RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Tomo XXXI (1982), pp. 125-136

# WHEN ARE $D + \mathfrak{M}$ RINGS LASKERIAN ?

VALENTINA BARUCCI — MARCO FONTANA

Si studia il passaggio della proprietà di laskerianità in alcuni tipi di anelli ottenuti per prodotto fibrato, estendendo alcuni risultati recenti di I. Armeanu. Si caratterizzano gli anelli di pseudo-valutazione che sono anelli (fortemente) laskeriani. Si costruiscono varie classi di anelli (fortemente) laskeriani non noetheriani.

## 0. Introduction and summary.

Several classes of « Noetherian-like » rings have been studied by many authors. In particular, the classes of Laskerian rings and strongly Laskerian rings gave rise to particular interest. Let us recall that a ring $R$ is *Laskerian* (resp. *strongly Laskerian*) if every ideal of $R$ is a finite intersection of primary (resp. strongly primary) ideals (cf., for example, [4; Ch. 4, § 2, Ex. 23 and Ex. 28, pp. 172-175], [12] and [14]). In order to construct examples of strongly Laskerian or Laskerian (possibly, non-Noetherian) rings, knowledge of the stability properties of such rings, under standard algebraic operations, is essential. Preservation of the Laskerian and strongly Laskerian properties when passing to the polynomial and power series rings has been studied by E. G. Evans [6] and by R. W. Gilmer and W. Heinzer [9]. In this paper, we shall study the transfer of the Laskerian and strongly Laskerian properties in $(D + \mathfrak{M})$-type constructions (cf. R. W. Gilmer [8; Appendix 2, p. 558]). In particular, our main result (cf. Th. 6) is a characterization theorem of Laskerian and strongly Laskerian rings $A$ which arise from a cartesian diagram of the following form:

$$(\Delta)\qquad \begin{array}{ccc} A = \varphi^{-1}(D) & \xrightarrow{\varphi|_A} & D \\ \downarrow i & & \downarrow j \\ R & \xrightarrow{\varphi} & K \end{array}$$

Work performed under the auspices of C.N.R. (*Gruppo Nazionale per le Strutture Algebriche, Geometriche e loro Applicazioni*).

where $R$ is a ring, $K$ a field, $D$ a subring of $K$ and $\varphi$ a surjective homomorphism such that the mapping $\varphi|_{U(R)}: U(R) \rightarrow K\setminus\{0\}$ is onto ($U(R)$ being the set of all the units of $R$). Note that the latter condition holds, for instance, if $R$ is local and $K$ is its residue field or if $K$ is a retract of $R$. Preliminarly (cf. Prop. 5), using some general results of M. Fontana [7], we simplify the proof of a recent theorem of I. Armeanu [2], concerning the case in which ring $D$, in diagram ($\Delta$), is a field.

Finally, we apply the results obtained here to construct several examples. These include (non-Noetherian) Laskerian or strongly Laskerian rings and domains either integrally closed or not, of any dimension. Among them, we obtain, once again, as a very special case, the classical example of W. Krull [11; p. 670] of an integrally closed local domain of dimension 1 which is not a valuation ring.

* * *

Throughout this paper, rings will be all commutative rings with units and homomorphisms will mean unitary ring-homomorphisms.

Let $R$ be a ring and $M$ a $R$-module. For $x \in M$, $x \neq 0$, we set $\mathrm{Ann}(x) = \{r \in R: rx = 0\}$. If $M = R/\mathfrak{R}$, for some ideal $\mathfrak{R}$ in $R$, and if $\bar{x} = x + \mathfrak{R} \in M$, $x \in R\setminus\mathfrak{R}$, then we set $(\mathfrak{R} : x)$ instead of $\mathrm{Ann}(\bar{x})$. We denote by $\mathrm{Ass}_R(M)$ the set of the prime ideals of $R$, which are minimal in the set of all prime ideals containing $\mathrm{Ann}(x)$, for some $x \in M$, $x \neq 0$.

Let us also recall the following well-known characterization of the Laskerian rings:

(*) *A ring $R$ is Laskerian if, and only if, for every ideal $\mathfrak{R}$ of $R$, $\mathrm{Ass}_R(R/\mathfrak{R})$ is finite and, for every prime ideal $\mathfrak{P} \in \mathrm{Ass}_R(R/\mathfrak{R})$, there exists $x \in R\setminus\mathfrak{R}$ such that $(\mathfrak{R} : x)$ is a $\mathfrak{P}$-primary ideal.* (Cf., for instance, [3; Th. 4.5 and Rk. 1, p. 52]).

