorphism.*

Proof. The diagram

$$\begin{array}{ccc} R[[X_1, \ldots, X_n]] & \longrightarrow & T[[X_1, \ldots, X_n]] \\ \downarrow & & \downarrow{\scriptstyle \bar{\varphi}} \\ k[[X_1, \ldots, X_n]] & \longrightarrow & K[[X_1, \ldots, X_n]] \end{array}$$

is a pullback, where $\bar{\varphi}|_T = \varphi$ and $\bar{\varphi}(X_i) = X_i$. Let $A = k[[X_1, \ldots, X_n]]$ and $B = K[[X_1, \ldots, X_n]]$. (If $K^{p^e} \subset k$, then $B^{p^e} \subset A$, and hence the contraction map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homeomorphism by [1, Theorem 2.1].) On the other hand, $R[[X_1, \ldots, X_n]] = T[[X_1, \ldots, X_n]] \times_B A$ and, according to [14, Theorem 2.4], $\mathrm{Spec}(R[[X_1, \ldots, X_n]])$ is homeomorphic to $\mathrm{Spec}(T[[X_1, \ldots, X_n]]) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$. As $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are canonically homeomorphic,

$$\mathrm{Spec}(T[[X_1, \ldots, X_n]]) \amalg_{\mathrm{Spec}(B)} \mathrm{Spec}(A) \approx \mathrm{Spec}(T[[X_1, \ldots, X_n]])$$

(see [13, Chapter 6, 6-1]). □

Corollary 3.5. *With the same hypotheses as in Theorem 3.4, we have for any integer $n \geq 1$ that $R[[X_1, \ldots, X_n]]$ is catenarian if and only if $T[[X_1, \ldots, X_n]]$ is catenarian.* □

Example 3.6. For each integer $m \geq 1$, there is an m-dimensional non-Noetherian integral domain R such that (i) $R[[X_1, \ldots, X_n]]$ is catenarian for all integers $n \geq 1$, and (ii) $\dim(R[[X_1, \ldots, X_n]]) = n + m$.

Proof. Let $k \subset K$ be fields of characteristic $p > 0$ such that $K^p \subset k$ and $[K : k] = \infty$. Let

$$T = K[[Y_1, \ldots, Y_m]] = K + (Y_1, \ldots, Y_m)K[[Y_1, \ldots, Y_m]]$$

and

$$R = k + (Y_1, \ldots, Y_m)K[[Y_1, \ldots, Y_m]] .$$

Then, by integrality, $\dim(R) = \dim(T) = m$. It is clear (cf. [2, Corollary 3.29]) that R is not Noetherian, and that $\dim(R[[X_1, \ldots, X_n]]) = \dim(T[[X_1, \ldots, X_n]]) = n + m = n + \dim(R)$ since $T[[X_1, \ldots, X_n]]$ is integral over $R[[X_1, \ldots, X_n]]$. The ring $T[[X_1, \ldots, X_n]]$ is catenarian for each integer $n \geq 1$, whence $R[[X_1, \ldots, X_n]]$ is catenarian for each integer $n \geq 1$ by Corollary 3.5. $\square$

4. Field extensions inducing formal power series rings with homeomorphic spectra

The parenthetical assertion in Theorem 3.4 suggests the following definition and question. We shall say that a field extension $k \subset K$ satisfies $(*)$ in case that the canonical map $\mathrm{Spec}(K[[X_1, \ldots, X_n]]) \to \mathrm{Spec}(k[[X_1, \ldots, X_n]])$ is a homeomorphism for each integer $n \geq 1$. How can we characterize the field extensions that satisfy $(*)$? According to Theorem 3.4, if $k \subset K$ is a purely inseparable extension of fields of characteristic $p > 0$ and is of finite exponent, then $k \subset K$ satisfies $(*)$. Our main interest in this section is to study the possible validity of the converse of this result. We show in Corollary 4.7 that if $k \subsetneq K$ satisfies $(*)$, then $k \subset K$ is purely inseparable. This leads, in Corollary 4.8, to a satisfactory answer to the above question in case K is a finitely generated field extension of k; and, in Corollary 4.9, to a new characterization of purely inseparable field extensions of finite exponent. For motivational purposes, we begin this section by treating analogous questions in the simpler context of polynomial rings. Throughout, for an extension $A \subset B$ of rings, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ always denotes the canonical contraction map.

