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Warfield Domains*

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INTRODUCTION

In his first paper on abelian groups [W_1], R. B. Warfield developed a duality theory for the class of torsionfree abelian groups of finite rank. He showed that, given a subgroup A of the additive group $\mathbb{Q}$ of the rational numbers, the functor $\mathrm{Hom}_{\mathbb{Z}}(-, A)$ defines a duality of the class of locally free $\mathrm{End}_{\mathbb{Z}}(A)$-modules of finite rank into itself.

Let us denote by ϕ_G the canonical homomorphism of an abelian group G into its bidual $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(G, A), A)$ and by E_G the ring $\mathrm{End}_{\mathbb{Z}}(G) \cap \mathbb{Q}$. Then Warfield duality relies on the fact [W_1, Theorem 2] that a torsionfree group of finite rank G is A-reflexive (i.e., ϕ_G is bijective) if and only if G is locally free as an E_G-module, $E_G = \mathrm{End}_{\mathbb{Z}}(A)$, and $Ot(G) \leq t(A)$ ($Ot(G)$ is the outer type of G, $t(A)$ is the type of A, see [W_1] or [A]).

Notice that an A-reflexive group G is obviously A-torsionless (i.e., ϕ_G is injective) and satisfies the condition $E_G \geq \mathrm{End}_{\mathbb{Z}}(A)$. The condition $Ot(G) \leq t(A)$ is equivalent to the fact that G is A-torsionless [W_1, Proposition 3] and it implies that $E_G \leq \mathrm{End}_{\mathbb{Z}}(A)$; moreover the additional condition $E_G = \mathrm{End}_{\mathbb{Z}}(A)$ ensures that G is a locally free E_G-module [A, Theorem 1.15]. Hence Warfield duality can be rephrased in the following way: given any subgroup A of $\mathbb{Q}$, a torsionfree abelian group G is A-reflexive if and only if it is A-torsionless and $E_G \geq \mathrm{End}_{\mathbb{Z}}(A)$ (i.e., it is an $\mathrm{End}_{\mathbb{Z}}(A)$-module).

It is worthwhile to remark that the implication, G is A-torsionless implies $E_G \leq \mathrm{End}_{\mathbb{Z}}(A)$, is typical in the context of abelian groups and holds for Dedekind domains (see [Goe]) but not in a more general setting.

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Recently, J. Reid [R_1] tried to extend Warfield duality to modules over commutative integral domains. From his own [R_2] and Lady's [L] results, he noted that Warfield duality holds for Dedekind domains. Moreover, he proved a remarkable result: Warfield duality holds for integral domains R satisfying the following two conditions for any fixed R-submodule A of Q (Q denotes the field of quotient of R):

(1) Given a submodule X of A with $\operatorname{End}_R(X) = \operatorname{End}_R(A)$, the induced homomorphism

$$\operatorname{Hom}_R(X, Q) \to t \operatorname{Hom}_R(X, Q/A)$$

is surjective (t denotes the torsion functor).

(2) All submodules X of A with $\operatorname{End}_R(X) = \operatorname{End}_R(A)$ are A-reflexive.

Notice that condition (1) is equivalent to saying that $\operatorname{Ext}^1_R(X, A)$ is torsionfree as an R-module. Remark that the assumption $\operatorname{End}_R(X) = \operatorname{End}_R(A)$ is too strong, since, over arbitrary domains, the class of A-torsionless modules satisfying this assumption is not closed under direct summands, as we will see from an example in Section 3, and must be substituted by the weaker assumption $\operatorname{End}_R(X) \geq \operatorname{End}_R(A)$, i.e., X is an $\operatorname{End}_R(A)$-module.

At this point it is convenient to introduce the following definition:

DEFINITION. A commutative integral domain R is called a *Warfield domain* if, given any R-submodule A of the field of quotients Q of R, all A-torsionless $\operatorname{End}_R(A)$-modules of finite rank are A-reflexive.

Reid wondered whether conditions (1) and (2) characterize Dedekind domains; in the above terminology Reid's question asks whether Warfield domains are Dedekind domains. It is easy to find non-noetherian valuation domains which are Warfield domains. Hence we reformulate Reid's question as follows: characterize Warfield domains.

In order to reach this characterization, Reid's conditions for the selected submodule A of Q have to be modified as follows:

(1′) Given any $\operatorname{End}_R(A)$-submodule X of A, $\operatorname{Ext}^1_E(X, A)$ is a torsionfree E-module, where $E = \operatorname{End}_R(A)$.

(2′) All $\operatorname{End}_R(A)$-submodules X of A are A-reflexive.

The relevant modification added in condition (1′), namely the hypothesis that $\operatorname{Ext}^1_E(X, A)$—and not $\operatorname{Ext}^1_R(X, A)$—is torsionfree, is unavoidable; in fact an example, appearing in Goeters' paper [Goe], shows that there exists a Warfield domain R such that $\operatorname{Ext}^1_R(X, A)$ has a non-trivial torsion submodule.

A partial answer to the above question was given very recently by Goeters [Goe], who restricted his investigation to the case of noetherian domains R whose integral closure is a finitely generated R-module. By using a famous result by H. Bass [Ba], Goeters proved that such a noetherian domain R is a Warfield domain if and only if every ideal of R can be generated by two elements.

The aim of this paper is to investigate Warfield domains in the general setting of integral domains. They are strictly connected with the reflexive and divisorial domains studied by E. Matlis [M_1] and W. Heinzer [H]. Recall that an integral domain is said to be reflexive (respectively divisorial) if all torsionless modules of finite rank (respectively all fractional ideals) are reflexive (in the above terminology, "torsionless" and "reflexive" mean, respectively, R-torsionless and R-reflexive).

In fact, some results of this paper can be viewed as generalizations and improvements of Matlis' and Heinzer's results in [M_1, H].

After some preliminary results in Section 1, in Section 2 we prove a crucial result concerning the torsionfreeness of some modules of extensions of two rank one modules over h-local domains.

In Section 3 we characterize A-reflexive domains by proving that they are exactly the domains satisfying the two conditions (1′) and (2′) for the fixed submodule A of Q.

Section 3 shows also that A-reflexivity is a property combining two different features: the first one is A-divisoriality, which is a typical property of commutative ring and ideal theory; the second one, namely the torsionfreeness of some modules of extensions, is a typical homological property of module theory, which is connected with some "injective type" properties of the factor module Q/A.

In Section 4 we investigate A-divisorial domains, generalizing and improving some results by Heinzer [H]. Then, fusing the results obtained in this and in previous sections, we reduce the investigation of A-divisorial and A-reflexive domains to the local case. As an easy consequence we deduce that R is a Warfield domain if and only if all of its overrings are h-local and every localization of R at a maximal ideal is a Warfield domain. This seems to be a decisive step, in view of the difficulties arising when dealing with the global case.

In Section 5, we continue the investigation of local divisorial domains using the results on rings with a unique minimal overring obtained in [GH_1].

Section 6 is devoted to the proof of one of the main results of the paper, namely that the class of Warfield domains coincides with the class of totally reflexive domains, i.e., of those domains whose overrings are all reflexive. The proof is based on the analysis of the structure of the integral closure of a totally divisorial domain, i.e., a domain whose overrings are all divisorial.

In Section 7 we extend Goeters' results to arbitrary noetherian domains, proving that a noetherian domain is a Warfield domain if and only if all its ideals are two generated. Here we use the powerful Bass–Matlis characterization of noetherian domains with two generated ideals [Ba, M_2].

Moreover, we characterize integrally closed Warfield domains: they are exactly those Prüfer domains which are almost maximal (in Brandal's sense [Br]) and strongly discrete (as defined in [FHP]). Hence this class of Prüfer domains can be viewed as the most appropriate generalization of the ring of the integers, in the present context.

The paper leaves the question of characterizing Warfield domains in the non-noetherian non-integrally closed case open; an example of such a domain is given at the end of the paper. The results obtained in Section 6 give a rather detailed description of properties of Warfield domains that could lead to a satisfactory characterization also in the general case.

1. PRELIMINARIES

We always denote by R a commutative integral domain with identity, and by Q its field of quotients. Max R denotes the set of the maximal ideals of R.

By an overring of R we will mean a ring S such that $R \subseteq S \subseteq Q$. If X and Y are R-submodules of Q, then $X : Y$ denotes the R-submodule of Q, defined as $\{q \in Q \mid qY \leq X\}$.

We will mainly deal with torsionfree R-modules. A torsionfree R-module M has a rank, which is simply the Q-dimension of $M \otimes_R Q$.

Rank one torsionfree R-modules are isomorphic to R-submodules of Q. If such a submodule A satisfies $qA \leq R$ for some $0 \neq q \in Q$, i.e., $R : A \neq 0$, then A is a fractional ideal of R.

A submodule N of a torsionfree R-module M is said to be pure in M if M/N is torsionfree.

We will use the following two facts.

(1) If M is a torsionfree R-module, then $M = \bigcap\{M_P \mid P \in \text{Max } R\}$. (The proof of this fact goes exactly as the proof of Theorem 4.10 in [G].)

(2) If M and N are R-torsionfree modules which are also S-modules for an overring S of R, then $\text{Hom}_R(M, N) = \text{Hom}_S(M, N)$.

An R-module B is bounded if $rB = 0$ for some $0 \neq r \in R$.

We introduce now the following definitions.

Given a fixed R-submodule A of Q, an R-module M is said to be *A-reflexive* (respectively *A-torsionless*) if the canonical homomorphism

$$\phi_M : M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M, A), A),$$

defined by $\phi_M(m)(f) = f(m)$ ($m \in M$, $f \in \operatorname{Hom}_R(M, A)$), is an isomorphism (respectively a monomorphism).

Obviously A-reflexive modules are A-torsionless and A-torsionless modules are torsionfree.

In order to adapt our terminology to the one used in the literature, R-torsionless modules will be simply called "torsionless."

We shall make frequent use of the long exact sequences containing the functors Hom_R and Ext_R^n ($n \geq 1$); for their properties we refer to [FS, Cap. III].

Some examples in Sections 2 and 3 will deal with abelian groups; we just recall that $\mathbb{Z}$ and $\mathbb{Q}$ will denote the ring of the integers and the field of rational numbers, respectively. Moreover, for the notion of type of a torsionfree rank one group and connected facts, we refer to [F].

We recall now some notions of commutative ring theory. A valuation domain is a domain whose ideals are totally ordered by inclusion. A valuation domain is almost maximal if every proper quotient is linearly compact. A Prüfer domain is a domain whose localizations at maximal (or prime) ideals are valuation domains.

A local domain is a domain with exactly one maximal ideal and it is not necessarily noetherian. A domain is h-local if every non-zero prime ideal is contained in a unique maximal ideal and every non-zero ideal is contained in finitely many maximal ideals.

