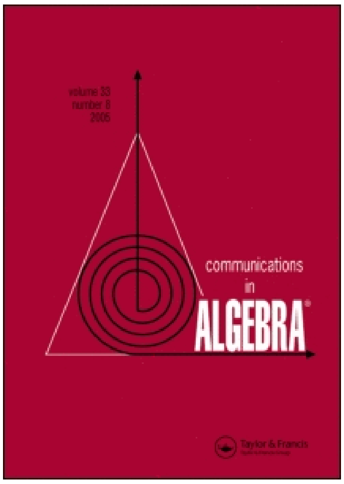

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Two sufficient conditions for universal catenarity

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TWO SUFFICIENT CONDITIONS FOR UNIVERSAL CATENARITY

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Abstract. Let R be a locally finite-dimensional integral domain. It is proved that $R[X_1, \ldots, X_n]$ is catenarian for each positive integer n if either gl. dim(R) = 2 or R is a going-down strong S-domain.

All rings considered below are (commutative integral) domains. A ring R is said to be catenarian in case, for each pair $P \subset Q$ of prime ideals of R, all saturated chains of primes from P to Q have a common finite length. Note that each catenarian R must be locally finite-dimensional (LFD), in the sense that each prime ideal of R has finite height. In [2, Lemma 2.3], we showed that if the polynomial ring R[X] is catenarian, then R is a strong S-domain, in the sense in [11]. We say that a (not necessarily Noetherian) ring R is universally catenarian if the polynomial rings $R[X_1, \ldots, X_n]$ are catenarian for each positive integer n . The most familiar examples of universally catenarian rings are arbitrary Cohen-Macaulay (C.M.) domains (cf. [13, Theorem 31]); the Noetherian domains of (Krull) dimension 1 (as a consequence of C.M., or by using Ratliff's result [15, (2.6)] that a Noetherian ring R is universally catenarian if and only if R[X] is catenarian); and the LFD Prüfer domains (cf. [14], [12], [1]). In [2], we axiomatically characterized the class of all universally catenarian domains and presented some new classes of universally catenarian rings. Several of these classes involved going-down (GD) domains, in the sense of [3]. Both one-dimensional domains and Prüfer domains are GD, and such a context is to be contrasted with a Noetherian setting, since the prime spectrum of any GD ring is a tree [3]. Our purpose here is to prove the following two results.

THEOREM 1. If R is a GD domain, then the following conditions are equivalent:

(1) R is an LFD strong S-domain;

(2) R[X] is catenarian;

(3) R is universally catenarian.

THEOREM 2. If R is a domain of global dimension 2, then R is universally catenarian if (and only if) R is LFD .

The implication (2) ⇒ (3) in Theorem 1 may be viewed as a GD-theoretic analogue of the above-mentioned result of Ratliff on Noetherian rings. The special case of Theorem 1 for which dim(R) = 1 has been obtained in [2, Corollary 6.3], along with several other equivalents. [2, Corollary 6.7] obtained the case of Theorem 1 for which R is an LPVD, in the sense of [6] . Finally, we note that Theorem 1 was motivated by work of Hedstrom-Houston [10, (2.4)-(2.6)] on finite-dimensional PVD's.

To place Theorem 2 in perspective, note first that domains of global dimension 1 are Dedekind, hence universally catenarian (by any of the three familiar criteria noted above). Moreover, Noetherian domains of finite global dimension are also universally catenarian, because they are locally regular and hence C.M. (cf. [11, Theorem 170]). Theorem 2 is plausible in view of the Vasconcelos-Greenberg (cf. [16]) structure theory for LFD quasi-local domains of global dimension 2. Any such is a pullback of a diagram each of whose vertices is universally catenarian (a field, a finite-dimensional valuation ring, a local Noetherian domain of global dimension 2). However, no proof of Theorem 2 can hinge solely on pullback considerations. Indeed, as explained in [2, Remark 6.9], the examples in [4] lead to a family F of one-dimensional PVD's such that (i) if $R \in F$, then gl. dim(R) = 3 and R is a pullback of a diagram each of whose vertices is universally catenarian (two fields and a one-dimensional valuation ring); and (ii) some, but not all, rings in F are universally catenarian.

