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Universally Catenarian Integral Domains

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The rings of the title are the (not necessarily Noetherian) integral domains R such that $R[X_1, ..., X_n]$ is catenarian for each positive integer n. It is proved that each such R must be a stably strong S-domain, in the sense of Malik–Mott. The class of all universally catenarian integral domains is characterized as the largest class of catenarian integral domains which is stable under factor domains and localizations and whose members R satisfy the altitude formula and $\dim_v(R) = \dim(R)$. Moreover, the following theorem is given, generalizing Ratliff's result that each one-dimensional Noetherian integral domain is universally catenarian. Let R be a locally finite-dimensional going-down domain; then R is universally catenarian if and only if the integral closure of R is a Prüfer domain. Other results and applications are also given.

1. INTRODUCTION

All rings considered in this article are commutative with identity, and all ring-homomorphisms are unital. A ring R is said to be *catenarian* in case, for each pair $P \subset Q$ of prime ideals of R, all saturated chains of primes from P to Q have a common finite length. We shall say that R is *universally catenarian* if the polynomial rings $R[X_1, ..., X_n]$ are catenarian for each

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positive integer n. (Note that this terminology differs from that of the French school [23, (5.6.1) and (5.6.2)] who require universally catenarian rings to be Noetherian as well. The variety of influential terminology has been surveyed by Ratliff [36, pp. 7–10].) This article contributes to the study of universal catenarity with two types of main results. First we axiomatically characterize the class of all universally catenarian (integral) domains (see Theorem 5.1(b)); second, we present several new subclasses of universally catenarian domains (Theorems 6.2 and 8.1, Corollaries 6.7 and 6.4).

We shall next describe three families of universally catenarian rings. The most familiar of these, of considerable importance in algebraic geometry, consists of all Cohen–Macaulay rings (cf. [29, Theorem 31]). In particular, arbitrary (commutative) affine algebras over a field and regular local rings are universally catenarian. The second family of examples consists of all Noetherian domains of (Krull) dimension 1. This may be seen as an application of Ratliff's result [35, (2.6)] that a Noetherian ring R is universally catenarian if (and only if) $R[X]$ is catenarian; the point is that two-dimensional domains are trivially catenarian. Before introducing the third family, we note that each catenarian domain must be *locally finite-dimensional* (for short, LFD), in the sense that each of its prime ideals has finite height. Now we can describe the third family of examples: it consists of all LFD Prüfer domains. Their universal catenarity was recently established independently—and in very different ways—by Malik–Mott [28, p. 256] and Bouvier–Fontana [4, Theorem 12], extending a number of earlier partial results (cf. [3, 16]). Two of our main results, Theorems 6.2 and 8.1, each simultaneously generalize the second and third families of examples.

A special case of Theorem 8.1 is developed earlier as Corollary 6.4. It states that each domain of valuative dimension 1 must be universally catenarian; thus Corollary 6.4 is strong enough to recover the second family of examples described above. (For background on valuative dimension, we recommend [2, 26].) Corollary 6.4 is a consequence of Theorems 6.1 and 6.2. The first of these is a descent result for universal catenarity suggested by a remark of Ratliff [36, p. 13] about descent of catenarity and employing a type of "going-down" hypothesis. Theorem 6.2 establishes that a going-down domain R (in the sense of [8]) is universally catenarian if and only if its integral closure is an LFD Prüfer domain. Much of Sections 6 and 7 consists of an intensive study of universal catenarity for going-down domains, with particular applications to the pseudovaluation domains of Hedstrom–Houston [24] (thereby extending results in [25, 19]) and, more generally, to the globalized contexts introduced in [13]. All such applications are examples of locally divided domains (in the sense of [12]) and lead naturally to the setting for the above-mentioned Theorem 8.1.

The work in Section 7 is possible thanks, in part, to Theorem 5.1. Among other things, this main result characterizes the class of universally catenarian domains as the largest class of rings satisfying certain conditions. These include the "altitude formula" and equality between dimension and valuative dimension. The necessity of these conditions is established, of course without Noetherian hypotheses, in Sections 3 and 4, by elaborating upon the "*-function" technique which had been introduced in [4] in order to study LFD Prüfer domains. As a curious by-product, universal catenarity leads, in Corollary 3.4, to yet another characterization of LFD Prüfer domains. Our debt to the approach in [28] is mostly motivational, although we begin by showing in Theorem 2.4 that our (not necessarily Noetherian) notion of universally catenarian domain admits a theory, implying the "stably strong S-" property introduced in [28]. As a result, Remark 2.5 recovers several results of [28].

As Example 8.3 reveals, Corollary 6.4 is the best possible, in that "$\dim(R)=2=\dim_v(R)$" for a domain R does not imply that R is universally catenarian. This counterexample is derived from a construction studied in [13] and depends ultimately on the pullback techniques in [17]. It points out the importance of the "coequidimensional" hypothesis in Theorem 8.1 and suggests the need to characterize the condition that dimension and valuative dimension agree at all localizations. The latter task is accomplished in Proposition 9.3.

It cannot be expected that the typical reader is conversant with all this article's references. Therefore, in order to shorten this introduction, we have chosen to recall relevant definitions and facts as needed throughout the article. Any unexplained material is standard, as in [22, 27, 32].

2. Universally Catenarian Domains Are Stably Strong S-Domains

Let R be a domain. Following [27], we say that R is an *S-domain* if, for each height 1 prime ideal P of R, the extended prime $PR[X]$ has height 1 in the polynomial ring $R[X]$; and R is said to be a *strong S-domain* if R/P is an S-domain for each prime ideal P of R. The most natural examples of strong S-domains are arbitrary Noetherian domains (cf. [27, Theorem 148]) and arbitrary valuation domains [27, Theorem 68]. It is known that $\dim(R[X])=\dim(R)+1$ for each strong S-domain R (cf. [27, Theorem 39]); and that the class of strong S-domains is stable with respect to localization [28, Corollary 2.4].

Despite the above material, the class of S-domains is not very stable, for instance with respect to polynomial extensions. Following [28], we say that R is a *stably strong S-domain* if $R[X_1, ..., X_n]$ is a strong S-domain for each nonnegative integer n. It was shown by means of a two-dimensional example in [5, p. 40] that a strong S-domain need not be a stably strong S-domain. Moreover, there exists an infinite-dimensional Krull domain which is not a stably strong S-domain [28, Example 4.17]. Theorem 2.4 will provide new examples of stably strong S-domains, by establishing the result announced in this section's title. For additional motivation, we recall that a catenarian domain need not be a strong S-domain: a one-dimensional pseudovaluation domain illustrating this phenomenon can be found using [25, Propositions 2.2 and 2.4].

PROPOSITION 2.1. *Let R be a domain, I an infinite index set satisfying $|R| \leqslant |I|$, and $\{X_i: i \in I\}$ a collection of algebraically independent indeterminates over R. Then $T = R[\{X_i: i \in I\}]$ is not a strong S-domain.*

Proof. *Case* 1. R is infinite. Let K denote the quotient field of R. With Y and Z denoting new variables over K, we consider $V = K(Y)[Z]_{(Z)}$, a DVR with maximal ideal $M = ZV$. Since $V = K(Y) + M$, it is natural to examine $B = K + M$. As is well known (cf. [21, Theorem A, p. 560]), B is a one-dimensional integrally closed non-Prüfer domain. Accordingly, by [37, Theorem 2] and [26, Corollaire 3, p. 61; and Corollaire 1, p. 67], $\dim(B[X]) = 3$. Thus, by the above remarks, B is not a strong S-domain. However, the usual laws of infinite cardinal arithmetic (assuming Zorn's lemma) guarantee that $|B| = |R|$. Thus, with D denoting $R[\{Y_b: b \in B\}]$, we obtain surjective R-algebra homomorphisms $T \to D$ and $D \to B$. (The former is available since $|B| \leqslant |I|$; the latter via $Y_b \mapsto b$.) By composition, B is an (R-algebra) homomorphic image of T. However, it is easy to see that each factor domain of a strong (resp., stably strong) S-domain is itself a strong (resp., stably strong) S-domain. The assertion about T therefore follows from the above observation about B.

Case 2. R is finite. Select $j \in I$. Set $A = R[X_j]$ and $J = I \setminus \{j\}$. Note that A is an infinite domain and $|A| = \aleph_0 \leqslant |I| = |J|$. Thus, by Case 1, $C = A[\{X_i: i \in J\}]$ is not a strong S-domain. Since $T = C$, the proof is complete.

COROLLARY 2.2. *Let R be a countable Noetherian domain and $\{X_i: i \in I\}$ a collection of algebraically independent indeterminates over R. Then $T = R[\{X_i: i \in I\}]$ is a strong S-domain if and only if I is finite.*

Proof. If I is finite, T is Noetherian by the Hilbert basis theorem, and so T is a strong S-domain by the above remarks. If I is infinite, $|R| \leqslant |I|$, and so Proposition 2.1 applies, completing the proof.

The above results show that $\mathbb{Q}[X_1, X_2, \ldots]$ is not a strong S-domain, with a similar conclusion for $\mathbb{R}[\{X_i: i \in I\}]$ whenever $|\mathbb{R}| \leqslant |I|$. But what about $\mathbb{R}[X_1, X_2, \ldots]$? In a related vein, does there exist a finite-dimensional

Krull domain (preferably a UFD) which is not a strong S-domain? We do not know the answers to these questions.

LEMMA 2.3. *Let R be a domain such that $R[X]$ is catenarian. Then R is a strong S-domain.*

Proof. If I is a prime ideal of R, then $T=(R/I)[X]$ is also catenarian since $T \cong R[X]/IR[X]$ is a homomorphic image of $R[X]$. Hence our task reduces to showing that R is an S-domain. To this end, consider any height 1 prime ideal p of R. We shall show that if P is a nonzero prime ideal of $R[X]$ contained in $pR[X]$, then $P=pR[X]$. Set $Q_1=XR[X]$ and $Q_2=pR[X]+Q_1$. We claim that $0 \subset Q_1 \subset Q_2$ is a saturated chain of primes in $R[X]$. Indeed, since $Q_1 \cap R=0$, it is well known (cf. [27, Theorem 38(b)]) that Q_1 has height 1. Moreover, Q_1 and Q_2 are adjacent since $Q_2/Q_1 \cong p$, proving the claim. By catenarity of $R[X]$, the chain $0 \subset P \subset pR[X] \subset Q_2$ therefore contains at most three distinct primes, so that $P=pR[X]$ by process of elimination.