We introduce the following definitions. If A is a fixed R-submodule of Q, we say that the domain R is *A-reflexive* (respectively *A-divisorial*), if every A-torsionless $\operatorname{End}_R(A)$-module of finite rank (resp. of rank one) is A-reflexive.

To avoid trivial cases, we will always assume $0 \neq A \neq Q$.

In order to adapt our terminology to the one used in the literature, R-reflexive and R-divisorial domains will be simply called "reflexive" and "divisorial," respectively.

Recall that an ideal I of the domain R is divisorial if $I = R : (R : I)$; thus, in our terminology, divisorial ideals are exactly the R-reflexive rank one R-modules.

A *Warfield domain* is a domain R which is A-reflexive for all $A \leq Q$. Some more definitions of particular domains connected with Warfield domains, as order reflexive and totally reflexive domains, will be introduced later on.

For unexplained terminology and notions of module theory and of commutative rings we refer to [FS] and to the books by Gilmer [G] and Kaplansky [K].

2. PROPERTIES OF h-LOCAL DOMAINS

The main result of this section is Proposition 2.5, in which we prove that over an h-local domain R, the module of extensions of two rank one modules, one included in the other, is torsionfree if and only if the module of extensions of the localizations of the two rank one modules is torsionfree for every localization at a maximal ideal of R.

We will make use of the following well known result (see [Goe]).

LEMMA 2.1. *Given two R-modules X and Y, $\mathrm{Ext}^1_R(X,Y)$ is torsionfree if and only if, for every $0 \neq r \in R$, every homomorphism $f\colon X \to Y/rY$ has a lifting homomorphism $g\colon X \to Y$ such that $f = \pi \circ g$, where $\pi\colon Y \to Y/rY$ is the canonical projection.*

LEMMA 2.2. *Let R be an h-local domain, X and A be torsionfree R-modules such that $\mathrm{Ext}^1_R(X, A)$ is a torsionfree R-module. Then $\mathrm{Ext}^1_{R_M}(X_M, A_M)$ is a torsionfree R_M-module, for every $M \in \operatorname{Max} R$.*

Proof. In view of the isomorphism of R-modules

$$\mathrm{Ext}^1_{R_M}(X_M, A_M) \cong \mathrm{Ext}^1_R(X, A_M)$$

(see [Bou, Chap. X, Sect. 6]), it is enough to prove that $\mathrm{Ext}^1_R(X, A_M)$ is a torsionfree R-module for every $M \in \operatorname{Max} R$. Fix an $M \in \operatorname{Max} R$ and consider a homomorphism $f\colon X \to A_M/rA_M$, where we can assume $r \in M$, otherwise $f = 0$. There is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & A/rA \\ {\scriptstyle\lambda_A}\downarrow & & \downarrow{\scriptstyle\rho} \\ A_M & \xrightarrow{\pi_M} & A_M/rA_M \end{array}$$

where π and π_M are the canonical projections, and ρ is the composite of the localization map $\lambda_{A/rA}$ at M and of the canonical isomorphism $(A/rA)_M \cong A_M/rA_M$. Since R is h-local, $A/rA \cong \bigoplus_N (A/rA)_N$, where $N \in \operatorname{Max} R$, hence there is a homomorphism $\varepsilon\colon A_M/rA_M \to A/rA$ such that $\rho \circ \varepsilon$ is the identity map of A_M/rA_M. $\mathrm{Ext}^1_R(X, A)$ is torsionfree, hence by Lemma 2.1, the homomorphism $\varepsilon \circ f\colon X \to A/rA$ can be lifted to a homomorphism $h\colon X \to A$ such that $\pi \circ h = \varepsilon \circ f$. Then, setting $g =$

$\lambda_A \circ h$, we have that $\pi_M \circ g = f$; in fact we have $\pi_M \circ g = \pi_M \circ \lambda_A \circ h = \rho \circ \pi \circ h = \rho \circ \varepsilon \circ f = f$. ∎

We have also the following lemma that we will often use in the sequel.

LEMMA 2.3. *Let R be h-local and $X \le A \le Q$. Then, for every $M \in$* Max *R, $(A : X)_M = A_M : X_M$.*

Proof. Since $A = \bigcap_{N \in \mathrm{Max}\, R} A_N$, we have that $A : X = (\bigcap_N A_N) : X = \bigcap_N (A_N : X) = \bigcap_N (A_N : X_N) = (A_M : X_M) \cap \bigcap_{N \neq M} (A_N : X_N)$. Therefore we have

$$(A : X)_M = (A_M : X_M) \cap \left[\bigcap_{N \neq M} (A_N : X_N) \right] R_M. \tag{1}$$

But $X \le A$ implies $X_N \le A_N$ for every $N \in \mathrm{Max}\, R$, hence $A_N : X_N \ge R_N$ for all N's. Thus we have

$$\left[\bigcap_{N \neq M} (A_N : X_N) \right] R_M \ge \left(\bigcap_{N \neq M} R_N \right) R_M = Q$$

by [M_2, Theorem 22]. Hence by (1) we deduce $(A : X)_M = A_M : X_M$. ∎

The following corollary is an immediate but very useful consequence.

COROLLARY 2.4. *Let R be h-local and A a submodule of Q. Then, for every $M \in$* Max *R,* $(\mathrm{End}\, A)_M = \mathrm{End}\, A_M$.

We can now prove the converse of Lemma 2.2 under the hypothesis that $X \le A \le Q$.

PROPOSITION 2.5. *Let R be h-local and $X \le A \le Q$. Then $\mathrm{Ext}^1_R(X, A)$ is torsionfree if and only if $\mathrm{Ext}^1_{R_M}(X_M, A_M)$ is a torsionfree R_M-module, for all maximal ideals $M \in$* Max *R.*

Proof. By Lemma 2.2, it is enough to prove that $\mathrm{Ext}^1_R(X, A)$ is torsionfree assuming that $\mathrm{Ext}^1_R(X, A_M)$ is torsionfree for all maximal ideals $M \in \mathrm{Max}\, R$. Let us consider a homomorphism $f\colon X \to A/rA$. By Lemma 2.1, we have to show that f lifts to a homomorphism $g\colon X \to A$ such that $\pi \circ g = f$, where $\pi\colon A \to A/rA$ is the canonical projection. First, let us assume that r is contained in a unique maximal ideal M of R, so that $A/rA \cong A_M/rA_M$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(X, rA) & \xrightarrow{\alpha} & \mathrm{Hom}_R(X, A) & \xrightarrow{\beta} & \mathrm{Hom}_R(X, A/rA) & & \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \cong} & & \\ 0 & \rightarrow & \mathrm{Hom}_R(X, rA_M) & \xrightarrow{\gamma} & \mathrm{Hom}_R(X, A_M) & \xrightarrow{\delta} & \mathrm{Hom}_R(X, A_M/rA_M) & \rightarrow & 0, \end{array}$$

where the vertical maps are the injections induced by the localizations at the maximal ideal M of rA, A, and A/rA, respectively. Since δ is surjective, in order to prove that β is surjective, it is enough to show that $\mathrm{Hom}_R(X, A_M) = \mathrm{Im}\,\gamma + \mathrm{Im}\, j$, or equivalently that $A_M : X = rA_M : X + A : X$. Since we are dealing with torsionfree modules, the above equality holds if the localizations at every maximal ideal N of R of the two-hand sides coincide. If $N = M$ we have, by Lemma 2.3, $(A_M : X)_M = A_M : X_M$ and

$$(rA_M : X + A : X)_M = rA_M : X_M + A_M : X_M = A_M : X_M;$$

if $N \neq M$ then, again by Lemma 2.3 and by [M_2, Theorem 19], we have that $(A_M : X)_N = Q : X_N = Q$ and

$$(rA_M : X + A : X)_N = Q : X_N + A_N : X_N = Q.$$

Let us assume now that $\{M_1, M_2, \dots, M_k\}$ is the set of the maximal ideals of R containing r, so that $A/rA \cong \bigoplus_{i=1}^{k} A_{M_i}/rA_{M_i}$. We have the following commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(X, rA) & \xrightarrow{\alpha} & \mathrm{Hom}_R(X, A) & \xrightarrow{\beta} & \mathrm{Hom}_R\left(X, \dfrac{A}{rA}\right) & & \\ & & {\scriptstyle i}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \cong} & & \\ 0 & \rightarrow & \bigoplus\limits_{i=1}^{k} \mathrm{Hom}_R(X, rA_{M_i}) & \xrightarrow{\gamma} & \bigoplus\limits_{i=1}^{k} \mathrm{Hom}_R(X, A_{M_i}) & \xrightarrow{\delta} & \bigoplus\limits_{i=1}^{k} \mathrm{Hom}_R\left(X, \dfrac{A_{M_i}}{rA_{M_i}}\right) & \rightarrow & 0, \end{array}$$

where the vertical maps i and j are the diagonal homomorphisms induced by the localizations of A and of rA at the M_i's.

As before, it is enough to prove that $\bigoplus_{i=1}^{k}\mathrm{Hom}_R(X, A_{M_I}) = \mathrm{Im}\,\gamma + \mathrm{Im}\, j$. Since all modules are torsionfree and $\mathrm{Hom}_R(X, A_{M_i})$ can be identified with $A_{M_i} : X$ we have to show that, for every maximal ideal N of R,

$$\bigoplus_{i=14sk} (A_{M_i} : X)_N = \bigoplus_{i=1}^{k} (rA_{M_i} : X)_N + (\mathrm{Im}\, j)_N. \tag{2}$$

From what we prove above in the case $k = 1$, it is easy to see that, if $N = M_i$ for some i, then the left hand side in (2) becomes $A_{M_i} : X_{M_i} \oplus \bigoplus_{j \neq i} Q_j$ and the right hand side becomes $[rA_{M_i} : X_{M_i} \oplus \bigoplus_{j \neq i} Q_j] + (\mathrm{Im}\, j)_{M_i}$ which is $[rA_{M_i} : X_{M_i} \oplus \bigoplus_{j \neq i} Q_j] + (\pi_i(\mathrm{Im}\, j)_{M_i})$, where π_i denotes the projection of $\bigoplus_{i=1}^{k} \mathrm{Hom}_R(X, A_{M_i})$ onto its ith-component; hence the equality in (2) holds since clearly $(\pi_i(\mathrm{Im}\, j)_{M_i}) = A_{M_i} : X_{M_i}$.