Any unexplained terminology is standard as in [3] or [4].

## 1. The theorem.

PROPOSITION 1. *Let $R$ be a ring, $\mathfrak{M}$ a maximal non-zero ideal of $R$, $K = R/\mathfrak{M}$, $\varphi: R \twoheadrightarrow K$ the canonical projection, $D$ a subring of $K$, $A = \varphi^{-1}(D)$, $T = A\setminus\mathfrak{M}$, $\tau: A \rightarrow T^{-1}A = A_{\mathfrak{M}}$ the canonical homorphism. We assume that $K$ is the field of quotients of $D$ and, for every $r \in R\setminus\mathfrak{M}$, that we have*

*$m \in \mathcal{M}$ in such a way that $r + m$ is a unit of $R$ (i.e. $\varphi|_{U(R)}: U(R) \to K \setminus \{0\}$ is onto). Then, $A$ is a Laskerian (resp. strongly Laskerian) ring if, and only if, one of the following equivalent conditions holds:*

(i) $\mathrm{Ker}(\tau) = \mathcal{M}$ *and $D$ and $R$ are Laskerian (resp. strongly Laskerian) rings;*

(ii) *The canonical map $A_{\mathcal{M}} \to K$ is an isomorphism and $D$ and $R$ are Laskerian (resp. strongly Laskerian) rings;*

(iii) *There exists a ring $R'$ and an isomorfism $\sigma: R \xrightarrow{\sim} K \times R'$, in such a way that $\varphi$ coincides with the composite homomorphism $R \xrightarrow{\sigma} K \times R' \overset{pr_K}{\twoheadrightarrow} K$ and, moreover, $D$ and $R'$ are Laskerian (resp. strongly Laskerian) rings.*

*Proof.* In the following cartesian diagram:

$$\begin{tikzcd} R \times_K D = A \arrow[r, two heads, "\varphi|_A"] \arrow[d, hook, "i"'] & D \arrow[d, hook, "j"] \\ R \arrow[r, two heads, "\varphi"] & K \end{tikzcd}$$

one can easily verify that $i$ is injective, because $j$ is injective. We identify, for the sake of simplicity, $A$ with its image into $R$ under $i$; then, it is straightforward that $\mathcal{M} \cap A = \mathcal{M}$ (cf. also [7; Prop. 2.2 (8)]). We shall at first show that, if $S$ is the set of all the elements of $A$ which are invertible as elements of $R$, then the canonical homomorphism $\alpha: S^{-1}A \to R$ (which is trivially injective) is also surjective. As a matter of fact, for each $r \in R$, we have $\varphi(r) = d\,d'^{-1}$, with $d, d' \in D$ and $d' \neq 0$; if $s \in S$ is such that $\varphi(s) = d'$, then clearly $a = rs \in R$ belongs to $A$ and, therefore, we conclude that $r = a s^{-1}$, with $a \in A$ and $s \in S$.

Now, we wish to prove that if $A$ is a Laskerian ring, then $\mathrm{Ker}(\tau) = \mathcal{M}$. Since $S^{-1}A = R$ and $\mathcal{M} \cap A = \mathcal{M}$ evidently, $T^{-1}A = A_{\mathcal{M}} = R_{\mathcal{M}}$. If, *ab absurdo*, $\mathrm{Ker}(\tau) \neq \mathcal{M}$, then $\mathrm{Ker}(\tau) \subsetneqq \mathcal{M}$. Let $m$ be an element of $\mathcal{M}$ such that $m \notin \mathrm{Ker}(\tau)$, let $d$ be a non-zero non-unit element of $D$ and let $s \in A$, such that $\varphi(s) = d$. The element $s$ is not a unit of $A$ but it is a unit of $R$, hence $s \in S$. Considering the principal ideal generated by $ms$, we claim that:

$$msR \cap A = mR \cap A \supsetneqq msA, \tag{1}$$

because $m \notin msA$. If not, there exists $a \in A$ such that $m = msa$, so $m(1 - sa) = 0$.