Proposition 4.1. *For a field extension $k \subset K$, the following conditions are equivalent*:

(1) $\mathrm{Spec}(K[X_1, \ldots, X_n]) \to \mathrm{Spec}(k[X_1, \ldots, X_n])$ *is a homeomorphism for each integer* $n \geq 1$;

(2) $\mathrm{Spec}(K[X_1, \ldots, X_n]) \to \mathrm{Spec}(k[X_1, \ldots, X_n])$ *is a homeomorphism for some integer* $n \geq 1$;

(3) $\mathrm{Spec}(K[X]) \to \mathrm{Spec}(k[X])$ *is an injection*;

(4) $k \subset K$ *is purely inseparable.*

Proof. (1)$\Rightarrow$(2) Trivial.

(2)$\Rightarrow$(3) This is an easy consequence of the following observation (cf. [12, proof of (iii)$\Rightarrow$(i) in Theorem 2.1]). If $A \subset B$ are commutative rings and $\mathrm{Spec}(B[X]) \to \mathrm{Spec}(A[X])$ is an injection, then so is $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

(3)$\Rightarrow$(4) This is established in the proof of [22, Theorem 3].

(4)$\Rightarrow$(1) Assume (4). Then $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is radiciel, in the sense of [17]. However, radiciel is a universal property [17]. Thus, for each integer $n \geq 1$, $\mathrm{Spec}(K[X_1, \ldots, X_n]) \to \mathrm{Spec}(k[X_1, \ldots, X_n])$ is also radiciel, and hence injective. This canonical continuous map is also closed and surjective, since $K[X_1, \ldots, X_n]$ is integral over $k[X_1, \ldots, X_n]$, and hence is a homeomorphism. □

Remark 4.2. The literature provides means for alternate proofs of much of Proposition 4.1. For instance, [12, Theorem 2.1] shows that (3) is equivalent to $\text{Spec}(K) \to \text{Spec}(k)$ being radiciel, which is clearly equivalent to (4). Also, the interested reader can develop arguments based on the concept of weak normalization, in the sense of [4].

We now turn to the context of formal power series rings. Lemma 4.3 collects some useful observations. According to its part (b), a one-variable condition is far from enough to characterize (*): contrast the situation in condition (3) of Proposition 4.1 for polynomial rings. The impact of a two-variable condition on formal power series rings is quite different; see Theorem 4.6 below.

Lemma 4.3. (a) *If $k \subset K$ are fields and $n \geq 1$ is an integer, then the canonical function* $\text{Spec}(K[[X_1, \ldots, X_n]]) \to \text{Spec}(k[[X_1, \ldots, X_n]])$ *is continuous and surjective.*

(b) *If $k \subset K$ are fields, then* $\text{Spec}(K[[X]]) \to \text{Spec}(k[[X]])$ *is a homeomorphism.*

(c) *Let $k \subset F \subset K$ be a tower of fields and $n \geq 1$ an integer. Then*

$$\alpha : \text{Spec}(K[[X_1, \ldots, X_n]]) \to \text{Spec}(F[[X_1, \ldots, X_n]])$$

and

$$\beta : \text{Spec}(F[[X_1, \ldots, X_n]]) \to \text{Spec}(k[[X_1, \ldots, X_n]])$$

are homeomorphisms if and only if

$$\beta\alpha : \text{Spec}(K[[X_1, \ldots, X_n]]) \to \text{Spec}(k[[X_1, \ldots, X_n]])$$

is a homeomorphism. Hence $k \subset K$ satisfies (*) if and only if $k \subset F$ and $F \subset K$ each satisfy (*).