Finally, if $N \neq M_i$ for every i, then both sides of (2) coincide with $\bigoplus_{i=1}^{k} Q_i$. ∎

Proposition 2.5 can be viewed as a generalization of the fact that, over an h-local domain R, the R-module Q/A is injective if and only if all its localizations $(Q/A)_M$ at maximal ideals M of R are injective R_M-modules (see [M_1, Theorem 2.4]). In fact, Q/A injective is equivalent to $\mathrm{Ext}^1_R(I, A) = 0$ for all ideals I of R, which is equivalent to saying that $\mathrm{Ext}^1_R(I, A)$ is torsionfree, since $\mathrm{Ext}^1_R(I, A)$ is always bounded.

It is worthwhile to remark that neither Lemma 2.3, nor Proposition 2.5 holds if the assumption $X \leq A$ is dropped, as the following example shows.

EXAMPLE 2.6. Consider $R = \mathbb{Z}$, X the subgroup of $\mathbb{Q}$ of type $(1, 1, \ldots, 1, \ldots)$ and $A = \mathbb{Z}$, hence $A \subsetneq X$. For every maximal ideal M of $\mathbb{Z}$, $\mathrm{Ext}^1_{\mathbb{Z}_M}(X_M, A_M) = 0$, since $X_M = \mathbb{Z}_M$. On the other hand, $\mathrm{Ext}^1_{\mathbb{Z}}(X, A)$ is not torsionfree, by [W_2, Theorem 3]. Moreover, $A : X = 0$ so that $(A : X)_M = 0$ and $A_M : X_M = \mathbb{Z}_M$ for every maximal ideal M of $\mathbb{Z}$.

3. A-REFLEXIVE DOMAINS

One of the main characterizations of reflexive domains was given by Matlis [M_1, Corollary 2.2]: they are exactly the divisorial domains R such that Q/R is an injective R-module. The last condition is equivalent to $\mathrm{Ext}^1_R(J, R) = 0$ for all ideals J of R. Recall that $\mathrm{Ext}^1_R(J, R)$ is always a bounded R-module, hence $\mathrm{Ext}^1_R(J, R)$ vanishes if and only if it is torsionfree.

The main goal of this section is to generalize Matlis' results to A-reflexive domains, where A is a fixed R-submodule of Q.

We start with some lemmas on A-torsionless R-modules.

Note that, if $E = \mathrm{End}_R(A)$ and M is an A-torsionless module, then M is not necessarily an E-module; however, if M is a torsionfree E-module, then it is A-torsionless as an R-module if and only if it is A-torsionless as an E-module, since $\mathrm{Hom}_R(M, A) = \mathrm{Hom}_E(M, A)$.

Moreover an A-reflexive module M satisfies the property $E_M \geq E$, where $E_M = \mathrm{End}_R(M) \cap Q$. This follows from the isomorphism

$\operatorname{Hom}_R(-, qA) \cong q \operatorname{Hom}_R(-, A)$ for all $q \in Q$; hence an A-reflexive module M is necessarily an E-module.

LEMMA 3.1. *Let A be an R-submodule of Q and let M be a torsionfree R-module of finite rank n. Then the following are equivalent*:

(1) *M is A-torsionless*;

(2) $\operatorname{Hom}_R(M, A)$ *has rank n*;

(3) *each rank one torsionfree quotient of M is embeddable into A*;

(4) *there is an embedding of M into A^n.*

Proof. (1) ⇒ (2). This is obvious, since in general $\operatorname{Hom}_R(M, A)$ has rank at most n.

(2) ⇒ (3). Assume, by way of contradiction, that N is a submodule of M such that M/N is a torsionfree rank one module not embeddable in A. Then $\operatorname{Hom}_R(M/N, A) = 0$, and we get the exact sequence

$$0 = \operatorname{Hom}_R(M/N, A) \to \operatorname{Hom}_R(M, A) \to \operatorname{Hom}_R(N, A)$$

which gives a contradiction, since the last term has rank at most $n - 1$.

(3) ⇒ (4). Let K_i $(1 \le i \le n)$ be pure submodules of M of corank one, such that $\bigcap_i K_i = 0$. Since each M/K_i is embeddable into A, we have the embeddings $M \rightarrowtail \bigoplus_i M/K_i \rightarrowtail A^n$.

(4) ⇒ (1). For each $0 \neq m \in M$ there is a canonical projection π_i: $A^n \to A$ such that $\pi_i(\epsilon(m)) \neq 0$, where ϵ: $M \to A^n$ is an embedding of M into A^n. This shows that the canonical homomorphism ϕ_M of M into $\operatorname{Hom}_R(\operatorname{Hom}_R(M, A), A)$ is injective. ∎

LEMMA 3.2. *Let A be an R-submodule of Q and let $E = \operatorname{End}_R(A)$. Let M be an A-torsionless E-module of finite rank and let N be a pure R-submodule of M. Then N is an E-submodule and both M/N and N are A-torsionless.*

Proof. The first claim is obvious. The last claim follows by considering ranks in the exact sequence

$$0 \to \operatorname{Hom}_E(M/N, A) \to \operatorname{Hom}_E(M, A) \to \operatorname{Hom}_E(N, A)$$

and by using Lemma 3.1(2). ∎

One could wonder whether the class of A-torsionless E-modules of finite rank is also closed under extensions. This is not true in general, as the following example shows.

EXAMPLE 3.3. Consider any non-free subgroup A of Q such that $\operatorname{End}_{\mathbb{Z}}(A) = \mathbb{Z}$. Then by Warfield [$W_2$, Theorems 2 and 3], $\operatorname{Ext}^1_{\mathbb{Z}}(A, A)$ is a non-zero torsionfree group. Hence there exists a non-splitting exact sequence

$$0 \to A \to X \to A \to 0.$$

The first part of the proof of Theorem 3.6 shows that if X is A-torsionless than the exact sequence splits.

The following result provides a sufficient condition for an extension of A-torsionless E-modules to be A-torsionless.

LEMMA 3.4. *Let A be an R-submodule of Q and let $E = \mathrm{End}_R(A)$. Let M be an E-module of finite rank and let N be an E-submodule of M, such that both N and M/N are A-torsionless. If the exact sequence*

$$0 \to N \to M \to M/N \to 0 \qquad \text{(a)}$$

represents a torsion element of $\mathrm{Ext}^1_E(M/N, N)$, *then M is A-torsionless.*

Proof. The exact sequence (a) is quasi-splitting (see [W_2]); i.e., there exist $0 \neq r \in E$ and an E-submodule K/N of M/N containing $r(M/N)$ such that the exact sequence (b) of E-modules

$$0 \to N \to K \to K/N \to 0. \qquad \text{(b)}$$

splits. Then $K \cong N \oplus (K/N)$, hence K is A-torsionless; since $rM \subseteq K$ we conclude that M is A-torsionless. ∎

An immediate consequence is the following.

COROLLARY 3.5. *Let A be a fractional ideal of R and let $E = \mathrm{End}_R(A)$. Then the class of A-torsionless E-modules of finite rank is closed under extensions.*

Proof. Let N be a pure submodule of the E-module M such that both N and M/N are A-torsionless. We can assume, without loss of generality, that $A \leq R$. Let m and n be the ranks of N and M/N, respectively. By Lemma 3.1, N is embeddable into A^m, hence into R^m, and M/N is embeddable into A^n, hence into R^n. So we have an exact sequence of R-modules

$$0 \to M/N \to R^n \to F \to 0,$$

where F is a finitely generated torsion module. We get the exact sequence

$$0 = \mathrm{Ext}^1_R(R^n, N) \to \mathrm{Ext}^1_R(M/N, N) \to \mathrm{Ext}^2_R(F, N) \to \mathrm{Ext}^2_R(R^n, N) = 0$$

which shows that $\mathrm{Ext}_R^1(M/N, N) \cong \mathrm{Ext}_R^2(F, N)$ is a torsion R-module. But $\mathrm{Ext}_E^1(M/N, N)$ is an R-submodule of $\mathrm{Ext}_R^1(M/N, N)$, hence it is torsion too, henceforth the claim follows by Lemma 3.4. ∎

We can now prove the main result of this section.

THEOREM 3.6. *Let A be an R-submodule of Q and let $E = \mathrm{End}_R(A)$. Then R is A-reflexive if and only if it is A-divisorial and $\mathrm{Ext}_E^1(X, A)$ is a torsionfree E-module, for all E-submodules X of A.*

Proof. Assume that R is A-reflexive. Then R is obviously A-divisorial. The proof that $\mathrm{Ext}_E^1(X, A)$ is a torsionfree E-module is partly similar to the proof of Theorem 2.1, (1) ⇒ (3), in [M_1]. Let us consider an exact sequence of E-modules

$$0 \to A \to B \to X \to 0, \tag{c}$$

where X is an E-submodule of A. If B is not A-torsionless, then Lemma 3.4 shows that (c) represents an element of infinite order of $\mathrm{Ext}_E^1(X, A)$. If B is A-torsionless, then the canonical homomorphism

$$\phi_B\colon B \to \mathrm{Hom}_R(\mathrm{Hom}_R(B, A), A)$$

is an isomorphism, as well as

$$\phi_X\colon X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, A), A).$$

The A-dual exact sequence of (c) gives rise to the exact sequence

$$0 \to \mathrm{Hom}_E(X, A) \to \mathrm{Hom}_E(B, A) \to K \to 0, \tag{d}$$

where K is the appropriate cokernel. Since B is A-torsionless, by Lemma 3.1, the middle term has rank two, hence K has rank one. We have the commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \longrightarrow & X & \to & 0 \\ & & & & \phi_B\downarrow & & \phi_X\downarrow & & \\ 0 & \to & K^* & \to & (B^*)^* & \xrightarrow{\beta} & (X^*)^*, & & \end{array} \tag{e}$$

where in (e), $(-)^*$ denotes the A-dual $\mathrm{Hom}_E((-), A)$. Since ϕ_B and ϕ_X are isomorphisms, the map β is onto and consequently there is an induced isomorphism of E-modules $\alpha\colon A \to \mathrm{Hom}_E(K, A)$. Thus K is an A-torsionless E-module, by Lemma 3.1; therefore

$$K \cong \mathrm{Hom}_E(\mathrm{Hom}_E(K, A), A) \cong \mathrm{Hom}_E(A, A) = E.$$

So the exact sequence (d) splits, thus also its A-dual sequence (e) splits, and the original exact sequence (c) splits too. This concludes the proof that $\mathrm{Ext}_E^1(X, A)$ is torsionfree.

The proof of the converse is essentially Reid's proof of Theorem 3.2 in [R_1], with some minor modifications.