Proof of Theorem 1. (3) ⇒ (2) ⇒ (1) by the above remarks (even without the GD assumption). As for (1) ⇒ (3), all the properties in question are local, and so we may assume that R is quasilocal, with maximal ideal M. Then $n = ht(M) < \infty$; as $Spec(R)$ is a tree, its members can be depicted as

$$0 = Q_n \subsetneq Q_{n-1} \subsetneq \cdots \subsetneq Q_1 \subsetneq Q_0 = M .$$

If the assertion fails, [2, Theorem 6.2] assures that the integral closure of R is not a Prüfer domain. Then, by combining [9, Theorem 5] and [5, Corollary 2.4], there exist u in the quotient field of R and distinct primes $P_0 \subset P$ of $R[u]$ such that $P_0 \cap R = M$ $(= P \cap R)$. Since $R \subset R[u]$ satisfies going-down, there exists a chain of primes in $R[u]$

$$0 = P_n \subset P_{n-1} \subset \cdots \subset P_1 \subset P_0$$

such that $P_i \cap R = Q_i$ for each i. Thus $\dim(R[u]) \geq ht(P) \geq n + 1$. Write $u = ab^{-1}$ with $a, b \in R$. Then $bX - a$ lies in Q, the kernel of the surjective R-algebra map $R[X] \to R[u]$ sending X to u. Thus $ht(Q) \geq 1$ and, since $R[u] \cong R[X]/Q$, $\dim(R[X]) \geq n + 2$. However, since R is a strong S-domain, [11, Theorem 39] gives $\dim(R[X]) = 1 + \dim(R) = 1 + n$, the desired contradiction.

Before proving Theorem 2, we record the following useful topological result. It will be convenient to let $C\ell_X(x)$ denote the closure of a point x in a space X.

LEMMA. Let Z be a closed subspace of a topological space X. Let $f: Z \to Y$ be an inclusion map for Z viewed as a subset of another space Y. Suppose that $X \cap Y = Z$. Set $S = X \cup_f Y$. Let $x \in X \setminus Z$ and $y \in Y \setminus Z$ such that $y \in C\ell_S(x)$ and $C\ell_X(x) \cap Z$ is nonempty. Then:

(a) $C\ell_S(x) \subset C\ell_X(x) \cup_{C\ell_X(x)\cap Z} C\ell_Y(C\ell_X(x) \cap Z)$.

(b) $y \in C\ell_Y(C\ell_X(x) \cap Z)$.

Proof of Lemma. (b) is a trivial consequence of (a) since $X \cap (Y \setminus Z) = \emptyset$. Moreover, (a) follows from the fact that, if X_1 is a closed subspace of X and Y_1 is a closed subspace of Y such that $(X_1 \cap Z) \cup Y_1$ is closed in Y, then $X_1 \cup_{f|X_1\cap Z} Y_1$ is a closed subspace of S.

Proof of Theorem 2. Universal catenarity is a local property and global dimension does not increase under localization. Hence we may assume that R is quasilocal, with maximal ideal M. Without loss of generality (cf. [16, Corollary 4.19]), R has a nonzero prime $p = pR_p$ such that $A = R/p$ is a regular local Noetherian two-dimensional domain, $V = R_p$ is a valuation domain, and $\text{p.d.}_A(k) = 1$, where $k = V/p$. Since A and V are each universally catenarian, applying [1, Lemma 1] to the pullback

$R \cong V x_k A$ reduces our task to the following: given primes $Q_1 \subset Q_2$ of $R[X_1,\ldots, X_n]$ such that

$$q_1 = Q_1 \cap R \subsetneq p \subsetneq q_2 = Q_2 \cap R ,$$

find a prime P of $R[X_1,\ldots, X_n]$ such that $Q_1 \subset P \subset Q_2$ and $P \cap R = p$.

We may reduce to $q_2 = M$. For suppose that $q_2 \neq M$. Then R_{q_2} is a quasilocal domain, of global dimension at most 2 , whose spectrum is a tree. Hence R_{q_2} is a valuation domain (cf. [16, Theorem 2.2]). Thus the inclusion map

$$R_{q_2}/q_1 R_{q_2} \to R[X_1,\ldots, X_n]_{Q_2}/Q_1 R[X_1,\ldots, X_n]_{Q_2}$$

is local and flat, and hence faithfully flat. Accordingly, the induced map on prime spectra is surjective, from which the existence of the required P follows easily.