As $m \notin \operatorname{Ker}(\tau) = \{a \in A \colon \exists\ t \in T$ such that $a t = 0\}$, we obtain that $1 - s a \in \mathfrak{M}$ and, hence, $1 = \varphi(1) = \varphi(s a) = \varphi(s) \varphi(a)$, that is $\varphi(s) = d$ is a unit of $D$, contradicting the choice of $d$. Therefore (1) holds. Let $m s A = \bigcap_{i=1}^{n} \mathfrak{Q}_i$ the primary irredundant decomposition of the ideal $m s A$ in the Laskerian ring $A$. We assume that $S \cap \mathfrak{Q}_j \neq \emptyset$ if, and only if, $1 \leq j \leq r \lneq n$. The ideals $\mathfrak{Q}_{r+1}, \mathfrak{Q}_{r+2}, \ldots, \mathfrak{Q}_n$ contain the element $m$, because $\mathfrak{M} \subset s A$ for every $s \in S$, and therefore $m \notin \mathfrak{Q}_j$ for some $j$, $1 \leq j \leq r$, since $m \notin m s A$. On the other hand, we know that $m s R \cap A = \bigcap_{j=1}^{r} \mathfrak{Q}_j$ (cf. [3; Prop. 4.9, p. 54]), hence $m \notin m s R \cap A = m A$, and that is a contradiction. Therefore, we have shown that $\operatorname{Ker}(\tau) = \mathfrak{M}$.

It is well-known that if $A$ is a Laskerian (resp. strongly Laskerian) ring, then $S^{-1} A = R$ is a Laskerian (resp. strongly Laskerian) ring (cf. for example [4; Ch. 4, § 2, Ex. 23 (e) and Ex. 28 (d)]); it is straightforward that $A/\mathfrak{M} = D$ is a Laskerian (resp. strongly Laskerian) ring if $A$ is the same. Thus, condition (i) holds.

LEMMA 2. *Let $R$ be a ring, $\mathfrak{M}$ a maximal ideal of $R$,*

$$\pi \colon R_{\mathfrak{M}} \twoheadrightarrow R_{\mathfrak{M}} / \mathfrak{M} R_{\mathfrak{M}} \cong R/\mathfrak{M}$$

*and $\mu \colon R \to R_{\mathfrak{M}}$ the canonical homomorphisms. Then $\pi$ is an isomorphism if (and only if)* $\operatorname{Ker}(\mu) = \mathfrak{M}$.

*Proof.* Straightforward.

LEMMA 3. *Let $R$ be a Laskerian ring an $\mathfrak{M}$ a maximal non-zero ideal of $R$. If $\pi \colon R_{\mathfrak{M}} \to R/\mathfrak{M} = K$ is an isomorphism, then:*

a) $\mathfrak{M} = \operatorname{Ann}(y)$, *for some* $y \in R \setminus \mathfrak{M}$;

b) $\mathfrak{M}$ *is a principal ideal generated by an idempotent element;*

c) *there exists a ring $R'$ and an isomorphism $\sigma \colon R \xrightarrow{\sim} K \times R'$ in such a way that the composite homomorphism $R \xrightarrow{\sigma} K \times R' \xrightarrow{pr_K} K$ coincides with the canonical projection $R \twoheadrightarrow R/\mathfrak{M} = K$.*

*Proof.* The ideal $\mathfrak{M}$ is a minimal prime ideal of $R$, therefore $\mathfrak{M} \in \operatorname{Ass}_R(R)$. By (*), there exists $y \in R$ such that $\operatorname{Ann}(y)$ is a $\mathfrak{M}$-primary ideal. But, by the wellknown one-to-one correspondence between the primary ideals of $R$ contained in $\mathfrak{M}$ and all the primary ideals of $R_{\mathfrak{M}} \simeq K$, $\mathfrak{M}$ does not contain any proper primary ideal. Therefore $\mathfrak{M} = \mathrm{Ann}(y)$ and $y \in R \setminus \mathfrak{M}$. This proves a). To prove b) and c), one need simply consider the ideal $\mathfrak{N} = (y)$. If $y$ is a unit element, $\mathfrak{M} = (0)$, hence $R$ is isomorphic to $K$. If not, the canonical homomorphism $\sigma: R \to R/\mathfrak{M} \times R/\mathfrak{N} = K \times R'$ is an isomorphism, because trivially $\mathfrak{M} + \mathfrak{N} = R$ and, moreover, $\mathfrak{M} \cap \mathfrak{N} = (0)$, being for each $m \in \mathfrak{M}$, $my = 0$.