Assume that, for all E-submodules X of A, the canonical homomorphism ϕ_X: $X \to \mathrm{Hom}_R(\mathrm{Hom}_R(X, A), A)$ is an isomorphism, and that $\mathrm{Ext}_E^1(X, A)$ is a torsionfree E-module. We must show that ϕ_M is an isomorphism for all A-torsionless E-modules M of finite rank. We induct on the rank of M, which is denoted by n. The case $n = 1$ is true by hypothesis. Suppose $n > 1$ and the result true for A-torsionless E-modules of smaller rank. Let N be any proper non-zero pure submodule of M. Then both N and M/N are A-torsionless by Lemma 3.2. From the exact sequence

$$0 \to N \to M \to M/N \to 0 \tag{f}$$

we obtain the commutative diagram,

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_E(M/N, A) & \longrightarrow & \mathrm{Hom}_E(M, A) & \longrightarrow & \mathrm{Hom}_E(N, A) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_E(M/N, Q) & \longrightarrow & \mathrm{Hom}_E(M, Q) & \longrightarrow & \mathrm{Hom}_E(N, Q) \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & t\,\mathrm{Hom}_E(M/N, Q/A) & \to & t\,\mathrm{Hom}_E(M, Q/A) & \to & t\,\mathrm{Hom}_E(N, Q/A) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where the third non-trivial row contains the torsion part of the Hom's, since the Hom's of the first row have the same rank as the corresponding Hom's of the second row. The C_i's are the appropriate cokernels of the respective columns. Since C_1 and C_3 are torsion submodules of $\mathrm{Ext}_E^1(M/N, A)$ and $\mathrm{Ext}_E^1(N, A)$, respectively, and these Ext's are torsionfree, by the inductive hypothesis, C_1 and C_3 are zero, hence C_2 is also zero. By the Snake Lemma, the last map of the first row is surjective, hence C_2 is also zero. By the Snake Lemma, the last map of the first row is surjective, hence the dualizing functor $\mathrm{Hom}_E(-, A)$ is exact on the sequence (f). Now we can conclude that the commutative diagram of E-modules with exact rows

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
 & & \phi_N \downarrow & & \phi_M \downarrow & & \phi_{M/N} \downarrow & & \\
0 & \rightarrow & (N^*)^* & \rightarrow & M(^*)^* & \rightarrow & ((M/N)^*)^* & \rightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

where $(-)^*$ denotes the A-dual $\mathrm{Hom}_E((-), A)$, has the central vertical map ϕ_M which is an isomorphism, by the Five Lemma. ∎

Remark. In the notation of Theorem 3.6, if $\mathrm{Ext}_R^1(X, A)$ is a torsionfree R-module, then $\mathrm{Ext}_E^1(X, A)$ is also a torsionfree E-module, since it is an R-submodule of $\mathrm{Ext}_R^1(X, A)$. This observation shows that the sufficiency in Theorem 3.6 holds under Reid's hypotheses (1) and (2) quoted in the Introduction, with the modification of the equality $\mathrm{End}_R(X) = \mathrm{End}_R(A)$ by the inclusion $\mathrm{End}_R(X) \geq \mathrm{End}_R(A)$. However, the next example which appears in Goeters' paper [Goe], shows that there exists an A-reflexive domain R such that $\mathrm{Ext}_R^1(X, A)$ has a non-trivial torsion part for a suitable E-submodule X of A.

EXAMPLE 3.7. Let $R = \mathbb{Z}_2 + i2\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the ring of integers localized at 2. R is a local ring, with maximal ideal $J = 2\mathbb{Z}_2[i]$, generated by 2 and $2i$ over R. Goeters proved that R is a Warfield domain, hence in particular, J-reflexive. An easy calculation shows that $\mathrm{Ext}_R^1(J, J)$ is a non-trivial torsion R-module. Notice that $E = \mathrm{End}_R(J) = \mathbb{Z}_2[i]$ is the integral closure of R and, according to Theorem 3.6, $\mathrm{Ext}_E^1(J, J) = 0$, since J is E-isomorphic to E.

Remark. The preceding Lemma 3.2 is needed to prove the sufficiency in Theorem 3.6. Given A, an R-submodule of Q, and $E = \mathrm{End}_R(A)$, Proposition 3.1 in [R_1] states that the class of A-torsionless R-modules B of finite rank, such that $E = E_B$ is closed under pure submodules and torsionfree quotients. This fact is not true, since this class is not closed even under summands, as the next example shows. This observation makes evident the reason why the assumption $\mathrm{End}_R(X) = \mathrm{End}_R(A)$ in Reid's condition (1) quoted in the Introduction must be substituted by the weaker assumption $\mathrm{End}_R(X) \geq \mathrm{End}_R(A)$.

EXAMPLE 3.8. Let R be a valuation domain with two different non-zero prime ideals $P_1 \lneq P$. Then the torsionfree R-module $B = P \oplus P_1$ is obviously P-torsionless, and $\mathrm{End}_R(P) = R_P = E_B$. On the other hand, P_1 is P-torsionless but $E_{P_1} = R_{P_1} \gneq R_P$.

Theorem 3.6 shows how A-divisorial domains are connected with A-reflexive domains. A-divisorial domains are investigated in the next section. The results of the next section together with Proposition 2.5 will enable us to reduce the investigation of A-reflexive domains, and consequently of Warfield domains, to the local case. Thus we must postpone a relevant result on A-reflexive domains, namely Theorem 4.8, to the end of the next section.

4. A-DIVISORIAL DOMAINS AND REDUCTION TO THE LOCAL CASE

In this section we will extend some of Heinzer's results in [H] to A-divisorial domains, where A is a fixed submodule of Q. Moreover we will prove that the properties of being A-divisorial, A-reflexive, and Warfield are local properties of a domain.

Let us denote by $\mathscr{F}(A)$ the class of the nonzero A-torsionless submodules of Q which are modules over the endomorphism ring of A. Obviously $\mathscr{F}(A)$ consists of those submodules X of Q such that $A : X \neq 0$ and $\mathrm{End}_R(X) \supseteq \mathrm{End}_R(A)$.

LEMMA 4.1. *Let R be an arbitrary domain and A an* R-submodule *of Q.*

(1) *For an arbitrary integral domain R, the class $\mathscr{F}(A)$ is a lattice under the usual operations of sum and intersection.*

(2) *If R is an A-divisorial domain, the correspondence $X \to A : X$ is a lattice anti-isomorphism of $\mathscr{F}(A)$.*

(3) *If $\{X_\alpha\}$ is a subset of $\mathscr{F}(A)$ such that $X = \bigcap_\alpha X_\alpha \neq 0$, then $X \in \mathscr{F}(A)$ and, if R is an A-divisorial domain, $A : X = \sum_\alpha A : X_\alpha$.*

Proof. (1) Let X_1 and X_2 be in $\mathscr{F}(A)$; then the claim follows from the relations

$$A : (X_1 + X_2) = (A : X_1) \cap (A : X_2)$$

$$A : (X_1 \cap X_2) \geq (A : X_1) + (A : X_2)$$

$$\mathrm{End}_R(X_1 + X_2) \geq \mathrm{End}_R(A) \qquad \text{and} \qquad \mathrm{End}_R(X_1 \cap X_2) \geq \mathrm{End}_R(A).$$

(2) This is straightforward.

(3) The first claim is obvious; the proof of the second one is similar to that of Lemma 2.1 in [H], observing that $\sum_\alpha A : X_\alpha$ is always contained into $A : X$, hence it belongs to $\mathcal{F}(A)$. ∎

LEMMA 4.2. *Let R be an arbitrary domain, A and R-submodule of Q, and* $E = \mathrm{End}_R(A)$. *Then every fractional ideal of E belongs to* $\mathcal{F}(A)$; *the converse is true if and only if A is a fractional ideal of E.*

Proof. Let J be a fractional ideal of E; then clearly $\mathrm{End}_R(J) \geq E$, and $E : J \neq 0$, so that $(A : J) : A \neq 0$ hence $A : J \neq 0$ and J is in $\mathcal{F}(A)$.

Conversely, let A be a fractional ideal of E, and let $J \in \mathcal{F}(A)$. Then $\mathrm{End}_R(J) \geq E$ ensures that J is an E-submodule of Q, and $A : J \neq 0 \neq E : A$ ensure that $E : J \neq 0$, so J is a fractional ideal of E.

As an example of a submodule A which is not a fractional ideal of $\mathrm{End}_R(A)$, consider the rational group R_1 of the fractions with squarefree denominators. Then $\mathrm{End}(R_1) = \mathbb{Z}$, and $\mathbb{Z} : R_1 = 0$.

LEMMA 4.3. *Let R be an A-divisorial domain and M a maximal ideal of* $\mathrm{End}_R(A) = E$. *Then* $(A : M)/A$ *is isomorphic, as an E-module, to* E/M, *hence* $A : M$ *properly contains A and* $A : M = A + Ex$, *for any* $x \in (A : M) \setminus A$.

Proof. $A : M$, A, and E clearly belong to $\mathcal{F}(A)$. M belongs to $\mathcal{F}(A)$ by Lemma 4.2. By Lemma 4.1, the set of the modules in $\mathcal{F}(A)$ between E and M is in bijective correspondence with those between $A : M$ and A, so $(A : M)/A$ is a simple E/M-module, hence it is isomorphic to E/M. ∎

LEMMA 4.4. *Let R be an A-divisorial domain,* $E = \mathrm{End}_R(A)$, *and B a non-zero proper ideal of E. Let M be a maximal ideal of E containing B. Then* $J = \bigcap\{J_\alpha \mid B \leq J_\alpha \leq E,\ J_\alpha \not\subseteq M\}$ *is not contained in M; hence* $B \subsetneq J$.

Proof. The proof is similar to that of Lemma 2.3 in [H]. Note that E is one of the J_α's; for each α, $J_\alpha \not\subseteq M$ implies $A : M \not\subseteq A : J_\alpha$, hence $(A : J_\alpha) \cap (A : M) = A$, by Lemma 4.3. Let $x \in (A : M) \setminus A$. We will show that $x \notin A : J$, so $A : M \not\subseteq A : J$ will imply that $J \not\subseteq M$, as we wanted to prove. Assume, by way of contradiction, that $x \in A : J = \sum_\alpha A : J_\alpha$ (recall Lemma 4.1(3)). Then $x \in \sum_{1 \leq i \leq n} A : J_{\alpha_i} = A : (J_{\alpha_1} \cap \cdots \cap J_{\alpha_n})$; but $J_{\alpha_1} \cap \cdots \cap J_{\alpha_n}$ is one of the J_α's, say J_{α_0}, thus $x \in (A : J_{\alpha_0}) \cap (A : M) = A$, a contradiction. ∎

We can now prove one of the main results of this section. Recall that an integral domain is said to be *h*-local if each non-zero prime ideal is contained in a unique maximal ideal, and each non-zero proper ideal is contained only in finitely many maximal ideals.