One also has the pullback description $R/q_1 \cong V/q_1 x_k A$, and (the valuation domain) V/q_1 has global dimension at most 2 (cf. [16, Theorem 2.1]). In view of the above information about A , V and k , [16, Corollary 4.19] yields gl. $\dim(R/q_1) = 2$. Replacing R by R/q_1 , we may therefore reduce to $q_1 = 0$.

A final reduction: $0 \neq Q_1 \not\subset p^{\#} = pR[X_1,\ldots, X_n]$. Otherwise, it suffices to choose $P = p^{\#}$.

The proof continues via the above lemma. Put

$Z = \mathrm{Spec}(k[X_1,\dots, X_n])$, viewed canonically inside $X = \mathrm{Spec}(V[X_1,\dots, X_n])$ and $Y = \mathrm{Spec}(A[X_1,\dots, X_n])$. Take $x = Q_1V$ $(= (Q_1)_{R/p})$ and $y = Q_2/p^{\#}$; evidently, $x \in X \setminus Z$ and $y \in Y \setminus Z$. We shall show next that $C\ell_X(x) \cap Z$ is non-empty; equivalently, that Q_1V $(= Q_1V[X_1,\dots, X_n])$ satisfies

$$Q_1V + p^{\#} \neq V[X_1,\dots, X_n] .$$

Indeed, if equality held, then

$$1 \in (Q_1V + p^{\#}) \cap R[X_1,\dots, X_n] ,$$

forcing $1 \in (Q_1V \cap R[X_1,\dots, X_n]) + p^{\#} \subset Q_1 + p^{\#} \subset Q_2$, the desired contradiction.

Next, observe that the pullback description $R \cong V \times_k A$ induces $R[X_1,\dots, X_n] \cong V[X_1,\dots, X_n] \times_{k[X_1,\dots, X_n]} A[X_1,\dots, X_n]$ (cf. [1, Lemma 2]) . Accordingly, S is canonically homeomorphic to $\mathrm{Spec}(R[X_1,\dots, X_n])$ and we identify these spaces (cf. [7, Theorem 1.4]). Then x and y are identified with Q_1 and Q_2 , respectively, and so the condition "$y \in C\ell_S(x)$" reduces just to the hypothesis $Q_1 \subset Q_2$. In particular, the lemma applies; so, by (b), $y \in C\ell_Y(C\ell_X(x) \cap Z)$.

It is straightforward to verify that $C\ell_X(x) \cap Z$ is the variety in Z of the ideal $(Q_1V + p^{\#})/p^{\#}$. Moreover, since Z is a Noetherian space, this variety has only finitely many minimal points, say $z_1,\dots, z_m$. Thus

$$y \in Cl_Y(Cl_Z(z_1) \cup \ldots \cup Cl_Z(z_m)) = \cup Cl_Y(Cl_Z(z_i)) = \cup Cl_Y(z_i),$$

the last containment holding since order-theoretic reasoning gives $Cl_Z(z) \subset Cl_Y(z)$ for each $z \in Z$. Hence $y \in Cl_Y(z_j)$ for some j. Viewing z_j as $P \in Z = \mathrm{Spec}(k[X_1,\ldots, X_n]) \subset S = \mathrm{Spec}(R[X_1,\ldots, X_n])$, we have $P \cap R = p$. Next, since y is in the closure of z_j, $P/p^\# \subset Q_2/p^\#$; that is, $P \subset Q_2$. Finally, since z_j is in the cited variety, $(Q_1V + p^\#)/p^\# \subset (P_{R \setminus p})/p^\#$, whence

$$Q_1 \subset (Q_1V + p^\#) \cap R[X_1,\ldots, X_n] \subset P_{R \setminus p} \cap R[X_1,\ldots, X_n] = P,$$

completing the proof.