THEOREM 2.4. *Let R be a universally catenarian domain. Then R is a stably strong S-domain and, in particular,* $\dim(R[X_1, ..., X_n]) = \dim(R)+n$ *for each nonnegative integer n.*

Proof. If $n \geqslant 0$ and $R_n=R[X_1, ..., X_n]$, then $R_n[X] \cong R[X_1, ..., X_{n+1}]$ is catenarian by hypothesis, and so R_n is a strong S-domain by Lemma 2.3. This proves the first assertion. Moreover, since each R_n is a strong S-domain, $\dim(R_{n+1})=\dim(R_n)+1$ by our earlier remarks, from which the second assertion follows by iteration.

Remark 2.5. (a) By combining Theorem 2.4 with either [28, p. 256] or [4, Theorem 1.2], we see that each LFD Prüfer domain is a stably strong S-domain. The "LFD" hypothesis can be removed using changes-of-rings and the going-down property, as in [28, p. 254, lines 16–27]. Thus we recover [28, Theorem 3.5]: each Prüfer domain is a stably strong S-domain. It is then also easy to recover [28, Proposition 2.5 and Corollaries 3.4 and 3.6].

(b) It is now simple to produce a stably strong S-domain which is neither Noetherian nor a Prüfer domain. Indeed, if D is an LFD Prüfer domain of dimension at least 2, then $D[X]$ has the asserted properties.

3. SOME NECESSARY CONDITIONS FOR UNIVERSAL CATENARITY

The first of the conditions referred to in this section's title concerns valuative dimension. As usual, if R is a domain, the *valuative dimension of*

R, denoted $\dim_v(R)$, is the supremum (possibly ∞) of $\dim(T)$ as T ranges over the set of overrings of R. (It is well known that attention may be restricted to valuation overrings T, which explains the terminology.) In general, $\dim(R) \leqslant \dim_v(R)$. Equality holds in several important cases, for instance if R is Noetherian or a Prüfer domain (cf. [22, Corollary 30.10]); however, $\dim_v(R) - \dim(R)$ may, in general, be arbitrarily large (cf. [21, pp. 572–573]). Equality is characterized in Lemma 3.1(a) below. For further motivation, note that the proof that $\dim(B[X]) = 3$ in Proposition 2.1 implicitly involved showing that $\dim_v(B) \neq \dim(B)$; and that condition (2) in Lemma 3.1(a) was the final assertion in Theorem 2.4.

LEMMA 3.1 (cf. Jaffard [26, Corollaire 1, p. 67]). *Let R be a domain. Then*:

(a) *The following two conditions are equivalent*:

(1) $\dim_v(R) = \dim(R)$;

(2) $\dim(R[X_1, ..., X_n]) = \dim(R) + n$ *for each nonnegative integer n.*

(b) $\dim_v(R) = n$ *if and only* $\dim(R[X_1, ..., X_n]) = 2n$.

As [28, Example 3.11] shows, the condition "$\dim_v(R) = \dim(R)$" is not strong enough to imply that R is a (stably) strong S-domain, and hence, by Theorem 2.4, does not imply that R is universally catenarian. However, pursuing this condition will ultimately lead, in Theorem 5.1(b), to a characterization of the class of all universally catenarian domains.

PROPOSITION 3.2. *Let $\mathbf{C}$ be a class of domains stable under localizations (at primes). Then the following conditions are equivalent*:

(1) $\dim_v(R) = \dim(R)$ *for each R in $\mathbf{C}$*;

(2) $\mathrm{ht}(pR[X_1, ..., X_n]) = \mathrm{ht}(p)$ *for each domain R in $\mathbf{C}$, prime ideal p of R of finite height, and nonnegative integer n*;

(3) $\mathrm{ht}(pR[X_1, ..., X_n]) = \mathrm{ht}(p)$ *for each quasilocal domain R in $\mathbf{C}$, prime ideal p of R of finite height, and nonnegative integer n.*

Proof. It is trivial that (2) ⇒ (3). Moreover, (3) ⇒ (1) follows by combining [5, Corollary 2] and Lemma 3.1(a). It therefore suffices to prove that (1) ⇒ (2). To this end, assume (1), and consider $R \in \mathbf{C}$ and $p \in \mathrm{Spec}(R)$ of finite height. Since $\mathbf{C}$ is stable under localizations and $q = pR_p$ satisfies $\mathrm{ht}(qR_p[X_1, ..., X_n]) = \mathrm{ht}(pR[X_1, ..., X_n])$ for each $n \geqslant 0$, we may evidently suppose R quasilocal, with maximal ideal p.

We show next by induction on $n \geqslant 0$ that $p^{\#} = pR[X_1, ..., X_n]$ is contained in some maximal ideal M_n of $R[X_1, ..., X_n]$ such that $\mathrm{ht}(M_n/p^{\#}) = n$. This is evident for $n = 0$, as $M_0 = p$ then suffices. For the induction step, take $n \geqslant 1$ and M_{n-1} a maximal ideal of $R[X_1, ..., X_{n-1}]$ containing $pR[X_1, ..., X_{n-1}]$ such that $\mathrm{ht}(M_{n-1}/pR[X_1, ..., X_{n-1}]) = n-1$. By Zorn's Lemma, the nonmaximal ideal $M_{n-1}[X_n]$ is contained in some maximal ideal P of $R[X_1, ..., X_n]$. Then

$$\mathrm{ht}(P/p^{\#}) \geqslant \mathrm{ht}(P/M_{n-1}[X_n]) + \mathrm{ht}(M_{n-1}[X_n]/p^{\#}) \geqslant 1 + (n-1) = n.$$

Thus, $\mathrm{ht}(P) \geqslant \mathrm{ht}(P/p^{\#}) + \mathrm{ht}(p^{\#}) \geqslant n + \mathrm{ht}(p) = n + \dim(R)$. On the other hand, by Lemma 3.1(a), it follows that

$$\mathrm{ht}(P) \leqslant \dim(R[X_1, ..., X_n]) = \dim(R) + n.$$

Hence $\mathrm{ht}(P/p^{\#}) = n$ [so that choosing $M_n = P$ completes the proof of the induction step] and $\mathrm{ht}(p^{\#}) = \dim(R) = \mathrm{ht}(p)$.

With the maximal ideal M_{n-1} now known to exist, we revisit the preceding argument, obtaining $\mathrm{ht}(p^{\#}) = \mathrm{ht}(p)$, for each $n \geqslant 1$. Thus (2) holds, completing the proof.

COROLLARY 3.3. *Let* $\mathbf{C}$ *denote the class of all universally catenarian domains. Then* $\mathbf{C}$ *is stable under factor domains and localizations; and, if* $R \in \mathbf{C}$, *then* R *is catenarian and* $\dim_v(R) = \dim(R)$.

Proof. The first two assertions follow by noting that if $p \in \mathrm{Spec}(R)$, then $(R/p)[X_1, ..., X_n] \cong R[X]/pR[X_1, ..., X_n]$ and $R_p[X_1, ..., X_n] \cong R[X_1, ..., X_n]_{R \setminus p}$. The third assertion is trivial, while the fourth is a consequence of Theorem 2.4 and Lemma 3.1(a).

Universal catenarity will be characterized in Section 5 with the aid of the necessary conditions listed in Corollary 3.3, the variants in Proposition 3.2, and another condition to be introduced in Section 4. We shall close this section with a result having the same moral as Theorem 2.4, namely that "universally catenarian" implies more than "catenarian" does. First, it is convenient to recall from [8] that a domain R is said to be a *going-down domain* if $R \subset T$ satisfies going-down (GD) for each domain T containing R; by [15, Theorem 1], it is enough to restrict the test extensions T to be valuation overrings of R. The most natural examples of going-down domains are arbitrary Prüfer domains, arbitrary one-dimensional domains, and certain pullbacks. The next result belongs to the *genre* (cf. [7, Corollary 4; 8, Proposition 2.7]) characterizing Prüfer domains within the class of integrally closed going-down domains. It is especially motivated by the universal catenarity of LFD Prüfer domains [28, 4].

COROLLARY 3.4. *For an LFD domain* R, *the following conditions are equivalent*:

(1) *R is an integrally closed, universally catenarian going-down domain;*

(2) *R is a Prüfer domain.*

Proof. In view of the above remarks, we need only prove $(1) \Rightarrow (2)$. Assume (1). It suffices to show that R_m is a valuation domain for each maximal ideal m of R. Observe that R_m is a quasilocal, integrally closed going-down domain. Accordingly, by [8, Proposition 2.7], it suffices to prove that $\dim_v(R_m) = \dim(R_m)$. However, this equality follows from the universal catenarity of R, by virtue of Corollary 3.3, and so the proof is complete.

Remark 3.5. One cannot replace "universally catenarian" with "catenarian" in the statement of Corollary 3.4. A one-dimensional illustration of this is provided by the ring B introduced in the proof of Proposition 2.1.

4. The Star Formula and the Altitude Formula

The main result of this technical section, Corollary 4.8, is another, and perhaps the most important, necessary condition for universal catenarity. It will be obtained after reexamining, in greater generality, the star operation introduced in [4].

It is convenient to use the following notation. If $p \in \operatorname{Spec}(A)$, then p^* and $p^{\#}$ denote $pA[X]$ and $pA[X_1, ..., X_n]$, respectively. Besides p^*, the other primes of $A[X]$ lying over p are known to be the uppers $\langle p, \alpha \rangle$ arising from monic irreducible polynomials α with coefficients in $\mathbf{k}(p) = A_p/pA_p$. (See [30, Theorem 1] or [27, p. 25] for information about uppers.)