THEOREM 4.5. *Let R be an A-divisorial domain. Then* $\mathrm{End}_R(A)$ *is h-local.*

Proof. The proof that each non-zero prime ideal of $\mathrm{End}_R(A)$ is contained in a unique maximal ideal is similar to the proof of Theorems 2.4 in [H]. Let $E = \mathrm{End}_R(A)$ and B be a non-zero proper ideal of E; assume that M_1 and M_2 are two different maximal ideals of E containing B. From Lemma 4.4 we know that $J = \bigcap_\alpha \{J_\alpha : B \leq J_\alpha \leq E,\ J_\alpha \nsubseteq M_1\}$ is not contained in M_1; choose $y \in J \setminus M_1$. Then $y^2 \notin M_1$, so $B + Ey^2$ is one of the J_α's, thus it contains y. Let $y = b + dy^2$ $(b \in B,\ d \in E)$. Then $b = y(1 - dy) \in B$, with $y \notin B$ and $1 - dy \notin M_2$ (since $y \in B \subseteq M_2$); therefore B cannot be a prime ideal. Let $\{M_\alpha\}$ be the family of the maximal ideals of E containing B; the proof that this family is finite goes exactly as in Theorem 2.5 in [H], since it depends only on the property of E proved in Lemma 4.4. ∎

One could conjecture that, if R is an A-divisorial domain, then $\mathrm{End}_R(A)$ is a divisorial domain; if this were true, our Theorem 4.5 would follow from Heinzer's results. This is not the case: for instance, as we will show later, a valuation domain R is L-divisorial for every non-zero prime ideal L, but $R_L = \mathrm{End}_R(L)$ is a diviorial domain if and only if $L \cong R_L$, by Theorem 5.1 in [H].

We give now a generalization of Proposition 2.5(1), proved by Matlis in [M_1], that will not be used later, but which is of independent interest.

PROPOSITION 4.6. *Let R be an A-divisorial domain and $E = \mathrm{End}_R(A)$. Then Q/A is an essential extension of $\bigoplus_{M \in \max E}(A : M)/A$, as an E-module.*

Proof. We know, from Lemma 4.3, that $(A : M)/A \cong E/M$, for all $M \in \mathrm{Max}\ E$, hence $\bigoplus_{M \in \mathrm{Max}\ E}(A : M)/A$ is the socle of Q/A as an E-module. To show that this socle is essential in Q/A, let $0 \neq x = a/b + A \in Q/A$, with $a, b \in E$; since $0 \neq x$, a does not belong to Ab. Consider the ideal J of E defined by $J = \{q \in E\colon qa \in Ab\}$; this is a proper ideal, since $a \notin Ab$. Let M be a maximal ideal of E containing J; we show now that the cyclic E-module generated by x has a non-zero multiple in $(A{:}M)/A$ i.e., that $(E(a/b) + A) \cap (A : M)$ properly contains A. Assume, by way of contradiction, that $(E(a/b) + A) \cap (A : M) = A$. Then dualizing with respect to A and using repeatedly Lemma 4.1, we get $[(A : E(a/b)) \cap E] + M = E$; but the first term in this equality coincides with J, hence we obtain that $M = E$, a contradiction. ∎

Our next goal is to prove that A-divisoriality is a local property.

THEOREM 4.7. *Let A be an R-submodule of Q with* $\mathrm{End}_R(A) = E$. *The following are equivalent*:

(1) *R is A-divisorial.*

(2) *E is A-divisorial.*

(3) *E is h-local and E_M is A_M-divisorial for every maximal ideal M of E.*

Proof. (1) ⇔ (2). It is clear since R is *A-divisorial* if and only if $A:(A:X) = X$ for every A-torsionless E-submodule of Q.

(2) ⇒ (3). In Theorem 4.5 we proved that E is h-local and by Corollary 2.4 we have that $\mathrm{End}_E(A_M) = E_M$ and clearly $\mathrm{End}_R(A_M) = \mathrm{End}_E(A_M)$. Thus it remains to prove that if Y is an A_M-torsionless E_M-submodule of Q, then

$$A_M : (A_M : Y) = Y.$$

Since there is an embedding of Y into A_M, we may assume that $Y \leq A_M$. Now let $X = Y \cap A$, then $X_M = Y_M \cap A_M = Y \cap A_M = Y$. X is an E-submodule of A and, by the A-divisoriality of E, we have that $A:(A:X) = X$. Clearly, E is embeddable in X, hence $A:X$ is embeddable in A; thus, by Lemma 2.3, we obtain $X_M = A_M : (A:X)_M$ and again by Lemma 2.3, $(A:X)_M = A_M : X_M$, since $X \leq A$. This proves that Y is A_M reflexive.

(3) ⇒ (2). We must show that, if X is an A-torsionless E-module, then $A:(A:X) = X$. We may assume $X \leq A$: moreover E is embeddable in X which yields $A:X$ embeddable in A. By hypothesis E is h-local, thus we may apply Lemma 2.3 twice to get

$$[A:(A:X)]_M = A_M : (A:X)_M = A_M : (A_M : X_M).$$

for every maximal ideal M of E.

Now the hypothesis that E_M is A_M-divisorial ensures that $A_M : (A_M : X_M) = X_M$, for every maximal ideal M of E, hence $A:(A:X) = X$. ∎

Combining Theorem 4.7 with the results of Sections 2 and 3, we may show that also the A-reflexivity is a local property.

THEOREM 4.8. *Let A be an R-submodule of Q with* $\mathrm{End}_R(A) = E$. *The following are equivalent*:

(1) *R is A-reflexive.*

(2) *E is A-reflexive.*

(3) *E is h-local and E_M is A_M-reflexive for every maximal ideal M of E.*

Proof. (1) ⇔ (2). This is clear, since A-reflexivity must be checked on A-torsionless E-modules of finite rank. (2) ⇔ (3). This follows by applying Theorems 4.7, 3.6, and Proposition 2.5. ∎

As an easy consequence of Theorem 4.8, we may prove that the property of being a Warfield domain is a local property.

PROPOSITION 4.9. *R is a Warfield domain if and only if every overring of R is h-local and every localization of R at a maximal ideal is a Warfield domain.*

Proof. The necessity is clear, since it is easy to check that every overring of a Warfield domain is a Warfield domain.

For the sufficiency, let A be an R-submodule of Q and let $E = \mathrm{End}_R(A)$. We must prove that R, or equivalently E, is A-reflexive. E is h-local by hypothesis, thus by Theorem 4.8, we may assume that E is a local domain. Let N be the maximal ideal of E, then $N \cap R$ is a prime ideal P of R and $R_P \subseteq E$. If M is a maximal ideal of R containing P, then $R_M \subseteq R_P \subseteq E$. Thus E is an overring of R_M, hence it is a Warfield domain and in particular it is A-reflexive. ∎

5. SOME MORE RESULTS ON DIVISORIAL AND TOTALLY DIVISORIAL DOMAINS

Recall that an integral domain is said to be divisorial if every fractional ideal I of R concides with its double inverse $(I^{-1})^{-1}$, where $I^{-1} = R : I$. In [H], Heinzer characterized the integrally closed divisorial domains and, in [M_1], Matlis characterized noëtherian divisorial domains.

In the general case there are only partial results (see [GH_1]).

The aim of this section is to present some results on general divisorial domains which will help in characterizing Warfield domains. The next two lemmas are easy, but very useful.

LEMMA 5.1. *Let I be an ideal of the divisorial domain R, and let $T = I : I$. Then $R : T$, the conductor of T in R, concides with II^{-1}.*

Proof. We have $R : II^{-1} = (R : I^{-1}) : I$ and $R : I^{-1}$ coincides with I, since R is divisorial, hence $R : II^{-1} = I : I = T$. This says that T is the inverse of II^{-1} and, since T is clearly a fractional R-module, we obtain the wanted conclusion. ∎

LEMMA 5.2. *Let I be an ideal of the divisorial domain R. Then $I : I = R$ if and only if I is invertible.*

Proof. Assume I is invertible and let $T = I : I$. By Lemma 5.1, $R : T = R$ hence $T = R$. Conversely, let $I : I = R$, then, by Lemma 5.1, $R : R = R = II^{-1}$, hence I is invertible. ▮

The next result follows immediately by Lemma 5.2, recalling that invertible ideals over local domains are principal.

COROLLARY 5.3. *Let I be an ideal of the divisorial local domain R. Then* $I : I = R$ *if and only if I is principal.*

As far as we know, the following immediate consequence of Theorem 4.7 does not appear in the literature.

PROPOSITION 5.4. *A commutative domain R is divisorial if and only if it is h-local and* R_M *is divisorial for every maximal ideal M of R.*

Thus the problem of characterizing divisorial domains can be reduced to the local case.

In [H, M_1] it is proved that if R is a divisorial local domain with maximal ideal M, then M^{-1}/R is isomorphic to R/M and it is an essential submodule of Q/R. We can prove the following:

LEMMA 5.5. *Let R be a local domain with maximal ideal M such that* M^{-1}/R *is isomorphic to* R/M *and essential in* Q/R. *Then*:

(1) *M is invertible if and only if R is a valuation divisorial domain.*

(2) *If R is integrally closed, then R is a valuation divisorial domain.*

Proof. (1) It is well known (see [H] or [M_1]) that a valuation domain is divisorial if and only if its maximal ideal is invertible. Thus we have only to show that, if M is invertible, namely $M = aR$, for some $a \in R$, then R is a valuation domain. Assume $q^{-1} \in Q \setminus R$; we will prove that $q \in R$.

$q^{-1}(R + qR) \supsetneq R$ and thus, by hypothesis, $q^{-1}(R + qR) \supseteq a^{-1}R = M^{-1}$. Now, $a(R + qR) \supseteq qR$ implies $q = ar + qas$, with r and s in R, hence $q(1 - as) = ar$, thus $q \in R$ since $1 - as$ is a unit in R.

(2) By part (1), it is enough to prove that M is invertible. Assume it is not; then $MM^{-1} = M$ hence $M^{-1} = M : M = R_1$ is an overring of R and, as proved in [H, Lemma 2.2] or [M_1, Lemma 2.3], $R_1 = R + R\alpha$ for every element $\alpha \in R_1 \setminus R$, hence $R_1 = R[\alpha]$ is integral over R, a contradiction. ▮

Remark. Lemma 5.5(1) shows that, in the investigation of divisorial domains, the interesting case that remains to be studied is the case of a local domain with non-principal maximal ideal M. Then, as noticed in the proof of Lemma 5.5(2), M^{-1} coincides with $M : M$, the endomorphism ring of M. Let us denote $M:M$ by R_1; then R_1 satisfies the following properties:

(1) $R_1 = R + R\alpha$ for every element $\alpha \in R_1 \setminus R$, hence R_1 is integral over R.