REMARK. An interesting alternate proof of Theorem 2 is available in case $n = 1$. It does not appeal to [7]. It begins with the initial four paragraphs of the earlier proof and continues as follows.

Let K denote the quotient field of R. As Q_1 lies over 0, $Q_1 = fK[X] \cap R[X]$ for some polynomial f in $R[X]$ which is irreducible in $K[X]$. Viewing matters over V, we may assume that some coefficient of f is 1, whence

$$Q_1V[X] = Q_1 R_p[X] = fK[X] \cap R_p[X] = f R_p[X],$$

the last equality via [8, Corollary 34.9].

Now set $\tilde{Q}_1 = (Q_1 + p^\#)/p^\#$, viewed as an ideal of $R[X]/p^\# \cong A[X]$. We claim $\mathrm{grade}_{A[X]}(\tilde{Q}_1) = 1$. To see this, consider elements $\overline{G}_i = G_i + p^{\#} \in \tilde{Q}_1$ (with $G_i \in Q_1$ for $i = 1,2$) . Choose $g_i \in R[X]$ and $c_i \in R \setminus p$ such that $G_i = fg_i c_i^{-1}$. Since $\overline{g}_2\overline{c}_1\overline{G}_1 = \overline{g}_1\overline{c}_2\overline{G}_2$, it will follow that $\{\overline{G}_1, \overline{G}_2\}$ is not a regular sequence. To see this, we offer an indirect proof. Suppose $\{\overline{G}_1, \overline{G}_2\}$ <u>is</u> regular. Then $\overline{g}_2\overline{c}_1 = \overline{G}_2\overline{H}$ for some suitable $H \in R[X]$, and so

$$\overline{g_2c_1c_2} = \overline{G_2H\ c_2} = \overline{f}\ \overline{g_2}\ \overline{c_2^{-1}}\ \overline{H}\ \overline{c_2} = \overline{g_2}\ \overline{f}\ \overline{H} .$$

By the supposed regularity, $\overline{G_2} \neq \overline{0}$, whence $\overline{g_2} \neq \overline{0}$ and cancellation leads to $\overline{c_1c_2} = \overline{fH}$, a nonzero constant. Thus, by comparing degrees, $\deg(\overline{f}) = 0$; that is, $f \in R + p^{\#}$. As we have seen that some coefficient of f is a unit in V , it follows that the constant coefficient, say r , of f lies in $R \setminus p$. Then

$$r^{-1} f \in r^{-1}(r + Xp\ R[X]) \in 1 + Xp\ R_p[X] = 1 + Xp\ R[X] \subset R[X] ,$$

whence $r^{-1}f \in K[X]f \cap R[X] = Q_1$, so that $1 \in Q_1 + XpR[X] \subset Q_2$, the desired contradiction. This proves the claim.

Now, since A is regular local, the classical unmixedness theorem obtains in $A[X]$ (cf. [<u>13</u>, Theorem 31]), and so $\operatorname{rank}(\tilde{Q}_1) = 1$ (cf. [<u>11</u>, Theorem 136]) . Hence $\tilde{Q}_2 = Q_2/p^{\#}$ must contain a height 1 prime $\tilde{Q} = Q/p^{\#}$ which is minimal over $\tilde{Q}_1$. Clearly $Q_1 \subset Q \subset Q_2$.

If $\tilde{Q}$ is not extended from A , then $\tilde{Q} \cap A = 0$, in which case $Q \cap R = p$, so that choosing $P = Q$ suffices. It remains only to treat the case of extended $\tilde{Q} = (q/p)\,A[X]$, where $q \in \mathrm{Spec}(R)$ properly contains p . In fact, given the above reductions, this case cannot arise. To see this, it will suffice to show that R_q is a valuation domain, for then $Q_1 R_q \subset (q\, R_q) R_q[X]$ would lead via [11, Theorem 68] to $Q_1 R_q = 0$, $Q_1 = 0 \subset p^{\#}$, the desired contradiction. To see that R_q is valuation, note first that $\mathrm{ht}(q/p) \leq \mathrm{ht}(\tilde{Q})$; then R_q is a quasilocal domain with treed spectrum and global dimension at most 2 , whence another appeal to [16, Theorem 2.2] completes the proof.

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