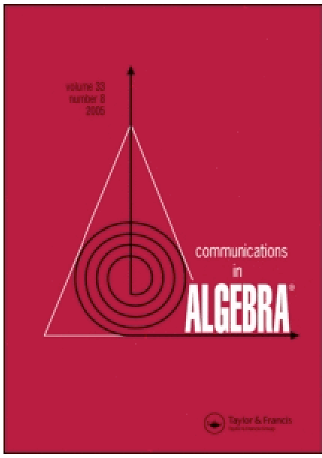

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Gorenstein conducive domains

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GORENSTEIN CONDUCIVE DOMAINS

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Let R be a (commutative integral) domain with quotient field K. As in [11], we say that R is a *conducive domain* in case the conductor $(R : V) := \{u \in K : uV \subset R\}$ is nonzero for each overring $V \neq K$ of R. (Conducive domains were introduced implicitly in [8, Theorem 4.5] and studied subsequently in [11] and [6].) As shown in [11], familiar examples of conducive domains include the classical $D+M$ construction (in the sense of [18, Appendix II]) and pseudo-valuation domains (in short, PVD's), in the sense of [20]. Noetherian conducive domains are particularly tractable. Indeed, if R is Noetherian and conducive, then R is local and of (Krull) dimension at most 1 ([11, Corollary 2.7]; see [5, Theorem 2.2] for a generalization to the Archimedean case). Moreover, if R is Noetherian and conducive, then each overring of R is Noetherian and conducive. In particular (apart from the trivial case $R = K$), the integral closure of R is a DVR which is a finitely generated R-module (cf. [6, Theorem 6 and Corollary 7]); and each overring of R is analytically irreducible. This last fact allows us to apply work of Kunz [23] on symmetric (value-)semigroups, with which we assume familiarity. Recently, basing their work on [23], Fröberg-Gottlieb-Häggkvist [17] have characterized the Gorenstein domains in a particular class of Noetherian conducive domains. *Our chief interest here* is to generalize [17, Theorem].

The authors of [17] considered the following type of Noetherian conducive domain R : the integral closure of R is $R' = k[[X]]$ where k is a field, $k \subset R$, and $(R : R') = X^{2n}k[[X]]$ for some positive integer n ; they proved that *R is Gorenstein if and only if R is maximal with the given conductor.* The examples in [17] (and [23, Theorem]) reveal the importance of their third assumption. While also retaining the second assumption (in the form that R contains the residue class field of R'), we obtain the desired generalization in Theorem 4. By using completions and flatness (at the level of [9]) in conjuction with [17, Theorem], we show how to generalize to the case in which R' is an arbitrary (not necessarily complete) DVR.

It is, however, not possible to generalize [17, Theorem] to arbitrary Noetherian conducive domains. Indeed, Remark 9 (b) shows, in the absence of the hypothesis $k \subset R$, that a Noetherian PVD need not satisfy the conclusion of Theorem 4. This is developed as an application of Proposition 6. In the latter result, we characterize the Gorenstein domains among the Noetherian conducives, as an application of work of Ferrand and Olivier [15] on minimal homomorphisms. Moving beyond the local case to what might be termed the "locally conducive" Gorenstein domains case, we show in Proposition 10 that quadratic orders of algebraic integers are Gorenstein and determine which of these are GPVD's. (GPVD's are a type of locally conducive domain introduced in [12] and [13].) We close with a number of relevant nonlocal examples.

Now ***we fix notation from this point until the end of Remark 5***, as follows: *R is a Noetherian conducive domain with (unique) maximal ideal M ; $V := R'$ (is a* DVR) *with maximal ideal $N := \pi V$; $k := V/N$; and $I := (R : V)$.*

We begin by stating a result of Kunz, contained in the first paragraph of the proof of [23, Theorem]. First, recall from [17] that if S is a numerical semigroup, then $g(S)$ denotes the largest integer which is not contained in S ; and $v(R)$

denotes the value-semigroup of R, where v is a (discrete rank 1) valuation associated to V.

LEMMA 1 (Kunz [23]). *Assume that* $k \subset R$. *If* $n := g(v(R))$, *then* $I = N^{n+1}$. ∎

The next result accomplishes half our main goal of generalizing [17, Theorem]. First, we recall two results. Since R is Noetherian and conducive, it is analytically irreducible, so that [23, Theorem] applies to it; the upshot in case $k \subset R$ is that R is Gorenstein if and only if $v(R)$ is a symmetric semigroup. Also, by a result of Roquette [28] (which is also given in [23, page 748]), if R is Gorenstein, then $\ell_R(V/I) = 2\ell_R(R/I)$ (Roquette's proof depends on the standard fact [22, Theorem 222] that R is Gorenstein if and only if each of its nonzero ideals is divisorial.)