REMARK. If the ideal $\mathfrak{M}$ is finitely generated, then one may remove from Lemma 3 the hypothesis of Laskerianess on the ring $R$. In fact, in this case, if $\mathfrak{M} = (m_1, m_2, \ldots, m_s)$ and if $y_i \in R \setminus \mathfrak{M}$ is such that $m_i y_i = 0$, $1 \leq i \leq s$, then $\mathfrak{M} = \mathrm{Ann}(y)$, for $y = \prod_{i=1}^{s} y_i$.

LEMMA 4. *Let* $\{A_i : 1 \leq i \leq n\}$ *be a finite family of rings. Then* $A = \prod_{i=1}^{n} A_i$ *is a Laskerian (resp. strongly Laskerian) ring if, and only if, for every* $i$, $A_i$ *is a Laskerian (resp. strongly Laskerian) ring.*

*Proof.* It is the mere consequence of the fact that, for every ideal $\mathfrak{A}$ of $A$, there exists an ideal $\mathfrak{A}_i$ of $A_i$, $1 \leq i \leq n$, in such a way that:

$$\mathfrak{A} = (\mathfrak{A}_1 \times A_2 \times \ldots \times A_n) \cap (A_1 \times \mathfrak{A}_2 \times \ldots \times A_n) \cap \ldots \cap (A_1 \times \ldots \times A_{n-1} \times \mathfrak{A}_n).$$

REMARK. Note that for an infinite family of rings $\{A_i : i \in I\}$ the «if» part of Lemma 4 does not hold anymore. For instance, the ring $A = \prod_{n \in \mathbf{N}} \mathbf{Z}_n$ with $\mathbf{Z}_n = \mathbf{Z}$, for every $n \in \mathbf{N}$, is not a Laskerian ring, because it has a non-Noetherian spectrum (cf. [9]). One can also see directly that $A$ is not a Laskerian ring, showing for example that the ideal (0) is not primary and it has no primary finite decomposition in $A$.

*End of the Proof of Prop.* 1. By Lemma 2, (i) $\Rightarrow$ (ii) and, by Lemma 3, (ii) $\Rightarrow$ (iii). Finally, from condition (iii), it follows that $A$ is a Laskerian (resp. strongly Laskerian) ring, being

$$A = D \times_K R \simeq D \times_K (K \times R') \simeq D \times R'$$

(cf. also Lemma 4).

PROPOSITION 5. (I. Armeanu [1]). *Let $R$ be a ring, $\mathfrak{M}$ a maximal non-zero ideal of $R$, $K = R/\mathfrak{M}$, $\varphi : R \twoheadrightarrow K$ the canonical projection, $k$ a subfield of $K$ and $A = R \times_K k$. If $\varphi|_{U(R)} : U(R) \to K \setminus \{0\}$ is onto, then $A$ is a Laskerian (resp. strongly Laskerian) ring if, and only if, $R$ is a Laskerian (resp. strongly Laskerian) ring.*

*Proof.* As usual, we identify $A$ with its canonical image into $R$. We know that:

(a) The continuous canonical map $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ is a homeomorphism, which, restricted to $\mathrm{Spec}(R) \setminus \{\mathfrak{M}\}$, establishes a scheme-isomorphism between $\mathrm{Spec}(R) \setminus \{\mathfrak{M}\}$ and $\mathrm{Spec}(A) \setminus \{\mathfrak{M}\}$.

(b) For every $h \geq 1$, $\mathfrak{M}^h \cap A = \mathfrak{M}^h$.

(c) For every $\mathfrak{P}' \in \mathrm{Spec}(R)$, $\mathfrak{P}' \neq \mathfrak{M}$, if $\mathfrak{P} = \mathfrak{P}' \cap A$, then the map $\mathfrak{Q}' \mapsto \mathfrak{Q}' \cap A$ establishes an order-preserving bijection between the set of all the $\mathfrak{P}'$-primary ideals in $R$ and the set of all the $\mathfrak{P}$-primary ideals in $A$. (Cf. [7; Th. 1.4 and Cor. 1.5 (2)]).

It is clear that:

(d) If $\mathfrak{Q}$ is a $\mathfrak{M}$-primary ideal in $A$, then $\mathfrak{Q}R$ is a $\mathfrak{M}$-primary ideal in $R$.

Furthermore, one can easily check that:

(e) For every $r \in R$, there exists $a \in A$ and $u \in U(R)$ in such a way that $r = ua$ (cf. [2; Lemma 1]).

From statement (e), we immediately deduce that:

(e′) For every ideal $\mathfrak{R}$ in $R$ and for every $r = ua \in R$, with $u \in U(R)$ and $a \in A$, then $(\mathfrak{R} : r) = (\mathfrak{R} : a)$.