The key definition, given $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$, is of a certain integer, $*Q$. In detail, set $q^{(k)} = Q \cap R[X_1, ..., X_k]$ for $1 \leqslant k \leqslant n$ and $q^{(0)} = Q \cap R$. Since $q^{(k)*} = q^{(k)}[X_{k+1}] \subset q^{(k+1)}$ whenever $0 \leqslant k \leqslant n$, it is natural to consider the cardinality

$$*Q = *_R Q = |\{k : 0 \leqslant k < n \text{ and } q^{(k)*} \neq q^{(k+1)}\}|.$$

Evidently, $0 \leqslant *Q \leqslant n$. Analysis of this star operation often involves quasilocal rings, since one readily checks that $*_{R_{q^{(0)}}} Q = *_R Q$, a fact used below in the proof of Corollary 4.2.

Proposition 4.1. *Let (R, m) be a quasilocal domain and let $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$ be such that $Q \cap R = m$. Then $*Q = \operatorname{ht}(Q/m^{\#})$.*

Proof. Case $n = 1$. Then either $Q = m^* (= m^{\#})$ or Q is an upper $\langle m, \alpha \rangle$. Observe that $*(m^*) = 0 = \operatorname{ht}(m^*/m^*)$ and $*(\langle m, \alpha \rangle) = 1 = \operatorname{ht}(\langle m, \alpha \rangle / m^*)$.

Induction (on n) step: consider $n \geqslant 2$. As $k = R/m$ is a field, $A = k[X_1, ..., X_n]$ is catenarian, whence

$$\mathrm{ht}_A(Q/m^{\#}) = \mathrm{ht}_A(Q/q^{(n-1)*}) + \mathrm{ht}_A(q^{(n-1)*}/m^{\#}).$$

By the definition of $*Q$, it therefore suffices to show that $*(q^{(n-1)}) = \mathrm{ht}_A(q^{(n-1)*}/m^{\#})$. Consider the (Noetherian) domain $B = R[X_1, ..., X_{n-1}]/m[X_1, ..., X_{n-1}]$ canonically identified with $k[X_1, ..., X_{n-1}]$. By the induction hypothesis, $*(q^{(n-1)})$ coincides with $\mathrm{ht}_B(q^{(n-1)}/m[X_1, ..., X_{n-1}])$ which, since B is a strong S-domain, reduces by virture of [27, Theorem 39] to $\mathrm{ht}_{B[X_n]}((q^{(n-1)}/m[X_1, ..., X_{n-1}])^*) = \mathrm{ht}_A(q^{(n-1)*}/m^{\#})$. The proof is complete.

COROLLARY 4.2. *Let* **C** *be a class of domains stable under localizations such that* $\dim_v(A) = \dim(A)$ *for each* A *in* **C**. *Then* $\mathrm{ht}(Q) = *Q + \mathrm{ht}(q^{(0)})$ *for each* R *in* **C** *and* $Q \in \mathrm{Spec}(R[X_1, ..., X_n])$.

Proof. By the observation preceding Proposition 4.1, we may replace R with $R_{q^{(0)}}$, and hence suppose that $q^{(0)}$ is the unique maximal ideal m of R. By [5, Theorem 1], $\mathrm{ht}(Q) = \mathrm{ht}(m^{\#}) + \mathrm{ht}(Q/m^{\#})$. The proof concludes by observing that $\mathrm{ht}(m^{\#}) = \mathrm{ht}(m)$ by Proposition 3.2 [(1)$\Rightarrow$(2)] and $\mathrm{ht}(Q/m^{\#}) = *Q$ by Proposition 4.1.

It will be convenient to say that a class **C** of domains *satisfies the star formula* in case the conclusion of Corollary 4.2 holds.

COROLLARY 4.3. *The class of all universally catenarian domains satisfies the star formula.*

Proof. Combine Corollaries 3.3 and 4.2.

We shall say that a class **C** of domains *satisfies the altitude formula* if $\mathrm{ht}(Q) = \mathrm{ht}(q) + t \cdot d_R(T) - t \cdot d_{R/q}(T/Q)$ whenever $R \subset T$ is an inclusion of domains such that R is in **C**, T is a finite-type R-algebra, $Q \in \mathrm{Spec}(T)$, and $q = Q \cap R$ is of finite height in R. (This terminology is a slight variant of the usual one: cf. [36, pp. 6–7].) It is well known (cf. [29, Corollary 14.D]) that a Noetherian domain R is universally catenarian if and only if R is catenarian and the class $\{R/p: p \in \mathrm{Spec}(R)\}$ satisfies the altitude formula. (See also [38] in this regard.) Our next five results will culminate in the fact that the class of universally catenarian domains satisfies the altitude formula. Along the way, we must exercise care since, for instance, a class of domains satisfying the hypotheses of Corollary 4.2 need not satisfy the altitude formula. To see this, consider the class of catenarian Noetherian domains, and recall that there exist noncatenarian three-dimensional Noetherian domains of the form $A[X]$: see the analysis by Zariski–Samuel

[39, pp. 328–329] of a famous construction of Nagata [31]. (Confere also [32, Example 2, p. 203]. Incidentally, Ogoma [34] has constructed a non-catenarian three-dimensional integrally closed Noetherian domain.)

LEMMA 4.4. *Let R be a domain, $p \in \operatorname{Spec}(R)$, and $P = \langle p, \alpha \rangle$ an upper of p. Then the residue field $K = \mathbf{k}(P)$ is an algebraic extension of the residue field $k = \mathbf{k}(p)$.*

Proof. By the construction of uppers, P is the kernel of the composite homomorphism

$$R[X] \twoheadrightarrow (R/p)[X] \subsetneq k[X] \twoheadrightarrow k[X]/\alpha k[X] = k(x),$$

where x denotes $X + \alpha k[X]$. Thus we have injective homomorphisms of domains,

$$R/p \to R[X]/P \to k(x),$$

inducing a tower of quotient fields, $k \subsetneq K \subsetneq k(x)$. Since x is a root of α, $k(x)$ is algebraic over k and, a fortiori, so is K, completing the proof.

Next, we need to broaden the definition of the star function, by taking into account the possibility of rearranging the indeterminates X_i. Given $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$ and a permutation $\sigma \in S_n$, set

$$q^{(\sigma(k))} = Q \cap R[X_{\sigma(1)}, ..., X_{\sigma(k)}], \qquad \text{for } 1 \leqslant k \leqslant n,$$

$q^{(\sigma(0))} = Q \cap R (= q^{(0)})$, and consider the cardinality

$$*_\sigma Q = *_{\sigma, R} Q = |\{k : 0 \leqslant k < n \text{ and } q^{(\sigma(k))*} = q^{(\sigma(k))}[X_{\sigma(k+1)}] \neq q^{(\sigma(k+1))}\}|.$$

Notice that if σ is the identity permutation, then $*_\sigma Q$ coincides with $*Q$ as defined earlier. In fact, we have in general

LEMMA 4.5. *If a domain R is LFD and $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$, then $*_\sigma Q = *Q$ for each $\sigma \in S_n$.*

Proof. By the proof of Corollary 4.2, $\operatorname{ht}(Q) = *Q + \operatorname{ht}(q^{(0)\#})$. Working inside the polynomial ring viewed as $R[X_{\sigma(1)}, ..., X_{\sigma(n)}]$, we similarly obtain $\operatorname{ht}(Q) = *_\sigma Q + \operatorname{ht}(q^{(\sigma(0))\#})$. Since $q^{(\sigma(0))\#} = q^{(0)\#}$ has finite height (by the LFD property of R), the conclusion follows easily.

We come next to the main technical result of this section.

LEMMA 4.6. *If a domain R is LFD and $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$, then there exists $\sigma \in S_n$ such that*

$$*q^{(\sigma(i))} = 0, \qquad \textit{for } 1 \leqslant i \leqslant n - *Q$$

and

$$*q^{(\sigma(i+1))} = 1 + *q^{(\sigma(i))}, \qquad \textit{for } n - *Q \leqslant i \leqslant n-1.$$

Proof. Set $t = *Q$. If $t = 0$ then each $\sigma \in S_n$ has the asserted properties, and so we may suppose that $t \geqslant 1$. The proof will proceed by induction on $n \geqslant 1$. The case $n = 1$ is simple, for then $t = 1$ and the only available σ suffices. We pass therefore to the induction step, taking $n \geqslant 2$. If $q^{(n-1)}[X_n] \neq Q(=q^{(n)})$, apply the induction hypothesis to $q^{(n-1)}$, obtaining a certain $\tau \in S_{n-1}$; extending τ by $n \mapsto n$ then produces a permutation $\sigma \in S_n$ with the asserted properties, the point being that $n - *Q = n - 1 - (*q^{(n-1)})$. We may henceforth suppose that $q^{(n-1)}[X_n] = Q$; in particular, $t = *q^{(n-1)}$.

Applying the induction hypothesis to $q^{(n-1)}$ now produces $\pi \in S_{n-1}$ such that

$$*q^{(\pi(i))} = 0, \qquad \text{for } 1 \leqslant i \leqslant n-1-t$$

and

$$*q^{(\pi(i+1))} = 1 + *q^{(\pi(i))}, \qquad \text{for } n-1-t \leqslant i \leqslant n-2.$$

Consider the ring $T = R[X_{\pi(1)}, ..., X_{\pi(n-2)}]$ and set $Y_1 = X_{\pi(n-1)}$, $Y_2 = X_n$, $q = Q \cap T$, and $q_1 = Q \cap T[Y_1]$. (Since $T[Y_1] = R[X_1, ..., X_{n-1}]$, $q_1 = q^{(n-1)}$; however, considering polynomials with coefficients in T, we see from the definition of q_1 that $q_1 = q^{(1)}$. Since the $q^{(i)}$ notation suppresses reference to the base ring, it is ambiguous in regard to q_1, and henceforth avoided.) Since $q_1[X_n] = Q$, $*_T Q$ is just $*_T q_1$ which, by taking $i = n-2$ above, is 1. Express q_1 as an upper, $q_1 = \langle q, \alpha \rangle$, where $\alpha \in \mathbf{k}_T(q)[Y_1]$ is a monic irreducible polynomial. Essentially by the lemma of Gauss, α remains irreducible in $\mathbf{k}_T(q)(Y_2)[Y_1]$, which may be viewed as $\mathbf{k}_{T[Y_2]}(qT[Y_2])[Y_1]$. Thus we may consider the prime ideal $P = \langle qT[Y_2], \alpha \rangle$ of $T[Y_2, Y_1](= R[X_1, ..., X_n])$. We shall show next that $*_T P = 1$.