Proof. (a) Continuity is standard [8, Proposition 13, p. 101]. Moreover, $K[[X_1, \ldots, X_n]]$ is faithfully flat over $k[[X_1, \ldots, X_n]]$ (cf. [8, Exercise 17(b), p. 250]), whence surjectivity follows.

(b) Put $B = K[[X]]$ and $A = k[[X]]$; and let $\alpha : \text{Spec}(B) \to \text{Spec}(A)$ be the canonical map. Since $\alpha(0) = 0$ and $\alpha(XB) = XA$, we see that α is a bijection. As the closed sets of $\text{Spec}(B)$ (resp., $\text{Spec}(A)$) are $\text{Spec}(B)$ (resp., $\text{Spec}(A)$), $\{XB\}$ (resp., $\{XA\}$), and $\emptyset$, the above definition of α shows that α is a closed map. Hence, by (a), α is a homeomorphism.

(c) The 'only if' assertion holds since any composition of homeomorphisms is a homeomorphism. Conversely, suppose that $\beta\alpha$ is a homeomorphism. It follows that $\beta\alpha$ is an injection, and so α is an injection. But, by (a), both α and β are surjective. Hence, α is a bijection, and thus so is $(\beta\alpha)\alpha^{-1} = \beta$. Since α is continuous and surjective and $\beta\alpha$ is open, we see that β is open and, hence, a homeomorphism. Hence, so is $\beta^{-1}(\beta\alpha) = \alpha$. In view of the definition of property (*), the final assertion now follows by universal quantification on n. □

Remark 4.4. Condition (3) in Proposition 4.1 showed, in that result's context, that the topological condition being studied there could be characterized without topology, i.e., set-theoretically. One might ask if the same is true for the property (*). Theorem 4.6 and Corollaries 4.8 and 4.9 provide some positive evidence. Meanwhile, in this regard, we note, by the above proof, the validity of the analogue of Lemma 4.3(c) in which 'homeomorphism' is replaced throughout with 'injection'.

It is convenient next to introduce some notation. If $\sigma : A \to B$ is a homomorphism of commutative rings and $n \geq 1$ is an integer, we let $\bar{\sigma} = \bar{\sigma}_n$ denote the ring-homomorphism $A[[X_1, \ldots, X_n]] \to B[[X_1, \ldots, X_n]]$ defined by

$$\bar{\sigma}\Big(\sum a_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}\Big) = \sum \sigma(a_{i_1 \cdots i_n}) X_1^{i_1} \cdots X_n^{i_n}\,, \quad a_{i_1 \cdots i_n} \in A\,.$$

The $\bar{\sigma}$ construction will be very useful in the proof of our main result. For the sake of clarity, we next isolate that fragment of the argument. As usual, if $F \subset G$ are fields, then $\mathrm{Gal}(G/F)$ will denote the group of F-algebra automophisms of G.

Lemma 4.5. (a) *Let $F \subset G$ be fields, $\alpha \in G - F$, $\sigma \in \mathrm{Gal}(G/F)$ such that $\sigma(\alpha) \neq \alpha$, and $n \geq 2$ an integer. Put $A = F[[X_1, \ldots, X_n]]$ and $B = G[[X_1, \ldots, X_n]]$. Then $X_1 + \alpha X_2$ and $X_1 + \sigma(\alpha)X_2$ are nonassociated irreducible elements of the unique factorization domain B and $(X_1 + \alpha X_2)B \cap A = (X_1 + \sigma(\alpha)X_2)B \cap A$. Hence $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not an injection.*

(b) *If $k \subset K$ is a Galois field extension and $\mathrm{Spec}(K[[X_1, X_2]]) \to \mathrm{Spec}(k[[X_1, X_2]])$ is an injection, then $K = k$.*

Proof. (a) Since B is a power series ring in a finite number of variables over a field, it is a unique factorization domain (cf. [8, Proposition 8, p. 511] and [8, Corollary 3, p. 533]). Of course, neither $Y = X_1 + \alpha X_2$ nor $Z = X_1 + \sigma(\alpha)X_2$ is a unit of B, since their constant terms are not units of G. We show next that Y is irreducible in B; the proof for Z is similar and hence omitted. Consider any factorization $Y = b_1 b_2$ in B. By an easy order argument, at least one of the b_i has a unit constant coefficient. Thus, b_i is a unit of B; therefore, Y is irreducible, as asserted.