(e″) For every ideal $\mathfrak{R}$ in $R$, $(\mathfrak{R} \cap A)R = \mathfrak{R}$.

Finally, we claim that:

(f) For every ideal $\mathfrak{A}$ in $A$ and for every element $a \in A$, then

$$V((\mathfrak{A} : a)R) \setminus \{\mathfrak{M}\} = V((\mathfrak{A}R : a)) \setminus \{\mathfrak{M}\}.$$

Hence, the contraction to $A$ establishes a one-to-one correspondence between $\mathrm{Ass}_R(R/\mathfrak{A}R) \setminus \{\mathfrak{M}\}$ and $\mathrm{Ass}_A(A/\mathfrak{A}A) \setminus \{\mathfrak{M}\}$.

(1) We simplify the original proof given by Armeanu [2], using some results of [7].

As a matter of fact, it is clear that $V((\mathfrak{A}R : a))\setminus\{\mathfrak{M}\} \subseteq \subseteq V((\mathfrak{A} : a)R)\setminus\{\mathfrak{M}\}$. For the reverse inclusion, we suppose that $\mathfrak{P}' = \mathfrak{P}R \in V((\mathfrak{A} : a)R)$, $\mathfrak{P}' \neq \mathfrak{M}$ (i.e. $\mathfrak{P} \in V((\mathfrak{A} : a))$, $\mathfrak{P} \neq \mathfrak{M}$). Let $r \in (\mathfrak{A}R : a)$, then $ra \in \mathfrak{A}R$; if $m \in \mathfrak{M}\setminus\mathfrak{P}'$, then $ram \in \mathfrak{A}$, thus $rm \in (\mathfrak{A} : a) \subseteq \mathfrak{P} \subseteq \mathfrak{P}'$, hence $r \in \mathfrak{P}'$.

From points (a) - (f), we easily deduce the statement of Prop. 5. In fact, if $R$ is a Laskerian ring, then it can be shown that $A$ is also a Laskerian ring, using the characterization (*). Let $\mathfrak{A}$ be an arbitrary ideal in $A$. From (f), it follows that $\mathrm{Ass}_A(A/\mathfrak{A})$ is finite. Furthermore, if $\mathfrak{P} \in \mathrm{Ass}_A(A/\mathfrak{A})$, $\mathfrak{P} \neq \mathfrak{M}$, then $\mathfrak{P}R \in \mathrm{Ass}_R(R/\mathfrak{A}R)$, therefore there exists $a \in A$ in such a way that $(\mathfrak{A}R : a)$ is a $\mathfrak{P}R$-primary ideal of $R$ (cf. (e')). In view of (c), if $(\mathfrak{A} : a)$ is not a $\mathfrak{P}$-primary ideal of $A$, we can «enlarge» this ideal (resorting to a technique similar to the one employed in the proof of (f)), multiplying $a$ by an element $m \in \mathfrak{M}\setminus\mathfrak{P}$, in such a way that $(\mathfrak{A} : am)$ becomes a $\mathfrak{P}$-primary ideal in $A$. If $\mathfrak{P} = \mathfrak{M} \in \mathrm{Ass}_A(A/\mathfrak{A})$, then there exists $a \in A$ such that $\mathfrak{M}$ is a minimal prime ideal over $(\mathfrak{A} : a)$. We may always assume that $\mathfrak{M}$ is the unique minimal prime ideal over $(\mathfrak{A} : a)$, after multiplying, in case, element $a$ by an opportune non-zero element of $A$, determined by a standard method based on the finiteness of the set of minimal prime ideals over $(\mathfrak{A} : a)$. Since $\mathfrak{M}$ is maximal, $(\mathfrak{A} : a)$ is a $\mathfrak{M}$-primary ideal.

Conversely, if $A$ is a Laskerian ring, then, from (c), (d) and (e''), it follows that $R$ is also a Laskerian ring; more precisely, for every ideal $\mathfrak{R}$ of $R$, one can obtain a primary decomposition of $\mathfrak{R}$, by extending to $R$ a primary decomposition of $\mathfrak{R} \cap A$.

The statement concerning the strongly Laskerian property follows directly from (b).