Observe that $P \cap T = q$. Hence, by Lemma 4.5, each permutation λ of $\{1, 2\}$ satisfies $*_{\lambda, T} P + *_R q = *_R Q$. It follows that $*_T P = *_{(12), T} P$. As P is an upper of $P \cap T[Y_2] = qT[Y_2]$, which in turn is an extended prime, it is evident that $*_{(12), T} P = 1$. Hence $*_T P = 1$, as asserted.

By the construction of uppers, $\langle q, \alpha \rangle \subset \langle qT[Y_2], \alpha \rangle (= P)$; thus, $Q = q_1[Y_2] \subset P$. Moreover,

$$*_R P = *_T P + *_R q = 1 + *_R q = *_T Q + *_R q = *_R Q.$$

Thus, by two applications of the variant of Corollary 4.2 noted in the proof of Lemma 4.5, $\text{ht}(P) - \text{ht}(p^{(0)\#}) = \text{ht}(Q) - \text{ht}(q^{(0)\#})$. However, $p^{(0)} = P \cap R = q \cap R = q^{(0)}$, whence $\text{ht}(P) = \text{ht}(Q)$, which is of course finite. As $Q \subset P$, we have $Q = P$.

Finally, consider the prime $q_2 = Q \cap R[X_{\pi(1)}, ..., X_{\pi(n-2)}, X_n]$ $(= P \cap T[Y_2] = qT[Y_2])$. As P is an upper of q_2,

$$t = *P = 1 + *_R q_2.$$

Moreover, applying the induction hypothesis to q_2 allows us to rewrite the set $\{\pi(1), ..., \pi(n-2), n\}$ as $\{v(1), v(2), v(3), ..., v(n-1)\}$ in such a way that

$$*q^{(v(i))} = 0, \qquad \text{for } 1 \leqslant i \leqslant n-1-*q_2(=n-t)$$

and

$$*q^{(v(i+1))} = 1 + *q^{(v(i))}, \qquad \text{for } n-t \leqslant i \leqslant n-2.$$

Then setting $v(\eta) = \pi(n-1)$ leads to $v \in S_n$ with the desired properties, since

$$\begin{aligned} 1 + *q^{(v(\eta-1))} &= 1 + *(Q \cap R[X_{\pi(1)}, ..., X_{\pi(n-2)}, X_n]) = 1 + *q_2 \\ &= t = *Q = *q^{(v(n))}. \end{aligned}$$

The proof is complete.

If $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$ and $\sigma \in S_n$, it will be convenient to let $R^{(Q,\sigma(i))}$ denote $R[X_{\sigma(1)}, ..., X_{\sigma(i)}]/q^{(\sigma(i))}$.

PROPOSITION 4.7. *Let a domain R be LFD, $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$, $q = Q \cap R$, and $s = n - *Q$. Then there exists $\sigma \in S_n$ such that $R^{(Q,\sigma(i))} \cong (R/q)[X_{\sigma(1)}, ..., X_{\sigma(i)}]$ if $1 \leqslant i \leqslant s$ and $R^{(Q,\sigma(i+1))}$ is algebraic over $R^{(Q,\sigma(i))}$ if $s \leqslant i \leqslant n-1$. Moreover, $*Q = n - t \cdot d_{R/q}(R[X_1, ..., X_n]/Q)$.*

Proof. Let σ be as in the conclusion of Lemma 4.6. Then $q^{(\sigma(i))} = qR[X_{\sigma(1)}, ..., X_{\sigma(i)}]$ for $1 \leqslant i \leqslant s$; and, if $s \leqslant i \leqslant n-1$, $q^{(\sigma(i+1))}$ is an upper of $q^{(\sigma(i))}$. Note that the "increasing" sequence $\{R^{(Q,\sigma(i))}: i = 1, 2, ..., s\}$ of extensions of R/q is identified with the sequence of polynomial rings $\{(R/q)[X_{\sigma(1)}, ..., X_{\sigma(i)}]: i = 1, 2, ..., s\}$, so that $t \cdot d_{R/q}(R^{(Q,\sigma(s))}) = s$. As $R^{(Q,\sigma(n))} = R[X_1, ..., X_n]/Q$, it therefore suffices to prove the algebraicity assertion. Let k_i denote the quotient field of $R^{(Q,\sigma(i))}$ and then observe via Lemma 4.4 that k_{i+1} is algebraic over k_i for $s \leqslant i \leqslant n-1$.

COROLLARY 4.8. *The class of universally catenarian domains satisfies the altitude formula.*

Proof. Consider domains $R \subset T$ and $\bar{Q} \in \operatorname{Spec}(T)$ such that R is universally catenarian and T is a finite-type R-algebra. Of course, $q = \bar{Q} \cap R$ is of finite height in R since R is LFD. Our task is to show that $\operatorname{ht}(\bar{Q}) = \operatorname{ht}(q) + t \cdot d_R(T) - t \cdot d_{R/q}(T/\bar{Q})$. To this end, first express T as $R[X_1, ..., X_n]/P$, where P is a prime such that $P \cap R = 0$; next, express $\bar{Q}$ as Q/P for a suitable prime Q. It is easy to see that $q = Q \cap R$. Since $R[X_1, ..., X_n]$ is catenarian, $\mathrm{ht}(Q) = \mathrm{ht}(\bar{Q}) + \mathrm{ht}(P)$. In conjunction with Corollary 4.3 and Proposition 4.7, this leads to

$$\begin{aligned}\mathrm{ht}(\bar{Q}) &= ({*Q} + \mathrm{ht}(q)) - ({*P} + \mathrm{ht}(P \cap R)) \\ &= n - t \cdot d_{R/q}(R[X_1, ..., X_n]/Q) + \mathrm{ht}(q) - (n - t \cdot d_R(R[X_1, ..., X_n]/P)).\end{aligned}$$

As $T/\bar{Q} \cong R[X_1, ..., X_n]/Q$, the proof is complete.

We close this section with another application of Proposition 4.7. It is a normalization lemma somewhat in the spirit of Noether's. A complete analogy, replacing "algebraic" below with "integral," is not available: consider, for instance, $R = \mathbb{Z}$ and $T = \mathbb{Z}\,[2/3]$.

COROLLARY 4.9. *Let* $\mathbf{C}$ *be a class of domains stable under localizations such that* $\dim_v(A) = \dim(A)$ *for each* A *in* $\mathbf{C}$. *Let* R *in* $\mathbf{C}$ *be LFD and let* T *be a domain containing* R *such that* T *is a finite-type* R*-algebra. Then there exists in* T *a finite subset* $\{x_1, ..., x_s\}$ *of algebraically independent indeterminates over* R *such that* T *is algebraic over* $R[x_1, ..., x_s]$. *If* T *is actually integral over* $R[x_1, ..., x_s]$, *then* $\dim(T) = \dim(R) + s = \dim(R) + t \cdot d_R(T)$.

Proof. Express T as $R[X_1, ..., X_n]/Q$, where Q is a prime satisfying $Q \cap R = 0$. Proposition 4.7 supplies a permutation $\sigma \in S_n$ and canonical injections $R \subsetneq R[X_{\sigma(1)}, ..., X_{\sigma(s)}] \subsetneq T$ such that $s = n - {*Q}$ and T is algebraic over $A = R[X_{\sigma(1)}, ..., X_{\sigma(s)}]$. Setting $x_i = X_{\sigma(i)}$, we then have the first assertion (without yet using the hypotheses on $\mathbf{C}$). Of course $s = t \cdot d_R(T)$ and $\dim(A) = \dim(R) + s$, the latter by virture of Lemma 3.1(a). The final assertion now follows because integrality preserves dimension.

5. A CHARACTERIZATION OF CLASSES OF UNIVERSALLY CATENARIAN DOMAINS

As noted following Corollary 4.3, it is well known how to characterize universal catenarity for Noetherian domains. We next address the situation for arbitrary domains with the aid of Sections 3 and 4.

THEOREM 5.1. (a) *Let* $\mathbf{C}$ *be a class of domains stable under localizations. Then the following two conditions are equivalent*:

(1) *Each* A *in* $\mathbf{C}$ *is catenarian and satisfies* $\dim_v(A) = \dim(A)$. *Moreover,* $\{R/p\colon R \in \mathbf{C}$ *and* $p \in \mathrm{Spec}(R)\}$ *satisfies the altitude formula*;

(2) *Each* R *in* $\mathbf{C}$ *is universally catenarian.*

If, in addition, **C** *is stable under factor domains, then the above conditions are also equivalent to*

(3) *Each* A *in* **C** *is catenarian and satisfies* $\dim_v(A) = \dim(A)$. *Moreover,* **C** *satisfies the altitude formula.*

(b) *The class of universally catenarian domains is the largest class* **C** *of catenarian domains such that* $\dim_v(A) = \dim(A)$ *for each* A *in* **C**, **C** *is stable under localizations and factor domains and* **C** *satisfies the altitude formula.*

Proof. By Corollaries 3.3 and 4.8, it evidently suffices to show that $(1) \Rightarrow (2)$. Assume (1). We must prove that each A in **C** is universally catenarian. For this, it is enough to show that if $P \subset Q$ are prime ideals of $A[X_1, ..., X_n]$ such that $\mathrm{ht}(Q/P) = 1$, then $\mathrm{ht}(Q) = 1 + \mathrm{ht}(P)$. (To be sure, $\mathrm{ht}(Q)$ is finite. Indeed, (1) guarantees that $q = Q \cap A$ has finite height. Then $A_q[X_1, ..., X_n]$ is finite-dimensional, as is each of its rings of fractions; in particular, $A[X_1, ..., X_n]_Q$ is finite-dimensional.)