(2) M is the conductor $R:R_1$ of R in R_1.

(3) R_1/R is a simple R-module and it is an essential R-submodule of Q/R; thus, in the terminology used in [GH_1], R_1 is a unique minimal overring of R.

(4) R_1 has at most two maximal ideals [GH_1, Corollary 2.2].

The next results will illustrate another interesting property of the domain R_1. Its proof appears, under different hypotheses, in the proof of implication (2) ⇒ (3) of Theorem 57 of [M_2], we give it for the sake of completeness.

LEMMA 5.6. *Let R be a local divisorial domain with non-principal maximal ideal M and let $R_1 = M : M$. An ideal I of R is an R_1-ideal if and only if I is not principal over R.*

Proof. Let I be a non-principal ideal of R, then $II^{-1} \subseteq M$, hence $R : II^{-1} = (R : I^{-1}) : I \supseteq R_1$ which yields $R_1 \subseteq I : I$. ▮

We collect in the following theorem some facts about R_1 which will be used in the sequel; some of these facts are known, but we sketch their proofs for convenience.

THEOREM 5.7. *Let R be a local divisorial domain with non-principal maximal ideal M and let $R_1 = M : M$. Then one and only one of the following cases can occur*:

(1) *R_1 is local with maximal ideal M_1 properly containing M. In this case M_1/M is simple both as an R-module and as an R_1-module, moreover $M_1^2 \subseteq M$ and $R : M_1 = M_1$.*

(2) *R_1 has exactly two maximal ideals M_1 and M_2. In this case $M = M_1 \cap M_2$, M_i/M is simple both as an R-module and as an R_1-module, for $i = 1, 2$; $R : M_i = M_j$, for $i \neq j$. Moreover R_1 is a Prüfer domain, intersection of two valuation domains V_1 and V_2 whose maximal ideals N_1 and N_2 satisfy $M = N_1 \cap N_2$.*

(3) *R_1 is local with maximal ideal M. In this case R_1 is a valuation domain.*

Proof. By (4) in the preceding Remark, one of the above cases has to occur.

(1) M_1/M is a non-zero R/M module properly contained in the two dimensional R/M-module R_1/M. Thus clearly M_1/M is simple as an R_1-module, consequently $M_1^2 \subseteq M$ and $R : M_1 \supseteq M_1$. But $R : M_1$ is contained in $R : M = R_1$ and by the hypothesis of divisoriality of R, it is properly contained in R_1. Hence it coincides with M_1.

(2) The first statement is proved in [GH_1, Corollary 2.2]. The proof that M_i/M is simple both as an R and as an R_1 module, is the same as in (1) above. Clearly $M_i \subseteq R : M_j$ and the equality holds since $R : M_j \subsetneq R_1$. By Proposition 2.5 in [GH_1] the integral closure $\overline{R}$ of R (or of R_1) is a Prüfer domain, intersection of two valuation domains V_1 and V_2 with maximal ideals N_1 and N_2 such that $N_1 \cap N_2 = M$. We prove now that in these hypothesis $\overline{R} = R_1$. In fact, since $N_1 \cap N_2 = M$ we have that M is an $\overline{R}$-ideal, hence $\overline{R} \subseteq M : M = R_1$ and the claim follows.

(3) By Proposition 2.6 in [GH_1] the integral closure $\overline{R}$ of R (or of R_1) is a valuation domain V with maximal ideal M. Then, again, M is an $\overline{R}$-ideal and thus $\overline{R} = R_1$. ∎

Notice that in the cases (2) and (3) considered in Theorem 5.7, the integral closure $\overline{R}$ of R is finitely generated as an R-module. The remaining case, namely when R_1 is local with maximal ideal properly containing M is more difficult to handle and will be studied in the next section, but under the stronger hypothesis that R is totally divisorial.

6. WARFIELD AND TOTALLY REFLEXIVE DOMAINS

The aim of this section is to prove that the class of Warfield domains coincides with the class of totally reflexive domains, whose definition will be given below. Recall that an order of a domain R is an overring of R which is finitely generated as an R-module.

DEFINITION. (1) A domain R is said to be *totally divisorial* (resp. *order divisorial*) if every overring (resp. every order) of R is a divisorial domain.

(2) A domain R is said to be *totally reflexive* (resp. *order reflexive*) if every overring (resp. every order) of R is a reflexive domain.

In view of Proposition 5.4, we will still consider the local case. The next two results are improvements of Theorem 5.7 under a stronger hypothesis on R and will be useful tools in the investigation of Warfield domains. Their proofs must be compared with the proof of Theorem 57 in [M_2], which characterizes commutative integral domains whose ideals are two generated.

PROPOSITION 6.1. *Let R be a local order divisorial domain with non-principal maximal ideal M and let $R_1 = M : M$. Let $\overline{R}$ be the integral closure of R. Then either $\overline{R}$ is a Prüfer order of R with at most two maximal ideals, or $\overline{R} \supseteq \bigcup_n R_n$, where $\{R_n\}_{n \in \mathbb{N}}$ is a chain of overrings of R satisfying the following properties*:

(1) *For every n, $R_n \subsetneq R_{n+1}$ and R_{n+1}/R_n is a simple R-module.*

(2) *For every n, R_n is local with maximal ideal M_n and there exists an element $a \in R$ independent of n, such that $M_n = aR_{n+1}$.*

(3) *$T = \bigcup_n R_n$ is a local domain with principal maximal ideal aT.*

Proof. If R_1 is as in case (2) or (3) of Theorem 5.7, then $\overline{R} = R_1$ and the first statement is proved. Otherwise R_1 is local with maximal ideal M_1 properly containing M. By hypothesis R_1 is a divisorial domain too and, by Lemma 5.5 we may assume M_1 is not principal, otherwise $\overline{R} = R_1$. Define R_2 as $R_1 : M_1 = M_1 : M_1$; applying Proposition 5.7 to the pair (R_1, R_2), we obtain that either R_2 coincides with the integral closure of R_1 (hence of R) and is a Prüfer order of R_1 (hence of R) with at most two maximal ideals, or R_2 is local with maximal ideal M_2 strictly bigger than M_1. Again as above, we may assume that M_2 is not principal. Going on we find that either at a certain nth step R_n is the integral closure of R and is a Prüfer order of R with at most two maximal ideals or, by induction, we obtain a strictly increasing chain of overrings

$$R \subsetneq R_1 \subsetneq R_2 \subsetneq \cdots \subsetneq R_n \subsetneq \cdots$$

satisfying the following properties:

(a) $R_{n+1} = R_n : M_n$ is a local order of R with non-principal maximal ideal M_{n+1} properly containing M_n, for every n.

(b) $M = aR_1$ and $M_n = aR_{n+1}$ for every n.

(c) R_{n+1}/R_n is a simple R-module for every n.

(a) This is clear by the above construction.

(b) Since R_1 is divisorial and $M : M = R_1$ we get, by Corollary 5.3, that $M = aR_1$ for some element $a \in R$. By induction, assume $M_n = aR_{n+1}$ for every n. We have $R_{n+2} = R_{n+1} : M_{n+1}$, then $R_{n+2} = R_n : M_n M_{n+1} = R_n : aM_{n+1}$. Now R_n is finitely generated over R, hence divisorial and thus, By Lemma 5.6(1), $R_n : aM_{n+1}$ coincides with $a^{-1}M_{n+1}$, namely $M_{n+1} = aR_{n+2}$.

(c) If $n = 0$ it is clear that R_1/R is a simple R-module. Let $n \geq 1$; consider the exact sequence

$$0 \to R_{n-1}/M_{n-1} \to R_n/M_{n-1} \to R_n/R_{n-1} \to 0.$$

By (3) in the Remark before Lemma 5.6, the first term is isomorphic (as an R_{n-1}-module) to R_n/R_{n-1} which is a simple R-module by inductive hypothesis. The middle term is an R/M-module since $MR_n = aR_n = M_{n-1}$, hence it is a two dimensional R/M-module. Consider now the exact sequence

$$0 \to M_n/M_{n-1} \to R_n/M_{n-1} \to R_n/M_n \to 0.$$

M_n/M_{n-1} is a non-zero proper submodule of the two dimensional R/M-module R_n/M_{n-1}, thus both M_n/M_{n-1} and R_n/M_n are R-isomorphic to R/M. By construction, R_{n+1}/R_n is isomorphic to R_n/M_n as an R_n-module, hence it is a simple R-module.

Properties (a), (b), and (c) clearly prove statements (1) and (2) in the proposition. Let now $T = \bigcup_n R_n$ and let $N = \bigcup_n M_n$. Then, by (b) above, $N = \bigcup_n aR_{n+1} = aT$ and it is clear that T is a local domain with maximal ideal N. ∎

If R is totally divisorial, then condition (3) in the preceding proposition can be improved:

PROPOSITION 6.2. *Let R be a local totally divisorial domain. Then the integral closure $\overline{R}$ of R is a Prüfer domain with at most two maximal ideals. Moreover, either:*

(1) *$\overline{R}$ is an order of R, or*

(2) *$\overline{R}$ is a valuation domain with principal maximal ideal; $\overline{R} = \bigcup_n R_n$, where $\{R_n\}_{n \in \mathbb{N}}$ satisfies properties (1) and (2) of Proposition 6.1.*

Proof. If the maximal ideal M of R is principal, then the conclusion follows by Lemma 5.5. If M is not principal, then by Proposition 6.1, we only have to show that the domain T there defined is a valuation domain. By the hypothesis of totally divisoriality of R, we get that T is divisorial, hence by Lemma 5.5(1), T is a valuation domain. ∎

Remark. Proposition 6.2 says that if the integral closure $\overline{R}$ of the totally divisorial domain R is not finitely generated over R then, in the terminology used in [GH_2, HL], $\overline{R}$ is a Jónsson-extension of R.

Next Proposition 6.5 is crucial to prove our main result, namely that a totally reflexive domain is Warfield. For that we need some preparatory lemmas.

LEMMA 6.3. *Let R be a local totally divisorial domain and $\overline{R}$ the integral closure of R. Then either $\overline{R}$ is a fractional ideal of R or for every $x \in Q \setminus \overline{R}$, $x\overline{R} = xR + \overline{R}$.*

Proof. Assume $\overline{R}$ is not a fractional ideal of R. Then, by Proposition 6.2, $\overline{R}$ is a valuation domain and thus $x^{-1} = t \in \overline{R}$. If we show that $t\overline{R} + R = \overline{R}$, then the conclusion follows.