Suppose that Y and Z were associates in B. Then $Y = uZ$ for some unit $u \in B$. Equating coefficients of X_1, we see that the constant term of u must be 1. Then, equating coefficients of X_2, we have $\sigma(\alpha) = \alpha \cdot 1 = \alpha$, a contradiction. Thus, Y and Z are nonassociated in B.

By the above work, we know that YB and ZB are distinct prime ideals of B. It remains only to show that $YB \cap A \subset ZB \cap A$; for then the reverse inclusion would follow by replacing (α, σ) with $(\sigma(\alpha), \sigma^{-1})$. To this end, consider $h \in YB \cap A$. Write $h = Yf$, for some $f \in B$, and apply the ring-homomorphism $\bar{\sigma}$. Since $h \in A$, $\bar{\sigma}(h) = h$; also, $\bar{\sigma}(Y) = Z$. Thus

$$h = \bar{\sigma}(h) = \bar{\sigma}(Y)\bar{\sigma}(f) = Z\bar{\sigma}(f) \in ZB \cap A ,$$

as desired.

(b) In view of the hypothesis that $\mathrm{Spec}(K[[X_1, X_2]]) \to \mathrm{Spec}(k[[X_1, X_2]])$ is injective, we derive from (a), with $(F, G, n) = (k, K, 2)$, that there does not exist $\alpha \in K - k$, $\sigma \in \mathrm{Gal}(K/k)$ such that $\sigma(\alpha) \neq \alpha$. Since $k \subset K$ is Galois, this means that $K - k = \emptyset$, whence $K = k$. □

We can now give the main result of this section. Recall that, by convention, $k \subset k$ is purely inseparable of finite exponent, for any field k, regardless of its characteristic.

Theorem 4.6. *Let $k \subset K$ be fields such that $\mathrm{Spec}(K[[X_1, \ldots, X_n]]) \to \mathrm{Spec}(k[[X_1, \ldots, X_n]])$ is an injection for some integer $n \geq 2$. Then $k \subset K$ is purely inseparable.*

Proof. We begin with an observation motivated by the proof of Proposition 4.1. If $A \subset B$ are commutative rings and $\mathrm{Spec}(B[[X]]) \to \mathrm{Spec}(A[[X]])$ is an injection, then so is $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. (The underlying point is that if $Q \in \mathrm{Spec}(B)$, then $(Q, X) \cap A[[X]] = (Q \cap A, X)$.) It follows that we may suppose that $n = 2$.

Suppose the assertion fails because the field extension $k \subset K$ is not algebraic. Choose $X \in K$ such that X is transcendental over k. By Remark 4.4 (and the hypothesis), $\mathrm{Spec}(k(X)[[X_1, X_2]]) \to \mathrm{Spec}(k[[X_1, X_2]])$ is an injection. Consider the (linear fractional transformation) $\sigma \in \mathrm{Gal}(k(X)/k)$ such that $\sigma(X) = X^{-1}$. Since $\sigma(X) \neq X$, we can apply Lemma 4.5(a), with $(F, G, n) = (k, k(X), 2)$, to obtain a contradiction. Hence, K is algebraic over k.

Let L be the intermediate field between k and K consisting of all the elements of K which are separable over k. A standard consequence of the algebraicity of $k \subset K$ is the pure inseparability of $L \subset K$. Hence, it suffices to show that $L = k$. By Remark 4.4 (and the hypothesis), $\mathrm{Spec}(L[[X_1, X_2]]) \to \mathrm{Spec}(k[[X_1, X_2]])$ is an injection. Thus, without loss of generality, $K = L$; i.e., K is separable over k, and we must show that $K = k$.