Set $p = P \cap A$, $R = A/p$, and $T = A[X_1, ..., X_n]/P$. The altitude formula then rewrites $\mathrm{ht}_T(Q/P)$ as

$$1 = \mathrm{ht}_R(q/p) + t \cdot d_R(T) - t \cdot d_{A/q}(A[X_1, ..., X_n]/Q),$$

with the aid of the canonical isomorphisms $R/(q/p) \cong A/q$ and $T/(Q/P) \cong A[X_1, ..., X_n]/Q$. Note that $\mathrm{ht}(q/p) = \mathrm{ht}(q) - \mathrm{ht}(p)$ since A is catenarian. Moreover, two applications of Proposition 4.7 yield that $t \cdot d_R(T) = n - *P$ and $t \cdot d_{A/q}(A[X_1, ..., X_n]/Q) = n - *Q$. The displayed equation therefore simplifes to

$$1 = \mathrm{ht}(q) - \mathrm{ht}(p) - *P + *Q.$$

Now two applications of Corollary 4.2 yield that $\mathrm{ht}(Q) = \mathrm{ht}(q) + *Q$ and $\mathrm{ht}(P) = \mathrm{ht}(p) + *P$, so that the last-displayed equation gives $1 = \mathrm{ht}(Q) - \mathrm{ht}(P)$, completing the proof.

In Theorem 7.2, we shall show that a weak "going-down" type of hypothesis may replace the "altitude formula" conditions in Theorem 5.1. We must first, however, develop other going-down applications, specifically Corollary 6.4.

6. A Descent Result for Universal Catenarity and Some Going-Down Applications

In [36, p. 13], Ratliff remarks that if $R \subset T$ is an integral extension of domains, T is catenarian, and $\mathrm{ht}(M) = \mathrm{ht}(M \cap R)$ for each maximal ideal M of T, then R is catenarian. We next establish an analogous descent result for the universal catenarity property, with the aid of a slightly different condition on heights. For motivation, note that both the auxiliary conditions on heights hold if the integral extension $R \subset T$ satisfies GD.

THEOREM 6.1. *Let $R \subset T$ be an integral extension of domains such that T is universally catenarian. Suppose that* $\mathrm{ht}(q_1) = \mathrm{ht}(q_2)$ *whenever q_1, q_2 are prime ideals of T such that $q_1 \cap R = q_2 \cap R$. Then R is universally catenarian.*

Proof. It is enough to show that if $Q_1 \subset Q_2$ are prime ideals of $R[X_1, ..., X_n]$ such that $\mathrm{ht}(Q_2/Q_1) = 1$, then $\mathrm{ht}(Q_2) \leqslant 1 + \mathrm{ht}(Q_1)$. (To be sure, $\mathrm{ht}(Q_i) < \infty$. Indeed, by reasoning as in the parenthetical remark in the proof of Theorem 5.1, we need only show that $q_i = Q_i \cap R$ has finite height. But $\mathrm{ht}(q_i) = \dim(R_{q_i}) = \dim(T_{R \setminus q_i})$ by integrality. Moreover, by the going-up property (GU) and the hypotheses, $\dim(T_{R \setminus q_i}) = \mathrm{ht}(p)$ for each prime p of T contracting to q_i; and $\mathrm{ht}(p) < \infty$ since T, being universally catenarian, must be LFD.) We consider first the case $q_1 = q_2$ (=, say, q). By [5, Theorem 1], $\mathrm{ht}(Q_i) = \mathrm{ht}(q^*) + \mathrm{ht}(Q_i/q^*)$, and so the conclusion for this case will follow by showing $\mathrm{ht}(Q_2/q^*) = 1 + \mathrm{ht}(Q_1/q^*)$. This, in turn, will follow by viewing matters in $(R/q)[X_1, ..., X_n]_{(R/q)\setminus\{0\}}$, interpreted as the Cohen–Macaulay (and, hence, catenarian) ring $\mathbf{k}(q)[X_1, ..., X_n]$.

By replacing $R \subset T$ with $R_{q_2} \subset T_{R \setminus q_2}$, we may suppose that R is quasilocal with unique maximal ideal $q_2 \neq 0$. Then, reasoning as in the above parenthetical remark, $\dim(T) = d < \infty$, where d is the height of each maximal ideal of T. Let h denote $\mathrm{ht}(Q_1)$. By [27, Theorem 4.6], there exists a prime ideal P_1 of $T[X_1, ..., X_n]$ such that $P_1 \cap R[X_1, ..., X_n] = Q_1$ and $\mathrm{ht}(P_1) = h$. Moreover, by GU, there exists $P_2 \in \mathrm{Spec}(T[X_1, ..., X_n])$ such that $P_1 \subset P_2$ and $P_2 \cap R[X_1, ..., X_n] = Q_2$. Then the integrality of $R \subset T$ allows us to conclude that $m = P_2 \cap T$ is a maximal ideal of T, since $m \cap R = q_2$ is maximal in R.

Choose a maximal ideal N of $T[X_1, ..., X_n]$ containing P_2. It is easy to see that $N \cap T = m$. Since each maximal ideal in the affine algebra $(T/m)[X_1, ..., X_n]$ has height n, it follows that $\mathrm{ht}(N/m^*) = n$. Thus

$$\mathrm{ht}(N) \geqslant \mathrm{ht}(N/m^*) + \mathrm{ht}(m^*) \geqslant n + \mathrm{ht}(m) = n + d.$$

Of course $\mathrm{ht}(N) \leqslant \dim(T[X_1, ..., X_n]) = d + n$ by Lemma 3.1 and Corollary 3.3. Hence, $\mathrm{ht}(N) = n + d$.

Since $\mathrm{ht}(Q_2/Q_1) = 1$, the incomparability property (INC) yields that $\mathrm{ht}(P_2/P_1) = 1$ as well. Then, since $T[X_1, ..., X_n]$ is catenarian, $\mathrm{ht}(P_2) = \mathrm{ht}(P_1) + \mathrm{ht}(P_2/P_1) = h + 1$. Setting $k = \mathrm{ht}(N/P_2)$, we again invoke catenarity, obtaining

$$k = \mathrm{ht}(N) - \mathrm{ht}(P_2) = (n + d) - (h + 1).$$

Next, setting $M = N \cap R[X_1, ..., X_n]$, we again invoke INC, obtaining that $\mathrm{ht}(M/Q_2) \geqslant \mathrm{ht}(N/P_2) = k$. It follows that

$$\begin{aligned} n + d = \dim(R[X_1, ..., X_n]) &\geqslant \mathrm{ht}(M) \geqslant \mathrm{ht}(M/Q_2) + \mathrm{ht}(Q_2) \\ &\geqslant k + \mathrm{ht}(Q_2) = n + d - h - 1 + \mathrm{ht}(Q_2), \end{aligned}$$

whence $\mathrm{ht}(Q_2) \leqslant 1 + h$, completing the proof.

We next present a key result, having several applications. It may be viewed, in part, as an ascent result for universal catenarity and also reestablishes contact with the archetypes in [28, 4]. As usual, if A is a domain, then A' denotes the integral closure of A.

THEOREM 6.2. *Let R be a going-down domain. Then the following conditions are equivalent*:

(1) *R is universally catenarian*;

(2) $\dim_v(R_p) = \dim(R_p) < \infty$ *for each* $p \in \mathrm{Spec}(R)$;

(3) $\dim_v(R_m) = \dim(R_m) < \infty$ *for each maximal ideal m of R*;

(4) *R' is an LFD Prüfer domain*;

(5) *R' is universally catenarian.*

Proof. (1) $\Rightarrow$ (2). Since each universally catenarian domain must be LFD, this implication follows from Corollary 3.3.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Since $R \subset R'$ satisfies lying-over (LO), INC, and GD, it is easy to see that R' is LFD if and only if R is LFD. However, (3) assures that R is LFD. Thus, if the assertion fails, R' is not a Prüfer domain. It is well known (cf. [22, Theorem 19.15]) that R then has a valuation overring V such that $R \subset V$ does not satisfy INC; that is, there exist distinct prime ideals $q_1 \subset q_2$ of V such that $q_1 \cap R = q_2 \cap R (=$, say, $p)$. Choose a maximal ideal m of R containing p. It is harmless to replace $R \subset V$ with $R_m \subset V_{R \setminus m}$; in particular, we may assume that R is quasilocal with unique maximal ideal m. Let $h = \dim(R)$. As R is a quasilocal going-down domain, [8, Theorem 2.2] shows that the prime ideals of R are linearly ordered by inclusion, giving a chain of distinct primes $p_0 \subset \cdots \subset p_i \subset \cdots \subset p_h$, with $p_0 = 0$, $p_i = p$, and $p_h = m$. Since $R \subset V$ satisfies GD, V contains primes $0 = Q_0 \subset \cdots \subset Q_i = q_1$ such that $Q_j \cap R = p_j$ for $j = 0, ..., i$. Moreover, by [22, Corollary 19.7(2)], there exists a valuation overring W of R, $W \subset V$, containing a chain of primes $q_2 = Q_{i+1} \subset \cdots \subset Q_h$ such that $Q_j \cap R = p_j$ for $j = i+1, ..., h$. As $\{Q_j : 0 \leqslant j \leqslant h\}$ consists of $h+1$ distinct primes of W (cf. [24, Proposition 1.1]),

$$h + 1 \leqslant \dim(W) \leqslant \dim_v(R) = \dim_v(R_m) = \dim(R_m) = h,$$

the desired contradiction.