Putting together Propositions 1 and 5, we get the main result of this paper:

THEOREM 6. *Let $R$ be a ring, $\mathfrak{M}$ a maximal non-zero ideal of $R$, $K = R/\mathfrak{M}$, $\varphi : R \twoheadrightarrow K$ the canonical projection, $D$ a subring of $K$ having $k$ ($\subseteq K$) as field of quotients, $A = \varphi^{-1}(D)$ and $R_1 = \varphi^{-1}(k)$. We assume that $\varphi|_{U(R)} : U(R) \to \to K\setminus\{0\}$ is onto. Then $A$ is a Laskerian (resp. strongly Laskerian) ring if, and only if, one of the following conditions holds*:

($\alpha$) *$D = k$ and $R$ is a Laskerian (resp. strongly Laskerian) ring;*

(β) *There exists a ring $R'$ and an isomorphism $\sigma : R \xrightarrow{\sim} K \times R'$ such that the composite homomorphism $R \xrightarrow{\sigma} K \times R' \xrightarrow{pr_K} K$ coincides with $\varphi$ and, furthermore, $R$ (i.e. $R'$; cf. Lemma 4) and $D$ are Laskerian (resp. strongly Laskerian) rings.*

*Proof.* We consider the following commutative diagram:

$$\begin{tikzcd}
R \times_K D = A \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
R \times_K k = R_1 \arrow[r, two heads] \arrow[d, hook] & k \arrow[d, hook] \\
R \arrow[r, two heads] & K
\end{tikzcd}$$

and we apply Prop. 1 to the upper square and Prop. 5 to the lower square, remarking that $A \cong R_1 \times_k D$ and that $R \cong K \times R'$ if, and only if, $R_1 \cong k \times R'$ (cf. the point (a) of the proof of Prop. 5).

REMARK. Using Theorem 6 one may generalize the characterization theorem concerning the $(D + \mathfrak{M})$-constructions which give rise to Noetherian rings. More precisely, with the same notations and hypotheses of Th. 6, we may affirm that *$A$ is a Noetherian ring if, and only if, one of the following conditions holds*:

(α) *$D = k$, $[K : k] < \infty$ and $R$ is a Noetherian ring;*

(β) *There exists a ring $R'$ and an isomorphism $\sigma : R \xrightarrow{\sim} K \times R'$ such that the composite homomorphism $R \xrightarrow{\sigma} K \times R' \xrightarrow{pr_K} K$ coincides with $\varphi$ and, furthermore, $R$ (i.e. $R'$) and $D$ are Noetherian rings.*

COROLLARY 7. *Under the same hypotheses and notation of Theorem 6, $A$ is a Laskerian (resp. strongly Laskerian) integral domain if, and only if, $D = k$ and $R$ is a Laskerian (resp. strongly Laskerian) integral domain.*

*Proof.* The statement follows immediately from Th. 6 and from the well-known result ensuring that $A$ is an integral domain if, and only if, $R$ is an integral domain (cf. [7; Cor. 1.5 (7)]).

COROLLARY 8. *We preserve the same notations and hypotheses of Theorem* 6.

(a) *Let $R$ be an integrally closed domain. Then, the ring $A$ is an integrally closed Laskerian (resp. strongly Laskerian) domain if, and only if, $D = k$ is a field algebraically closed in $K$ and $R$ is a Laskerian (resp. strongly Laskerian) integral domain.*

(b) *$A$ is a completely integrally closed, Laskerian (resp. strongly Laskerian) domain if (and only if) $D = k = K$ and $R$ is a completely integrally closed, Laskerian (resp. strongly Laskerian) domain. In this case, trivially, $A = R$.*

*Proof.* The corollary follows immediately from Cor. 7 and [7; Cor. 1.5 ((5) and (7))].

Let us recall that a *PVD* (= *pseudo-valuation domain*) is an integral domain $R$ such that for every $\mathfrak{P} \in \operatorname{Spec}(R)$, if $xy \in \mathfrak{P}$, where $x$ and $y$ are elements of $F$, field of quotients of $R$, then $x \in \mathfrak{P}$ or $y \in \mathfrak{P}$ (cf. [10; § 1, p. 138]). One may easily prove that a PVD is necessarily a local ring.