Consider N, the normal closure of K/k. Suppose the assertion fails. Then $K \neq k$

and, a fortiori, $N \neq k$. Since N is Galois over k, Lemma 4.5(b) yields that $\operatorname{Spec}(N[[X_1, X_2]]) \to \operatorname{Spec}(k[[X_1, X_2]])$ is not an injection. Indeed, there exist $\alpha \in N - k$ and $\sigma \in \operatorname{Gal}(N/k)$ such that $\sigma(\alpha) \neq \alpha$, and $I = (X_1 + \alpha X_2)N[[X_1, X_2]]$, $J = (X_1 + \sigma(\alpha)X_2)N[[X_1, X_2]]$ are distinct prime ideals satisfying $I \cap k[[X_1, X_2]] = J \cap k[[X_1, X_2]]$. The hypothesis leads to $N \neq K$ and, by Lemma 4.5(a), we can arrange $\alpha \in K - k$ and $\sigma(\alpha) \in N - K$. The hypothesis also leads to $I \cap K[[X_1, X_2]] = J \cap K[[X_1, X_2]]$.

We claim that $I \cap K[[X_1, X_2]] = (X_1 + \alpha X_2)K[[X_1, X_2]]$. Indeed, it is plain that $I \cap K[[X_1, X_2]]$ contains the height 1 prime ideal generated by the irreducible element $X_1 + \alpha X_2$. If equality failed, $I \cap K[[X_1, X_2]]$ would be the maximal ideal of the two-dimensional local ring $K[[X_1, X_2]]$, whence $X_1, X_2 \in I$ and $I = (X_1, X_2)N[[X_1, X_2]]$, contradicting $\operatorname{ht}(I) = 1$. Thus, the claim is established.

It follows that $J \cap K[[X_1, X_2]] = (X_1 + \alpha X_2)K[[X_1, X_2]]$. In particular, $X_1 + \alpha X_2 \in J = (X_1 + \sigma(\alpha)X_2)N[[X_1, X_2]]$. As in the second paragraph of the proof of Lemma 4.5(a), analyzing the constant term of $(X_1 + \alpha X_2)(X_1 + \sigma(\alpha)X_2)^{-1}$ leads to a contradiction. □

Corollary 4.7. *If a field extension $k \subset K$ satisfies $(*)$, then $k \subset K$ is purely inseparable.*

Proof. By hypothesis, $\operatorname{Spec}(K[[X_1, X_2]]) \to \operatorname{Spec}(k[[X_1, X_2]])$ is a homeomorphism and, hence, an injection. Application of Theorem 4.6 completes the proof. □

Corollary 4.8. *For a finitely generated field extension $k \subset K$, the following conditions are equivalent:*

(1) *$k \subset K$ satisfies $(*)$, i.e., $\operatorname{Spec}(K[[X_1, \ldots, X_n]]) \to \operatorname{Spec}(k[[X_1, \ldots, X_n]])$ is a homeomorphism for each integer $n \geq 1$;*

(2) *$\operatorname{Spec}(K[[X_1, \ldots, X_n]]) \to \operatorname{Spec}(k[[X_1, \ldots, X_n]])$ is an injection for some integer $n \geq 2$;*

(3) *$k \subset K$ is purely inseparable (necessarily of finite exponent).*

Proof. It is trivial that (1)$\Rightarrow$(2). Theorem 4.6 gives that (2)$\Rightarrow$(3); and the parenthetical assertion in Theorem 3.4 yields (3)$\Rightarrow$(1), since any finitely generated purely inseparable field extension is finite-dimensional and, hence, of finite exponent. □

Corollary 4.9. *For a field extension $k \subset K$, the following conditions are equivalent:*