(4) $\Rightarrow$ (5). Apply [28, 4].

(5) $\Rightarrow$ (1). Since $R \subset R'$ satisfies both INC and GD, it is easy to see that $\operatorname{ht}(q) = \operatorname{ht}(q \cap R)$ for each $q \in \operatorname{Spec}(R')$. Thus Theorem 6.1 may be applied, completing the proof.

An early version of the Krull–Akizuki Theorem states that if R is a one-dimensional Noetherian domain, then R' is a Dedekind (and hence a Cohen–Macaulay) domain. Accordingly, Theorem 6.2 [(5) $\Rightarrow$ (1)] leads to a new proof of the result of Ratliff mentioned in the Introduction, namely that each one-dimensional Noetherian domain is universally catenarian. One should also note that Theorem 6.2 also generalizes (but uses in its proof) the universal catenarity of arbitrary LFD Prüfer domains [28, 4]. It is convenient to note here that the methods of [28] derive from the fact that universal catenarity is a local property and Nagata's proof [33] of universal catenarity for arbitrary finite-dimensional valuation domains.

Corollary 6.4 will present a sharp generalization of Ratliff's result, in the absence of ascending chain condition; this will be of help below (in Theorem 7.2). First, to prepare for a detailed analysis of the one-dimensional case, we recall some facts already implicit in Proposition 2.1. For a one-dimensional domain R, the following three conditions are equivalent: $\dim_v(R) = 1$; $\dim(R[X]) = 2$; and R' is a Prüfer domain (cf. [26, Corollaire 1, p. 67; 20, Theorem 6; also [22, Theorem 30.9 and Proposition 30.14].) One may regard [8, Proposition 2.7] as a generalization of these equivalences, forging a link with Corollary 3.4, a result now eclipsed by Theorem 6.2 [(1) $\Leftrightarrow$ (4)].

COROLLARY 6.3. *Let R be a one-dimensional domain. Then the following conditions are equivalent*:

(1) *R is universally catenarian;*

(2) *$R[X]$ is catenarian;*

(3) *R is a stably strong S-domain;*

(4) *R is a strong S-domain;*

(5) *R is an S-domain;*

(6) *If $p \in \operatorname{Spec}(R)$ and $Q \subset p^{\#}$ is a prime ideal of $R[X_1, ..., X_n]$, then $Q = (Q \cap R)^{\#}$;*

(7) *If $p \in \operatorname{Spec}(R)$ and $Q \subset p^{*}$ is a prime ideal of $R[X]$, then $Q = (Q \cap R)^{*}$;*

(8) $\dim_v(R) = 1$;

(9) $\dim(R[X]) = 2$;

(10) *R' is a Prüfer domain.*

Proof. (1) $\Rightarrow$ (3). Apply Theorem 2.4.

(1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (4). Apply Lemma 2.3.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (9). If (9) fails, then some upper of $\{0\}$ in $R[X]$ is contained in p^*, for some (height 1) $p \in \operatorname{Spec}(R)$; then $2 \leqslant \operatorname{ht}(p^*) \neq 1$ and so (5) fails. This establishes the contrapositive of the asserted implication.

(9) $\Leftrightarrow$ (10) $\Leftrightarrow$ (8). See the facts recalled prior to the statement of this corollary.

(8) $\Rightarrow$ (6). Without loss of generality, $Q \subsetneqq p^{\#}$. If the assertion fails, then $q = Q \cap R$ is such that $q^{\#} \subsetneqq Q$. Hence $q \subsetneqq p$ and so, since $\dim(R) = 1$, $q = 0$. Now, consider the chain of primes

$$0 \subset Q \subset p^{\#} \subset (p, X_1) \subset \cdots \subset (p, X_1, ..., X_n).$$

By Lemma 3.1, (8) assures that $\dim(R[X_1, ..., X_n]) = n + 1$, whence $Q = 0$ and, in particular, $Q = q^{\#}$, the desired contradiction.

(6) $\Rightarrow$ (7). Trivial.

(7) $\Rightarrow$ (9). As above, if (9) fails, then $Q \subset p^*$ for some suitable (nonzero) $p \in \operatorname{Spec}(R)$ and Q an upper of $\{0\}$; then $Q \cap R = 0$ and so, since $Q \neq 0 = 0^*$, (7) fails. This establishes the contrapositive of the asserted implication.

(10) $\Rightarrow$ (1). Since R is a going-down domain and $\dim(R') = 1 < \infty$, we may apply Theorem 6.2 [(4) $\Rightarrow$ (1)], completing the proof.

The significance of the next result was discussed above. Its proof, contained in the preceding result, really depended on only the classical work of Jaffard [26], the universal catenarity of LFD Prüfer domains, and Theorem 6.1.

COROLLARY 6.4. *If* $\dim_v(R) = 1$, *then R is universally catenarian.*

In general, the computation of valuative dimension is extremely difficult. For instance, if R is a one-dimensional coherent domain, it is not known whether $\dim_v(R)$ is also 1, that is, whether R' is a Prüfer domain. This equation has resisted the efforts of several workers for more than a decade. By Corollary 6.3, *an equivalent question* is whether each one-dimensional coherent domain must be universally catenarian.

We next give additional applications of Theorem 6.2. For the first of these, recall that a domain R is said to be a *universally going-down domain* in case $S \to S \otimes_R T$ satisfies GD for each domain T containing R and each R-algebra S. It is known [14, Corollary 2.3] that R is a Prüfer domain if and only if R is an integrally closed universally going-down domain. In this vein, we offer

COROLLARY 6.5. *If R is an LFD universally going-down domain, then R is universally catenarian.*

Proof. By [14, Theorem 2.4], R' is a Prüfer domain. Moreover, as noted in the proof of Theorem 6.2 [(3) $\Rightarrow$ (4)], R' inherits the LFD property from R. An application of Theorem 6.2 [(4) $\Rightarrow$ (1)] now completes the proof.

It is an open question whether there exists a domain R such that $R[X]$ is catenarian but $R[X, Y]$ is not catenarian. By the result of Ratliff [35, (2.6)], such an R could not be Noetherian. Nor could it be one-dimensional, thanks to Corollary 6.3 [(2) $\Rightarrow$ (1)]. We shall present a similar answer in the context of LPVD's, to which we now turn.

As in [24], a domain R is said to be a *pseudovaluation domain* (PVD) in case R has a (canonically associated valuation overring V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets. By [1, Proposition 2.6], PVD's are precisely the Cartesian products $V \times_k F$, arising from a valuation domain (V, m) and a field extension $F \subset k = V/m$. Following [13], we say that a domain R is a *locally pseudovaluation domain* (LPVD) in case R_p is a PVD for each $p \in \mathrm{Spec}(R)$. A wide variety of examples of LPVD's was developed in [13]. It is shown (cf. [10, Proposition 2.1]) that each LPVD is a going-down domain. A particularly tractable type of LPVD is a *globalized pseudovaluation* domain (GPVD). Each Prüfer domain is a GPVD; so is each PVD. Rather than recall the technical definition [13, pp. 155–156], we mention here only that if R is a GPVD, then R has a canonically associated Prüfer overring T such that the contraction map $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is a homeomorphism (with respect to the Zariski topologies).

COROLLARY 6.6. *Let (R, m) be a finite-dimensional PVD, with canonically associated valuation overring V. Then the following conditions are equivalent:*

(1) *R is universally catenarian;*

(2) *$R[X]$ is catenarian;*

(3) $\dim_v(R) = \dim(R)$;

(4) $R' = V$;

(5) *V/m is algebraic over R/m.*

Proof. Write $R = V \times_k F$ as above; then $F = R/m$. By basic results about pullbacks (cf. [18, Proposition 1.3(a), p. 69; 17, Theorem 2.4]), one sees readily that $\dim_v(R) = \dim(R) + \sup\{\dim(W)\colon W$ is a valuation domain of k containing $F\}$. Thus (3) holds if and only if each such W is a field, that is, if and only if k is algebraic over F. This proves that $(3) \Leftrightarrow (5)$. Another basic calculation with pullbacks [17, Proposition 2.2(10)] assures that $(4) \Leftrightarrow (5)$.

Since each PVD is a going-down domain, $(3) \Rightarrow (1)$ follows from the corresponding implication in Theorem 6.2. As $(1) \Rightarrow (2)$ trivially, it will suffice to prove that $(2) \Rightarrow (4)$. For this, one need only combine Lemma 2.3 with [25, Proposition 2.4 and Remark 2.6]. The proof is complete.

The literature on PVD's will provide the interested reader with a number of alternate proofs for various parts of Corollary 6.6, but something like Theorem 6.2 seems to be needed in any event. Here, we note only that condition (5) above is "R is an algebraic $\mathrm{P}^0\mathrm{VD}$" in the sense of [19, Définition 1.10, Théorème 2.2].

COROLLARY 6.7. *Let R be an LPVD which is LFD. Then the following conditions are equivalent*:

(1) *R is universally catenarian*;

(2) *$R[X]$ is catenarian*;

(3) *R' is a Prüfer domain*;

(4) *For each maximal ideal m of R, if n denotes the maximal ideal of the valuation overring canonically associated to (the PVD) R_m, then the field extension $\mathbf{k}(mR_m) \subset \mathbf{k}(n)$ is algebraic.*

Proof. Since R is a going-down domain and R' inherits the LFD property from R, Theorem 6.2 $[(1) \Leftrightarrow (4)]$ yields that $(1) \Leftrightarrow (3)$. However, (1) (resp., (2)) holds if and only if R_m is universally catenarian (resp. $R_m[X]$ is catenarian) for each maximal ideal m of R. As each R_m is a finite-dimensional PVD, Corollary 6.6 now yields $(1) \Leftrightarrow (2) \Leftrightarrow (4)$, completing the proof.