COROLLARY 9. *Let $(R, \mathfrak{M})$ be an integral local domain, which is not a field. $(R, \mathfrak{M})$ is a Laskerian (resp. strongly Laskerian) PVD if, and only if, $(R : \mathfrak{M})$ is a valuation ring of dimension* 1 *(resp. a discrete valuation ring), having $\mathfrak{M}$ as maximal ideal.*

*Proof.* It is known that every PVD $(R, \mathfrak{M})$ is isomorphic to the pull-back of the canonical projection of a valuation ring $V$ onto its residue field $K$, with the field homomorphism $R/\mathfrak{M} = k \hookrightarrow K$; in such a situation $V = (R :_F \mathfrak{M})$ (cf. [1; Prop. 2.6]). Since a valuation ring that is not a field is a Laskerian (resp. strongly Laskerian) ring if, and only if, $\dim(V) = 1$ (resp. $V$ is a discrete valuation ring) (cf. [4; Ch. 4, § 2, Ex. 19 and Ex. 29; Ch. 6, § 3, Ex. 8]), the statement follows easily from Cor. 7.

## 2. Some examples.

(E.1) Let $K$ be a field, $X$ an indeterminate over $K$, $R = K[X]/(X^2) = K[\varepsilon]$, where $\varepsilon = X + (X^2)$, and $\varphi : R \twoheadrightarrow K$ the canonical surjection, mapping $\varepsilon$ to 0. For every subring $D$ of $K$, we consider the subring $A = D + \varepsilon K[\varepsilon]$

of $K[\varepsilon]$. Take $K = \mathbf{C}$. If $D = \mathbf{Q}$, then $A$ is a strongly Laskerian, non-Noetherian, ring of dimension 0; if $D = \mathbf{R}$, then $A$ is a Noetherian ring of dimension 0; if $D = \mathbf{Z}$, then $A$ is a non-Laskerian ring of dimension 1; if $D = \mathbf{Z}_{(p)}$, then $A$ is a non-Laskerian local ring of dimension 1. If we take $K = k(Y)$, where $k$ is a field and $Y$ is an indeterminate over $k$, and $D = k$, then $A$ is a strongly Laskerian, non-Noetherian, 0-dimensional ring, which is integrally closed in its total ring of fractions (that is $R$).

(E.2) Let $K$ be a field, $X$ and indeterminate over $K$, $R = K[X]_{(X)}$, $D$ a subring of $K$ and $\varphi : R \twoheadrightarrow K$ the canonical homomorphism, mapping $X$ to 0. We consider now the ring $A = D + XK[X]_{(X)}$. Take $K = \mathbf{C}$. If $D = \mathbf{Q}$, $A$ is a strongly Laskerian, non-Noetherian, local integral domain of dimension 1; if $D = \mathbf{R}$, $A$, like $R$, is a Noetherian local integral domain of dimension 1; if D $= \mathbf{Z}$, $A$ is a non-Laskerian 2-dimensional integral domain; if $D = \mathbf{Z}_{(p)}$, $A$ is a non-Laskerian local 2-dimensional integral domain. Take $K = \mathbf{Q}$, if $D = \mathbf{Z}$, $A$ is a non-Laskerian 2-dimensional (integrally closed) Prüfer domain; if $D = \mathbf{Z}_{(p)}$, $A$ is a non-Laskerian 2-dimensional valuation ring. If we take $K = k(Y)$ (where $k$ is a field and $Y$ an indeterminate over $k$) and $D = k$, then $A$ is a strongly Laskerian, non-Noetherian, 1-dimensional integrally closed PVD. This is the example, given by W. Krull [11], to show that there exists an integrally closed local domain of dimension 1, which is not a valuation ring.

(E.3) Let $(\mathcal{O}, \mathfrak{M})$ be a local ring of a point of an $n$-dimensional algebraic variety over an algebraically closed field $k$. Let $\mathrm{gr}(\mathcal{O})$ be the graded $k$-algebra associated with $\mathcal{O}$ and $\varphi_A : \mathrm{gr}(\mathcal{O}) \to k$ (resp. $\varphi_B : \mathcal{O} \to k$) the canonical projection. If $k'$ is a subfield of $k$, then the $k'$-graded algebra $A = k' \oplus \oplus (\bigoplus_{h \geq 1} \mathfrak{M}^h/\mathfrak{M}^{h+1})$ (resp. the $k'$-algebra $B = \mathcal{O} \times_k k'$), which is a $k'$-graded subalgebra of $\mathrm{gr}(\mathcal{O})$ (resp. a $k'$-subalgebra of $\mathcal{O}$), is a strongly Laskerian, non-Noetherian, ring of dimension $n$ if, and only if, $k$ is a non-finite field extension of $k'$. Moreover, if the point considered is regular and $k'$ is algebraically closed in $k$, then $A$ (resp. $B$) is a strongly Laskerian, non-Noetherian, $n$-dimensional integrally closed graded domain (resp. domain).