$t\overline{R} + R$ is an overring of R contained in $\overline{R}$, hence if it is properly contained in $\overline{R}$ it must be one of the orders R_n's defined in Proposition 6.1. Now $t\overline{R} + R = R_n$ implies $t\overline{R} \subseteq R_n$, hence $\overline{R}$ is a fractional ideal of R_n and thus also of R. ∎

LEMMA 6.4. *Let R be a local totally divisorial domain such that the integral closure $\overline{R}$ of R is not finitely generated over R. Then, in the notations of Proposition* 6.1 *the following hold*:

(1) *For every n, $R_{n+1} = R + R\gamma_{n+1}$ for any $\gamma_{n+1} \in R_{n+1} \setminus R_n$.*

(2) *If $\overline{M}$ denotes the maximal ideal of $\overline{R}$, then $\overline{R}/\overline{M}$ is isomorphic to R/M and $\overline{R} = R + a^n\overline{R}$ for every n.*

(3) *$R_m : R_n = a^{n-m}R_n$, for every $m \leq n$.*

(4) *If A is an R-submodule of Q such that $\mathrm{End}_R(A) = R$, then $\mathrm{End}_{R_n}(AR_n) = R_n$, for every n.*

Proof. (1) By (c) in the proof of Proposition 6.1, $R_{n+1} = R_n + R\gamma_{n+1}$ for any $\gamma_{n+1} \in R_{n+1} \setminus R_n$. We induct on n. If $n = 0$ then $R_1 = R + R\gamma_1$. Consider the element $a\gamma_{n+1}$; by (b) in the proof of Proposition 6.1, $a\gamma_{n+1} \in M_n \subseteq R_n$ but $a\gamma_{n+1} \notin M_{n-1}$. Now, if it were $a\gamma_{n+1} \in R_{n-1}$ then $a\gamma_{n+1}$ would be a unit of R_{n-1} contradicting the fact that $a\gamma_{n+1} \in M_n$. Hence, by induction, $R_n = R + a\gamma_{n+1}R$ and thus $R_{n+1} = R + a\gamma_{n+1}R + R\gamma_{n+1} = R + \gamma_{n+1}R$.

(2) Clearly $M\overline{R} = a\overline{R} = \overline{M}$, hence $\overline{R}/\overline{M}$ is an R/M-module. Let now $\bar{u}$ be a unit of $\overline{R}$; then $\bar{u}$ is a unit in some R_n, hence, by Proposition 6.1(1), $\bar{u}$ is congruent with a unit u of R modulo M_n. Thus $\bar{u}$ is congruent to u modulo $\overline{M}$. We have so proved that $\overline{R} = R + a\overline{R}$ and by induction we easily get $\overline{R} = R + a^n\overline{R}$.

(3) This is proved by induction. The case $n = 0$ is clear ($R_0 = R$). Assume the statement true for every $m \leq n$ and consider $R_m : R_{n+1}$; the case $m = n + 1$ is obvious. Assume $m \leq n$, then $R_m : R_{n+1} = R_m : R_n R_{n+1} = a^{n-m}R_n : R_{n+1} = a^{n-m}M_n = a^{n+1-m}R_{n+1}$.

(4) Clearly $R_n \subseteq \mathrm{End}_{R_n}(AR_n) = E_n$. Assume $R_n \subsetneq E_n$; then E_n must contain R_{n+1}, since R_{n+1}/R_n is an essential R_n-submodule of Q/R_n. Thus there exists an element $\gamma_{n+1} \in R_{n+1} \setminus R_n$ such that $\gamma_{n+1}A \subseteq AR_n$, hence $a^n\gamma_{n+1}A \subseteq Aa^nR_n$. By (3), $a^nR_n \subseteq R$ and thus $a^n\gamma_{n+1}A \subseteq A$. Now,

since $\mathrm{End}_R(A) = R$, we obtain that $a^n\gamma_{n+1}$ belongs to R. Using part (1), we get $a^n\gamma R_{n+1} \subseteq R$, namely $a^n \in a^{n+1}R_{n+1}$ (by (3)) which yields the contradiction $a^{-1} \in R_{n+1}$. ▮

PROPOSITION 6.5. *Let R be a local totally divisorial domain. Let A be an R-submodule of Q with $\mathrm{End}_R(A) = R$. Then A is isomorphic to R.*

Proof. Assume first that $\overline{R}$ is a fractional ideal of R, then there exists an element $s \in R$ such that $s\overline{R} \subseteq R$. Consider the $\overline{R}$-module $A\overline{R}$. Clearly $A\overline{R}$ is properly contained in Q and, since $\overline{R}$ is a Prüfer domain with at most two maximal ideals (by Proposition 6.2), then $A\overline{R}$ is a fractional $\overline{R}$-ideal. Hence there exists $t \in \overline{R}$ such that $tA\overline{R} \subseteq \overline{R}$ and thus, $stA \subseteq R$.

If $\overline{R}$ is not a fractional ideal of R, then it is a valuation domain, which is a union of a strictly ascending chain of overrings R_n of R defined as in Proposition 6.1. Without loss of generality we may assume that $A \supsetneq R$. Consider the subset C of $\mathbb{N}$ defined as

$$C = \{n \geq 0 \mid A \cap R_n \subsetneq A \cap R_{n+1}\},$$

$0 \in C$, since R_1 is an essential extension of R. We claim that C is finite. In fact, for every $n \in C$, A contains an element $\gamma_{n+1} \in R_{n+1} \setminus R_n$, then $A \supseteq R + R\gamma_{n+1} = R_{n+1}$, by Lemma 6.4(1). Hence if C were infinite we would obtain $A \supseteq \overline{R}$. Lemma 6.3 ensures that $A + \overline{R}$ is an $\overline{R}$-module, hence we would get the contradiction $\mathrm{End}_R(A) \supseteq \overline{R}$.

Thus C is finite, hence there exists an index $n_0 \geq 1$ such that $A \cap R_{n_0} = A \cap \overline{R}$. We show now that $A + \overline{R}$ cannot be equal to Q.

Assume $A + \overline{R} = Q$; then, if a is the generator of the maximal ideal of $\overline{R}$, we have $a^{-1}A \subseteq A + \overline{R} = \bigcup_n(A + R_n)$. We claim that $a^{-1}A \subseteq A + R_m$ for some $m \in \mathbb{N}$.

In fact $\mathrm{End}_R(a^{-1}A) = R$ and repeating the arguments used above for the module A, we obtain that there exists an index $m \in \mathbb{N}$ such that $a^{-1}A \cap \overline{R} = a^{-1}A \cap R_m$. Now $a^{-1}A \subseteq \bigcup_n(A + R_n)$ means $a^{-1}A = \bigcup_n a^{-1}A \cap (A + R_n) = \bigcup_n A + (a^{-1}A \cap R_n)$, hence $a^{-1}A = A + (a^{-1}A \cap R_m)$ namely $a^{-1}A \subseteq A + R_m$. Now, clearly $A + R_m \subseteq AR_m$ and thus we get that $a^{-1} \in \mathrm{End}(AR_m)$ contradicting Lemma 6.4(4). Thus $A + \overline{R}$ is a proper $\overline{R}$-submodule of Q, hence it is a fractional ideal of $\overline{R}$. This means that there exists an element $r \in R$ such that $r(A + \overline{R}) \subseteq \overline{R}$.

Now $(A + \overline{R})/\overline{R}$ is R-isomorphic to $A/A \cap R_{n_0}$, hence r annihilates the module $A/A \cap R_{n_0}$ or equivalently, $rA \subseteq R_{n_0}$. The overring R_{n_0} is an order of R, hence it is a fractional ideal of R and thus A is fractional too. By Corollary 5.3 it follows that A is a principal R-module. ▮

We can now prove the main result of this section.

THEOREM 6.6. *Let R be a domain. Then R is a Warfield domain if and only if it is totally reflexive.*

Proof. We have only to prove the sufficient condition. Let A be an R-submodule of Q and let $E = \mathrm{End}_R(A)$. By hypothesis E is reflexive, hence h-local. By Theorem 4.8, we must prove that E is A-reflexive, where E can be assumed to be local. E is clearly totally divisorial, hence Proposition 6.5 yields that A is isomorphic to E, hence E is A-reflexive. ∎

Remark. A consequence of Theorem 6.6 is that, in order to prove that a domain R is Warfield, it is not necessary to verify that for every R-submodule A of Q with $E = \mathrm{End}_R(A)$, R is A-divisorial and that $\mathrm{Ext}^1_E(X, A)$ is a torsionfree E-module for every E-submodule X of A, but it is enough to check that every overring S of R is divisorial and $\mathrm{Ext}^1_S(J, S) = 0$ for every ideal J of S (or equivalently that Q/S is an injective S-module).

As noted in the Introduction, Theorem 1.15 in [A] shows that, fixed a rational group A, a torsionfree abelian group G of finite rank satisfying $E_G = E_A$ and $Ot(G) \leq t(A)$ (i.e,. G is an A-torsionless E_A-module) is necessarily E_G-locally free (i.e,. $G \otimes \mathbb{Z}_p$ is a free $\mathbb{Z}_p$-module, provided that $pG \neq G$). Using Proposition 6.5, we can suitably extend this result to A-reflexive modules over general Warfield domains.

PROPOSITION 6.7. *Let R be a Warfield domain, let A be a submodule of Q, and $E = \mathrm{End}(A)$. Then, given an A-reflexive R-module B of finite rank n, B_M is a submodule of a free E_M-module of rank n, for every $M \in \mathrm{Max}(E)$.*

Proof. If B is A-reflexive, then B is an E-submodule of A by Lemma 3.1. Then B_M is an E_M-submodule of $(A^n)_M \cong A^n_M$, for every $M \in \mathrm{Max}(E)$. But A_M is an E_M-submodule of Q such that $\mathrm{End}(A_M) = E_M$, by Corollary 2.4 and since E is h-local, by Proposition 4.9. Now Proposition 6.5 shows that $A_M \cong E_M$, hence B_M is an E_M-submodule of a free E_M-module of rank n. ∎

7. NOETHERIAN AND INTEGRALLY CLOSED WARFIELD DOMAINS

In the first part of this section we characterize the noetherian Warfield domains generalizing the result obtained by Goeters in the case of noetherian domain with finitely generated integral closure [Goe].

By Theorem 4.9 our investigation will be restricted mostly to the case of a local domain. First we notice the following.