COROLLARY 6.8. *Let R be a GPVD which is LFD, and let T be the canonically associated Prüfer overring of R. Then the following conditions are equivalent*:

(1) *R is universally catenarian*;

(2) *$R[X]$ is catenarian*;

(3) *For each maximal ideal M of R, if N denotes the (maximal) prime ideal of T contracting to M, then the field extension $\mathbf{k}(M) \subset \mathbf{k}(N)$ is algebraic*;

(4) $R' = T$.

Proof. As explained in [13, p. 156], the homeomorphism $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ assures that N is maximal; and T_N is the canonically associated valuation overring of (the PVD) R_M. Thus the equivalence of (1), (2), and (3) is a special case of Corollary 6.7, while (1) $\Leftrightarrow$ (4) follows easily from the preceding sentence and Corollary 6.6.

It is amusing to note (taking $R = T$) that the universal catenarity of LFD Prüfer domains is subsumed as a trivial case of Corollary 6.8, the point being that each (residue) field is an algebraic extension of itself!

Remark 6.9. The universal catenarity of arbitrary Dedekind domains is fundamental, and may be shown as a consequence of *each* of the three criteria recalled in the Introduction (not to mention via each of Theorem 6.2 and Corollaries 6.3, 6.4, 6.5, 6.7, and 6.8). Of course, Dedekind domains are just the domains of global dimension at most 1, and so it seems natural to study universal catenarity for domains of higher global dimension. Elsewhere, we shall treat the case of global dimension 2. Here, we record the diversity in the "next" case. Reference [9, Remark 3.3] showed how to work inside a particular type of valuation domain $k + M$ (with value group $\mathbb{Q}$ and maximal ideal $M \neq 0$, such that k is a countable field) to find suitable domains R of the form $F + M$ (with F a subfield of k such that $t \cdot d_F(k) \leqslant 1$). It was proved that $gl \cdot \dim(R) = 3$, each overring of R is a going-down domain, and R is not coherent. Since each such R is a one-dimensional PVD, Corollary 6.6 shows how to construct the desired diversity; indeed such R is then universally catenarian if and only if k is algebraic over F.

7. A Characterization of Classes of Universally Catenarian Going-Down Domains

With the help of Corollary 6.4, Theorem 7.2 will develop a useful companion for Theorem 5.1, that is, a sufficient condition for universal catenarity. This will lead, in Corollary 7.3(b), to a characterization of the class of universally catenarian going-down domains. Such a result is desirable since each LFD going-down domain R is at least catenarian, as a consequence of the fact [8, Theorem 2.2] that $\operatorname{Spec}(R)$, as a poset under inclusion, is a tree.

It is convenient to isolate the next result. We sketch its proof, which is a straightforward calculation using the definition of the star function.

Lemma 7.1. *Let $Q \in \operatorname{Spec}(R[X_1, ..., X_n])$ and let q_1 be a prime of R contained in $q = Q \cap R$. Put $\tilde{Q} = Q/q_1^{\#}$. Then $*_{R/q_1} \tilde{Q} = *_R Q$.*

Proof. Corresponding contributions to the star functions agree at each step. To see this, for instance at the first step, involves showing that

$$(q/q_1)^{(n-1)*} \subsetneqq (q/q_1)^{(n)} \Leftrightarrow q^{(n-1)*} \subsetneqq q^{(n)}.$$

This follows after verifying that $(q/q_1)^{(n-1)} = q^{(n-1)}/q_1 R[X_1, ..., X_{n-1}]$, with a similar formula for $(q/q_1)^{(i)}$, $0 \leqslant i < n$.

THEOREM 7.2. *Let* **C** *be a class of domains stable under localizations and factor domains such that each* A *in* **C** *is catenarian and satisfies* $\dim_v(A) = \dim(A)$. *Suppose moreover that* $\mathrm{ht}(Q) \geqslant \mathrm{ht}(Q \cap A)$ *whenever* $A \subset B$ *is an inclusion of domains such that* A *is in* **C**, B *is a finite-type* A*-algebra, and* $Q \in \mathrm{Spec}(B)$. *Then each ring in* **C** *is universally catenarian.*

Proof. We must prove that if R is in **C** and $Q_1 \subset Q_2$ are primes of $R[X_1, ..., X_n]$ such that $\mathrm{ht}(Q_2/Q_1) = 1$, then $\mathrm{ht}(Q_2) \leqslant 1 + \mathrm{ht}(Q_1)$. (To be sure, $\mathrm{ht}(Q_i) < \infty$: see the parenthetical remark in the proof of Theorem 5.1.) Put $q_i = Q_i \cap R$. There are three cases to consider.

Suppose first that $q_1 = q_2$ (=, say, q). We proceed in the spirit of the first paragraph of the proof of Theorem 6.1. Generically, we let $\bar{I}$ denote the ideal $I_{R \setminus q}$ of $R_q[X_1, ..., X_n]$, given any ideal I of $R[X_1, ..., X_n]$. Of course, $\mathrm{ht}(\bar{Q}_i) = \mathrm{ht}(Q_i)$ and $\mathrm{ht}(\bar{Q}_2/\bar{Q}_1) = 1$, and so it will suffice to show that $\mathrm{ht}(\bar{Q}_2) - \mathrm{ht}(\bar{Q}_1) = 1$. However, [5, Theorem 1] gives

$$\mathrm{ht}(\bar{Q}_i) = \mathrm{ht}(qR_q^{\#}) + \mathrm{ht}(\bar{Q}_i/qR_q^{\#}).$$

The assertion for this case now follows by noting that the primes $\bar{Q}_i/qR_q^{\#}$ are adjacent in the Cohen–Macaulay (and, hence, catenarian) domain $\mathbf{k}(q)[X_1, ..., X_n]$.

Suppose next that $0 = q_1 \neq q_2$. Set $B = R[X_1, ..., X_n]/Q_1$, and view $R \subsetneq B$ in the usual way. Since $(Q_2/Q_1) \cap R = q_2$, the hypotheses yield $\mathrm{ht}(Q_2/Q_1) \geqslant \mathrm{ht}(q_2)$; thus, $\dim(R_{q_2}) = 1$. As R_{q_2} is in **C**, $\dim_v(R_{q_2}) = 1$, and so Corollary 6.4 assures that $R_{q_2}[X_1, ..., X_n]$ is catenarian. In particular, $\mathrm{ht}(Q_2 R_{q_2}[X_1, ..., X_n]) = 1 + \mathrm{ht}(Q_1 R_{q_2}[X_1, ..., X_n])$; that is, $\mathrm{ht}(Q_2) = 1 + \mathrm{ht}(Q_1)$, as desired.

In the final case, $0 \neq q_1 \subsetneqq q_2$. Consider the adjacent primes $\tilde{Q}_i = Q_i/q_1^{\#}$ in $(R/q_1)[X_1, ..., X_n]$. As R/q_1 is in **C**, the second case gives $\mathrm{ht}(\tilde{Q}_2) = 1 + \mathrm{ht}(\tilde{Q}_1)$. Since $Q_i/q_1^{\#} \cap R/q_1 = q_i/q_1$, we now infer via two applications of the star formula (Corollary 4.2) that

$$*_{R/q_1} \tilde{Q}_2 + \mathrm{ht}(q_2/q_1) = 1 + *_{R/q_1} \tilde{Q}_1 + \mathrm{ht}(q_1/q_1).$$

By use of Lemma 7.1 and the catenarity of R, this becomes

$$*Q_2 + \mathrm{ht}(q_2) - \mathrm{ht}(q_1) = 1 + *Q_1,$$

that is,

$$*Q_2 + \mathrm{ht}(q_2) = 1 + (*Q_1 + \mathrm{ht}(q_1)).$$

Two more applications of the star formula yield $\mathrm{ht}(Q_2) = 1 + \mathrm{ht}(Q_1)$, completing the proof.

COROLLARY 7.3. (a) *Let* **C** *be a class of LFD going-down domains stable under localizations and factor domains such that each A in* **C** *satisfies* $\dim_v(A) = \dim(A)$. *Then each ring in* **C** *is universally catenarian.*

(b) *The class of universally catenarian going-down domains is the largest class of LFD going-down domains such that* **C** *is stable under localizations and factor domains and* $\dim_v(A) = \dim(A)$ *for each A in* **C**.

Proof. (a) As noted in the first paragraph of this section, the hypotheses imply that each A in **C** is catenarian. Moreover, if $A \subset B$ are domains with A in **C** then (whether or not B is algebra-finite over A), $A \subset B$ satisfies GD; then, in particular, $\mathrm{ht}(Q) \geq \mathrm{ht}(Q \cap A)$ for each $Q \in \mathrm{Spec}(B)$. The assertion therefore follows from Theorem 7.2.

(b) By [10, Remarks 2.11 and 3.2(a)], each factor domain of a going-down domain is itself a going-down domain; and so is each localization. The assertion therefore follows by combining Corollary 3.3 with part (a).

Remark 7.4. If R is an LFD going-down domain, then Corollary 7.3(a) implies that R is universally catenarian provided that $\dim_v(A) = \dim(A)$ for each A taking one or more of the forms R_p, R/P, and R_Q/PR_Q arising from primes $P \subset Q$ of R. (The point is that such rings A comprise a class stable under localizations and factor domains.) Although this result is not as strong as Theorem 6.2 [(3) $\Rightarrow$ (1)], it does recover Corollary 6.4. (However, the latter result was needed in the proof of Theorem 7.2). Theorem 8.1 will sharpen Theorem 6.2 [(3) $\Rightarrow$ (1)] for an important class of LFD going-down domains.

8. THE COEQUIDIMENSIONAL CASE

The next result, although just another corollary of Theorem 6.2, is very useful. It generalizes an earlier result on PVD's (Corollary 6.6, (1) $\Leftrightarrow$ (3)). It will be convenient to say that a domain R is *coequidimensional* in case all maximal ideals of R have the same height. (See [36] for variant terminology.)

THEOREM 8.1. *Let R be a coequidimensional going-down domain. Then R is universally catenarian if and only if* $\dim_v(R) = \dim(R) < \infty$.