(1) *$\operatorname{Spec}(K[[X_1, \ldots, X_n]]) \to \operatorname{Spec}(A)$ is a homeomorphism for each integer $n \geq 1$ and for each ring A such that $k[[X_1, \ldots, X_n]] \subset A \subset K[[X_1, \ldots, X_n]]$.*

(2) *There exists an integer $n \geq 2$ such that $\operatorname{Spec}(K[[X_1, \ldots, X_n]]) \to \operatorname{Spec}(A)$ is a bijection for each ring A such that $k[[X_1, \ldots, X_n]] \subset A \subset K[[X_1, \ldots, X_n]]$.*

(3) *$k \subset K$ is purely inseparable of finite exponent.*

Proof. (3)$\Rightarrow$(1) Without loss of generality, $\mathrm{char}(k) = p > 0$ and $K^{p^e} \subset k$ for some integer $e \geq 1$. As in the proof of Theorem 3.4, $K[[X_1, \ldots, X_n]]^{p^e} \subset A$ and an appeal to [1, Theorem 2.1] yields (1).

(1)$\Rightarrow$(2) Trivial.

(2)$\Rightarrow$(3) Assume (2). By Theorem 4.6, $k \subset K$ is purely inseparable. Without loss of generality, $K \neq k$. According to [15, Corollary 3.4], in order to prove (3), it suffices to show that $E = K[[X_1, \ldots, X_n]]$ is integral over $D = k[[X_1, \ldots, X_n]]$. Hence, by a well-known consequence of Zariski's Main Theorem (cf. [11, Remark 2.5]), it suffices to show that $R \subset S$ satisfies the incomparable and lying-over properties for any rings R, S such that $D \subset R \subset S \subset E$. This, in turn, follows from (2), since $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, viewed as the composite of the bijections $\mathrm{Spec}(S) \to \mathrm{Spec}(E)$ and $\mathrm{Spec}(E) \to \mathrm{Spec}(R)$, is itself bijective. □

Remark 4.10. (a) Apart from the case of a finitely generated field extension treated in Corollary 4.8, we have not settled the question whether a (necessarily purely inseparable) field extension that satisfies (*) must be of finite exponent. In view of the above methods, it seems that the following question should be studied further. If fields $k \subset K$ are such that $\mathrm{Spec}(K[[X_1, X_2]]) \to \mathrm{Spec}(k[[X_1, X_2]])$ is an injection, is $\mathrm{Spec}(K[[X_1, X_2]]) \to \mathrm{Spec}(A)$ a surjection for each ring A between $k[[X_1, X_2]]$ and $K[[X_1, X_2]]$? An affirmative answer to *this* question would assure that (*) implies finite exponent; a negative answer, would, in our opinion, make it unlikely that (*) implies finite exponent.

(b) Let $k \subset K$ be a field extension. For each integer $n \geq 1$, consider the canonical ring-homomorphisms

$$\beta_n : k[[X_1, \ldots, X_n]] \to K \otimes_k k[[X_1, \ldots, X_n]] ,$$

$$\gamma_n : K \otimes_k k[[X_1, \ldots, X_n]] \to K[[X_1, \ldots, X_n]] ,$$

and

$$\alpha_n = \gamma_n \circ \beta_n : k[[X_1, \ldots, X_n]] \to K[[X_1, \ldots, X_n]] .$$

If $k \subset K$ satisfies (*), then K is purely inseparable over k by Corollary 4.7; it then follows that $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is a universal homeomorphism (cf. [17, 12]), so that $\mathrm{Spec}(\beta_n)$ is a homeomorphism for each n. Hence, if $k \subset K$ satisfies (*), then $\mathrm{Spec}(\gamma_n)$ is a homeomorphism for each n. Conversely, if $k \subset K$ is purely inseparable and $\mathrm{Spec}(\gamma_n)$ is a homeomorphism for each n, then we see, similarly, that $k \subset K$ satisfies (*). In general, $k \subset K$ satisfies (*) if and only if $\mathrm{Spec}(\beta_n)$ and $\mathrm{Spec}(\gamma_n)$ are homeomorphisms for each n.

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