PROPOSITION 7.1. *Let R be a totally divisorial domain. R is of Krull dimension one if an donly if R is noetherian.*

Proof. It is well known that a divisorial noetherian domain is of Krull dimension one (see [M_1]). For the converse, notice that a totally divisorial domain is h-local and every localization at a maximal ideal is totally divisorial. Since an h-local domain with noetherian localizations is noetherian (see [M_2, Theorem 26]), we may assume that R is a local domain. Moreover, by Cohen's Theorem (if every prime ideal is finitely generated the ring is noetherian), it is enough to check that the maximal ideal M of R is finitely generated. Assume M is not finitely generated. Consider the integral closure $\overline{R}$ of R; M is an $\overline{R}$-ideal. In fact M cannot be principal over any one of the orders R_n's defined in Proposition 6.1, otherwise M would be finitely generated over R, hence, by Lemma 5.6, M is an $\overline{R}$-ideal. It follows that $R : \overline{R} \supseteq M$. Now $\overline{R}$ is a Prüfer divisorial domain with at most two maximal ideals and its dimension is the same as the dimension of R, which is one. Thus $\overline{R}$ is a noetherian domain and M is finitely generated over $\overline{R}$. This implies that $\overline{R}$ cannot be finitely generated over R, thus, by Proposition 6.2, $\overline{R}$ is a discrete valuation domain with maximal ideal $a\overline{R}$. By Lemma 6.4(3), $R : \overline{R} = \bigcap_n a^n R_n \subseteq \bigcap_n a^n \overline{R} = 0$, thus $M = 0$, a contradiction. ∎

The following lemma will be used to characterize the noetherian Warfield domains.

LEMMA 7.2. *Let R be an order reflexive domain. Then R_M is order reflexive for every maximal ideal M of R.*

Proof. To prove that R_M is order reflexive for every maximal ideal M of R, we follow the same arguments used in the proof of the implication (2) ⇒ (3) of Theorem 57 in [M_2].

Let T be an order of R_M, namely $T = R_M[\alpha_1, \dots, \alpha_n]$ where the elements α_i's are integral over R_M and thus, multiplying by an element $s \in R \setminus M$, we may assume that they are integral over R. Consider the R-order $S = R[\alpha_1, \dots, \alpha_n]$; then $T = S_M$ and, by hypothesis, S is reflexive, hence h-local. Thus $T = S_M$ is h-local too.

To prove that T is reflexive it is enough to show that T_L is reflexive for every maximal ideal L of T (Theorem 4.8). Since S is finitely generated over R, L is of the form N_M for a prime ideal N of S maximal with respect to $N \cap R \setminus M = \emptyset$. Since S is finitely generated over R, N is a maximal ideal of S, and, clearly $T_L = S_N$, hence T_L is reflexive. ∎

We state now the characterization of noetherian Warfield domains. The next result should be compared with [Goe, Theroem 2.1]; the drastic simplification of our proof is due to the fact that we can reduce to the local case.

Notice that the noetherian domains satisfying condition (1) in the next theorem, have been characterized by Bass and Matlis in [Ba, M_2], respectively.

THEOREM 7.3. *Let R be a noetherian domain. The following are equivalent*:

(1) *Every ideal of R is two generated.*

(2) *R is order reflexive.*

(3) *R is totally reflexive.*

(4) *R is a Warfield domain.*

Proof. (1) $\Leftrightarrow$ (2). This is proved in [M_2, Theorem 57].

(2) $\Rightarrow$ (3). R is a reflexive noetherian domain, hence every overring of R is noetherian, of Krull dimension one, and h-local (see [M_1, M_2, K]). Let S be an overring of R; to prove that S is reflexive, we may assume that S is a local domain, by Theorem 4.8. Let N be the maximal ideal of S; consider the prime ideal $P = R \cap N$ and a maximal ideal M of R containing P. Clearly $R_M \subseteq R_P \subseteq S$.

By Lemma 7.2, R_M is order reflexive, hence we may assume that R is a local order reflexive domain.

Consider the integral closure $\overline{R}$ of R. $\overline{R}$ is a Dedekind domain and, if $\overline{R}$ is an order of R, then $\overline{R}$ coincides with one of the rings R_n's defined in Proposition 6.1. If $\overline{R}$ is not finitely generated over R, then, even without the hypothesis of totally divisoriality of R, we get that $\overline{R}$ coincides with the domain $T = \bigcup_n R_n$ defined in Proposition 6.1. In fact, T is a local noetherian domain with principal maximal ideal, hence it is a valuation domain.

In any case, since each R_{m+1} is a unique minimal overring of R_m, we obtain that either S is one of the R_m's or $S \supseteq \overline{R}$. We have so proved that S is either an order of R or an overring of a Dedekind domain and thus in both cases S is reflexive.

(3) $\Rightarrow$ (4). This is by Theorem 6.6.

(4) $\Rightarrow$ (2). This is obvious. ∎

Remark. The Example 3.5 in [HL] in the case of characteristic $p = 2$, is an example of a noetherian local domain with ideals at most two generated (hence a Warfield domain) with non-finitely generated integral closure.

The second part of this section is devoted to characterize the integrally closed Warfield domains.

It is well known (see [H]) that an integrally closed divisorial domain is an h-local Prüfer domain. Moreover we remark also the following fact.

LEMMA 7.4. *Let R be an h-local Prüfer domain. Then every overring of R is h-local.*

Proof. The prime ideals contained in a fixed maximal ideal of R form a chain; moreover, by Theroem 26.1, Cap IV in [G], every maximal ideal of an overring of R is an extension of a prime ideal of R. These facts, together with the h-locality of R, easily give the conclusion. ∎

By the preceding proposition and Theorem 4.9, we can reduce our investigation to the case of a valuation Warfield domain.

First of all we prove some general facts about A-divisoriality of a valuation domain.

PROPOSITION 7.5. *Let R be a valuation domain and A an R-submodule of Q. R is A-divisorial if and only if A is isomorphic to a prime ideal of R.*

Proof. Since we consider proper R-submodules of Q, every such an A is a fractional ideal of R. Let R be A-divisorial. If $A \cong R$, then we know that the maximal ideal M of R is principal, hence $A \cong M$. Thus, without loss of generality we may assume that A is a proper ideal of R. Consider the prime ideal $A^{\#}$ defined in [FS, Cap I, n. 4]. We show that $A \cong A^{\#}$. In fact, if $A \ncong A^{\#}$, then, by Lemma 1.1 in [BFS], we obtain $A : A^{\#} = A$ and thus $A:(A:A^{\#}) = R_{A\#} \supsetneq A$. But $A^{\#}$ is an A-torsionless $\mathrm{End}_R(A)$-module, hence we have a contradiction to the A-divisoriality of R.

Conversely, let P a prime ideal of R; we must show that R is P-divisorial. By Theorem 4.7 this is equivalent to showing that $\mathrm{End}_R(P) = R_P$ is P-divisorial. Hence we may assume that P is the maximal ideal of R. Let I be any ideal of R; we must show that $P:(P:I) = I$. Clearly $I \subseteq P:(P:I)$. Let $r \notin I$, then $I \subsetneq rR$ implies $I \subseteq rP$, hence $r^{-1} \in P:I$. Now $r \notin P:(P:I)$, since $rr^{-1} \notin P$. ∎

DEFINITION [FHP]. A Prüfer domain is said to be *strongly discrete* if every non-zero prime ideal is not idempotent.

It is now easily to characterize the totally divisorial valuation domains.

PROPOSITION 7.6. *Let R be a valuation domain. The following are equivalent*:

(1) *R is A-divisorial for every R-submodule A of Q.*
(2) *Every ideal of R is isomorphic to a prime ideal of R.*
(3) *Every non-zero prime ideal P of R is principal over R_P.*
(4) *R is strongly discrete.*
(5) *R is totally divisorial.*

Proof. (1) $\Leftrightarrow$ (2). Every proper R-submodule of Q is isomorphic to an ideal of R, hence the equivalence between the two conditions follows by Proposition 7.5.

The equivalences (2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (4) easily follow by Lemma 4.8, Cap I in [FS]. (See also [FHP].)

(5) ⇔ (3). By [H, Lemma 5.2], a valuation domain is divisorial if and only if the maximal ideal is principal, hence R is totally divisorial if and only if every localization at a prime ideal P is divisorial. ∎

It is well known, after Matlis results, that a valuation domain R is reflexive if and only if the maximal ideal of R is principal and R is almost maximal. Moreover, an overring of an almost maximal valuation domain is almost maximal too. Thus, in view of the preceding results, the following proposition is immediate.

PROPOSITION 7.7. *Let R be an integrally closed local domain. The following are equivalent*

(1) *R is a Warfield domain.*

(2) *R is a strongly discrete almost maximal valuation domain.*

(3) *R is a totally reflexive domain.*

Proof. As already noticed, a divisorial integrally closed domain is a Prüfer domain, hence (1) ⇒ (2) and (2) ⇒ (3) follow by Proposition 7.6 and the remark preceding this Theorem.
(3) ⇒ (1). This follows by Theorem 6.6.

Brandal [Br] gives the following definition.

DEFINITION. A commutative ring is *almost maximal* if every proper homomorphic image of R is linearly compact.

Theorem 2.9 in [Br] states that an integral domain R is almost maximal if and only if it is h-local and every localization of R at a maximal ideal is almost maximal. Using this result we may state the following characterization of integrally closed Warfield domains.

THEOREM 7.8. *Let R be an integrally closed domain. The following are equivalent*:

(1) *R is a Warfield domain.*

(2) *R is h-local and R_M is an almost maximal strongly discrete valuation domain, for every maximal ideal M of R.*

(3) *R is an almost maximal strongly discrete Prüfer domain.*

(4) *R is a totally reflexive domain.*

Proof. (1) ⇒ (2). This follows by Theorem 4.9 and Proposition 7.7.
(2) ⇔ (3). This follows by Theorem 2.9 in [Br] and Proposition 5.3.5 in [FHP].
(3) ⇒ (4). Every overring S of R is h-local, by Lemma 7.4. To show that it is reflexive, it is enough to assume that S is local. But then S is a

localization of R at a prime ideal, hence S is clearly almost maximal and strongly discrete, thus it is reflexive, by Proposition 7.7.

(4) $\Rightarrow$ (1). This follows by Theorem 6.6. ∎

We give now an example of a non-noetherian and non-integrally closed Warfield domain.

EXAMPLE. Let $k_0 \subsetneq K$ be fields such that there are no intermediate fields between k_0 and K.

Let $V = K[[\Gamma]]$ be the ring of the formal power series of Γ over K, where $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ with the lexicographic order.

V is a maximal strongly discrete valuation domain of rank 2, hence V is a Warfield domain. Let $0 \subsetneq P \subsetneq M$ be the only prime ideals of V. Consider the domain $R = k_0 + M$. Then $R : M = V$ and $R : V = M$. Moreover V is the unique minimal overring of R and thus R is a Warfield domain if and only R is reflexive. It is easy to check that R is divisorial and using the fact that Q/V is an injective V-module and that V/R is a simple R-module, one can verify that Q/R is injective as an R-module, hence R is reflexive.

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