Theorem 8.1 represents another generalization of Corollary 6.4. Moreover, as was the case for Theorem 6.2, Theorem 8.1 simultaneously generalizes the universal catenarity of arbitrary one-dimensional Noetherian domains [35] and of arbitrary LFD Prüfer domains [28, 4]. Indeed, although Prüfer domains need not be coequidimensional, valuation domains *are*, and we have explained in Section 6 how the result in [28] is due to Nagata's work [33] on valuation domains.

Proof of Theorem 8.1. The "only if" half follows from Corollary 3.3. Conversely, suppose $\dim_v(R) = \dim(R)$ ($=$, say, d) $< \infty$. Since R is a going-down domain and $\dim(R_m) = \mathrm{ht}(m) = d$, Theorem 6.2 [(3) $\Rightarrow$ (1)] reduces our task to showing that $\dim_v(R_m) = d$. This is clear since

$$d \leqslant \dim_v(R_m) \leqslant \dim_v(R) = d.$$

The proof is complete.

Remark 8.2. (a) We shall, in (b), indicate another proof of Theorem 8.1 for a special case of considerable importance. First, the following background is provided.

Following [10, 12], we shall say that a domain R is *divided* in case $P = PR_P$ for each $P \in \mathrm{Spec}(R)$; and R is *locally divided* if R_Q is divided for each $Q \in \mathrm{Spec}(R)$. Each (L) PVD is (locally) divided [11, p. 560]; and each locally divided domain is a going-down domain [10, Remark 2.7(b)]. Neither of the preceding assertions has a valid converse [11, Remark 4.10(b); 10, Example 2.9].

(b) Let R be a domain which is locally finite-conductor (for instance, coherent), locally divided and coequidimensional, such that $\dim_v(R) = \dim(R) < \infty$. One may prove that R is universally catenarian, without appealing to Theorems 6.2 or 8.1, as follows. By Theorem 6.1, it is enough to show that $T = R'_{R \setminus M}$ $(= R_M{}')$ is a Prüfer domain for each maximal ideal M of R. By a fundamental ascent result [11, Theorem 3.2], T is a going-down domain; hence so is T_N for each maximal ideal N of T. Accordingly, by [8, Proposition 2.7], it is enough to show that $\dim_v(T_N) = \dim(T_N) < \infty$. This in turn follows easily from the facts that integrality preserves valuative dimension (cf. [22, Proposition 30.13]) and T inherits coequidimensionality from R.

As noted in Remark 6.9, universal catenarity fails in general for domains of "high" global dimension. In view of Corollaries 6.4 and 3.3, it is then natural to ask whether a domain R such that $\dim_v(R) = \dim(R) = 2$ must be universally catenarian. We canvass the Noetherian case first. If such an R is integrally closed, it must be Cohen–Macaulay (cf. [6, p. 52]) and hence universally catenarian. However, not every two-dimensional Noetherian domain is Cohen–Macaulay; nor need it be universally catenarian. In fact, as noted following Corollary 4.3, an example of Nagata shows that the answer is "no" in the Noetherian case.

We next ask the same question in case R is a going-down domain. One might expect an affirmative answer here since such domains have treed (hence "small") spectra, in contrast to Noetherian domains of dimension at least 2 (cf. [27, Theorem 144]). Affirmative answers are known in some subcases: R quasilocal and integrally closed (by [8, Proposition 2.7]); R a PVD (by Corollary 6.6); and, more generally, R coequidimensional (by Theorem 8.1). Despite this evidence, the answer is "no" in this case too. Put differently, the "coequidimensional" hypothesis in Theorem 8.1 cannot be deleted. Indeed, we present a counterexample, with $|\mathrm{Spec}(R)|$ minimal, in

EXAMPLE 8.3. There exists a going-down domain R such that $\dim_v(R) = \dim(R) = 2$ and R is not universally catenarian. It can be further arranged that R is an LPVD, in fact a GPVD (so that, by Corollary 6.8, $R[X]$ is not catenarian), and that R has precisely two maximal ideals Q_1 and Q_2, with $\mathrm{ht}(Q_1) = 1$, $\mathrm{ht}(Q_2) = 2$, and $\dim_v(R_{Q_1}) = 2$.

Our construction of a suitable R depends on [13]. First, take A to be a one-dimensional domain having valuative dimension 2; it can be arranged as in [21, Appendix 2, Example 2] (cf. also [19]) that $A = k + M_1$ where k is a suitable field and M_1 is the maximal ideal of a DVR, $V = k(X) + M_1$. Next, take $B = k(X^2) + M_2$ to be a two-dimensional valuation domain incomparable with, but having the same quotient field as, V. We shall show that $R = A \cap B$ has the desired properties.

By [13, Examples 2.5 and 3.2(a)], R is a GPVD with exactly two maximal ideals, say Q_1 and Q_2, such that $R_{Q_1} = A$ and $R_{Q_2} = B$. In view of Theorem 6.2 [(1)$\Leftrightarrow$(3)] (cf. also Corollary 3.3), it remains only to show that $\dim_v(R) = 2$. To see this, note that each valuation overring of R contains at least one of R_{Q_1} and R_{Q_2}, so that $\dim_v(R) = \max(\dim_v(A), \dim_v(B)) = \max(2, 2) = 2$.

9. A LOCAL STUDY OF VALUATIVE DIMENSION

Corollary 3.3 and Theorem 6.2 provide ample reason to study the condition "$\dim_v(R_P) = \dim(R_P)$ for each $P \in \mathrm{Spec}(R)$." Note that this condition was implied by another condition, "$\dim_v(R) = \dim(R)$," under the hypotheses of Theorem 8.1. However, no such implication held for the specific ring R in Example 8.3. Indeed, $\dim(R) = \dim_v(R) = \dim_v(R_{Q_1}) = 2$, although $\dim(R_{Q_1}) = 1$, in Example 8.3. Accordingly, we devote this final section to studying these two conditions.

PROPOSITION 9.1. *Let R be a domain such that* $\dim_v(R) < \infty$. *Suppose that each valuation overring* (V, N) *of R satisfies*

$$\mathrm{ht}(N) + t \cdot d_{\mathbf{k}(N \cap R)}(\mathbf{k}(N)) \leqslant \mathrm{ht}(N \cap R).$$

Then $\dim_v(R) = \dim(R)$.

Proof. Since $\dim_v(R)$ is finite, say d, we can choose a (necessarily minimal) valuation overring (W, M) of R such that $\dim(W) = d$. By the hypothesis applied to $V = W$, $\mathrm{ht}(M) \leqslant \mathrm{ht}(M \cap R)$; thus, $d = \mathrm{ht}(M) \leqslant \dim(R)$. The reverse inequality holds in general, and so the proof is complete.

Remark 9.2. The preceding proof did not seem to use the transcendence degree term in the hypothesis. (However, for the specific minimal W considered, that term vanishes: cf. [22, Exercise 8, p. 250].) The term in question will play a basic role in Proposition 9.3. Now, we shall show that (with the term in place), the converse of Proposition 9.1 is false.

To this end, consider once again the data in Example 8.3. We have seen that $\dim_v(R) = 2 = \dim(R)$. Moreover, if $(V, N) = k(X) + M_1$, then $N \cap R = (N \cap A) \cap R = Q_1$ since $A = R_{Q_1}$; thus, $\mathrm{ht}(N \cap R) = 1$. However, $\mathbf{k}(N \cap R) \subsetneq \mathbf{k}(N)$ identifies with the (transcendence degree 1) field extension $k \subsetneq k(X)$, and so the assertion just amounts to the falsity of $1 + 1 \leqslant 1$.

PROPOSITION 9.3. *Let R be a domain such that* $\dim_v(R) < \infty$. *Then the following conditions are equivalent*:

(1) $\mathrm{ht}_V(N) + t \cdot d_{\mathbf{k}_R(N \cap R)}(\mathbf{k}_V(N)) \leqslant \mathrm{ht}_R(N \cap R)$ *for each valuation overring* (V, N) *of* R;

(2) $\dim_v(R_P) = \dim(R_P)$ *for each* $P \in \mathrm{Spec}(R)$.

Proof: (1) $\Rightarrow$ (2). This implication follows from Proposition 9.1 and the observations that if $Q \subset P$ are prime ideals of R, then $\mathrm{ht}_{R_P}(QR_P) = \mathrm{ht}_R(Q)$ and $\mathbf{k}_{R_P}(QR_P) \cong \mathbf{k}_R(Q)$.

(2) $\Rightarrow$ (1). Deny. Let a valuation overring (W, M) be a case in point. Set $P = M \cap R$. Then $\mathrm{ht}(M) + t \cdot d_{\mathbf{k}(P)}(\mathbf{k}(M)) > \mathrm{ht}(P)$. Choose a nonnegative integer d such that

$$\mathrm{ht}(P) - \mathrm{ht}(M) < d \leqslant t \cdot d_{\mathbf{k}(P)}(\mathbf{k}(M));$$

then let $\{X_1, ..., X_d\}$ be a d-element subset of some transcendence basis of $\mathbf{k}(M)$ over $\mathbf{k}(P)$. It follows (cf. [22, Corollary 19.7(1)]) that $\mathbf{k}(P)[X_1, ..., X_d]$ is contained in some valuation domain T of $\mathbf{k}(M)$ such that $\dim(T) \geqslant d$. Consider the Cartesian product $V = W \times_{\mathbf{k}(M)} T$. Evidently, $R_P \subset V \subset W$. By standard material on pullbacks or a direct case analysis, one readily shows that V is a valuation domain, whence $\dim(V) \leqslant \dim_v(R_P) = \dim(R_P)$. However, by applying [17, Proposition 2.1(5)] to the definition of V, we have $\dim(V) = \dim(W) + \dim(T)$, so that

$$\dim(V) \geqslant \mathrm{ht}(M) + d > \mathrm{ht}(P) = \dim(R_P),$$

the desired contradiction. The proof is complete.

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