(E.4) Let $K$ be a field and $\{X_h : h \geq 1\}$ a set of indeterminates over $K$. Let $R = K[X_h : h \geq 1]$ and $\mathfrak{A}$ the ideal of $R$ generated by $\{X_2^2 - X_1; X_3^2 - X_2, \ldots ; \ldots ; X_h^2 - X_{h-1}, \ldots\}$. If $S = R/\mathfrak{A}$ and $x_h = X_h + \mathfrak{A}$, then the ring

$$S = K[x_h : h \geq 1]$$

is a 1-dimensional Prüfer domain, because

$$S = \bigcup_{h \geq 1} K[x_1, x_2, \ldots, x_h]$$

and, for every $h$, $K[x_1, x_2, \ldots, x_h]$ is a principal ideal domain (hence, in particular, a 1-dimensional Prüfer domain); cf. [5; (18.6), p. 260]. Moreover, the maximal ideal $\mathfrak{M} = (x_h : h \geq 1)$ of $S$ is not finitely generated, hence $S$ is a non-Noetherian domain. Therefore, $S_{\mathfrak{M}}$ is a 1-dimensional valuation ring that is not a discrete valuation ring. Let $\varphi : S_{\mathfrak{M}} \to K$ be the canonical projection, mapping $x_h$ to 0 for every $h \geq 1$ and let $k$ be a subfield (resp. an algebraically closed subfield) of $K$, then the ring $A = S_{\mathfrak{M}} \times_K k$ is a Laskerian PVD (resp. an integrally closed Laskerian PVD), which is not a strongly Laskerian domain (cf. Cor. 9).

**Added in proofs.** After this paper was submitted, other interesting work on the subject of Laskerian rings was published. H. A. Hussain (*Rev. Roum. Math. Pures Appl.* **25** (1980), 43-48) gives an example of a local strong Laskerian ring which does not satisfy the altitude theorem of Krull. N. Radu (*Proceedings of the week of algebraic geometry,* Bucharest 1980, Teubner, Band **40**, 1981) gives, among other results, some conditions for the Laskerianess of the rings obtained by pull-backs of a general type. W. Heinzer and D. Lantz (*J. Algebra* **22** (1981), 101-114) prove several new properties for Laskerian rings and give other examples of Laskerian non-Noetherian rings.

REFERENCES

[1] Anderson D. F. - Dobbs D. E., *Pairs of rings with the same prime ideals,* Can. J. Math., **32** (1980), 362-384.

[2] Armeanu I., *On a class of Laskerian rings,* Rev. Roum. Math. Pures Appl., **8** (1977), 1033-1036.

[3] Atiyah M. F. - Mac Donald I. G., *Introduction to commutative algebra,* Addison-Wesley, Reading 1969.

[4] Bourbaki N., *Algèbre commutative,* Hermann, Paris 1961-1965.

[5] Brewer J. W. - Rutter E. A., *$D+\mathfrak{M}$ constructions with general overrings,* Michigan Math. J., **23** (1976), 33-42.

[6] Evans E. G. Jr., *Zero divisors in Noetherian-like rings,* Trans. Am. Math. Soc., **155** (1971), 505-512.

[7] Fontana M., *Topologically defined classes of commutative rings,* Annali Mat. Pura Appl., **123** (1980), 331-355.

[8] Gilmer R. W., *Multiplicative ideal theory,* Queen's Papers Pure Appl. Math. N. 12. Queen's University, Kingston 1968. Rev. Ed. Dekker, New York 1972.

[9] Gilmer R.W. - Heinzer W., *The Laskerian property for power series rings.* (To appear).

[10] Hedstrom J. R. - Houston E. G., *Pseudo-valuation domains,* Pac. J. Math., **75** (1978), 137-147.

[11] Krull W., *Beiträge zur Arithmetik kommutativer Integritätsbereiche, II. v - Ideale and vollständig ganz abgeschlossens Integritätsbereiche,* Math. Z., **41** (1936), 665-679.

[12] Krull W., *Über Laskersche Ringe,* Rend. Circ. Mat. Palermo **7** (1958), 155-166.

[13] Lafon J. P., *Algèbre commutative: Langages géométrique et algébrique,* Hermann, Paris 1977.

[14] Radu N., *Sur la decomposition primaire des modules,* Bull. Soc. Math. RSR, **10** (1966), 143-149.

Pervenuto il 6 gennaio 1981

*Istituto Matematico « G. Castelnuovo »*
*Università di Roma*
*00185 Roma*