PROPOSITION 2. *Assume that* $k \subset R$. *If* R *is Gorenstein, then there is no strictly larger ring* T *such that* $R \subset T \subset V$ *and* $(T : V) = I$.

Proof. Since R is Gorenstein, $v(R)$ is symmetric. Now suppose that $(T : V) = I$ for some ring T such that $R \subset T \subset V$. By the earlier remarks, T is Noetherian and conducive. Hence, since $(T : V) = (R : V)$, it follows from Lemma 1 that $g(v(T)) = g(v(R))$. Therefore, by an observation in [17, page 1622], we may conclude $v(R) = v(T)$, in view of the symmetry of $v(R)$ and the inclusion $v(R) \subset v(T)$. In particular, $v(T)$ is symmetric, and so T is Gorenstein. The above-cited result of Roquette now gives the equations:

$$\ell_R(V/I) = 2\ell_R(R/I) \quad \text{and} \quad \ell_T(V/I) = 2\ell_T(T/I).$$

Let $d := \dim_{R/M}(T/N \cap T)$. Since $k \subset R/M \subset T/(N \cap T) \subset V/N = k$ canonically, we have $d = 1$. Hence:

$$\ell_R(V/I) = \ell_{R/I}(V/I) = d \cdot \ell_{T/I}(V/I) = \ell_T(V/I)$$

and, similarly, $\ell_R(T/I) = \ell_T(T/I)$. It follows that:

$$2\ell_R(R/I) = \ell_R(V/I) = \ell_T(V/I) = 2\ell_T(T/I) = 2\ell_R(T/I)$$

and so $\ell_R(R/I) = \ell_R(T/I)$. Thus:

$$\ell_R(T/R) = \ell_R((T/I)/(R/I)) = \ell_R(T/I) - \ell_R(R/I) = 0,$$

and so $T/R = 0$, whence $T = R$, to complete the proof. ∎

Before giving our main result, we collect some useful information and techniques. Henceforth, $\widehat{\ }$ denotes completion in the M-adic topology.

LEMMA 3. *$\widehat{R}$ is a Noetherian conducive domain with maximal ideal $\widehat{M} = M\widehat{R}$; the integral closure of $\widehat{R}$ is $\widehat{V}$, which is a complete DVR; and $(\widehat{R} : \widehat{V}) = \hat{I}$. If $I = \pi^{n+1}V$, then $\hat{I} = \pi^{n+1}\widehat{V}$.*

Proof. By [9, Proposition 8 (i), page 204], $\widehat{R}$ is Noetherian with maximal ideal $\widehat{M} = M\widehat{R}$. We turn now to $\widehat{V}$. It is well known that the completion of V in the N-adic topology is a complete DVR with maximal ideal generated by π. We note next that the canonical map from $\widehat{V}$ into this completion is an isomorphism; that is,

$$\varprojlim V/M^{m}V \longrightarrow \varprojlim V/N^{m}$$

is an isomorphism. This follows via a cofinality argument:

$$(N^{m})^{n+1} = I^{m} \subset M^{m}V .$$

In the same way, we may identify the completions of I in the M-adic topology and in the N-adic topology; a similar comment holds for $\widehat{V}/\hat{I}$. Moreover, we have canonically that:

$$\hat{I} = \varprojlim I/M^{m}I = \varprojlim ((\pi^{n+1}V)/(M^{m}\pi^{n+1}V)) \cong \pi^{n+1}(\varprojlim V/M^{m}V) = \pi^{n+1}\widehat{V} .$$

We return to the assertion about $\widehat{R}$. Consider the pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & V \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & V/I \end{array}$$

as in [6, Theorem 6]. Since $\widehat{R}$ is a faithfully flat R-module [9, Proposition 9, page 206], appying the functor $\widehat{R} \otimes_R -$ yields another pullback diagram. By [9, Theorem 3, page 203; and Corollary 1, page 204], we see that the vertices of the new pullback diagram are canonically $\widehat{R}$, $\widehat{V}$, $\widehat{R/I} = \widehat{R}/\hat{I}$, and $\widehat{V/I} = \widehat{V}/\hat{I}$; that $(\widehat{R} : \widehat{V}) = \widehat{(R : V)} = \hat{I}$; and that $\widehat{V} = \widehat{R}V$, a finitely generated $\widehat{R}$-module. In particular $\widehat{V} = \widehat{R}'$. Since $\hat{I}$ canonically contains I , it is nonzero, and so applying [6, Theorem 6] to the new pullback diagram yields that $\widehat{R}$ is a conducive domain. The proof is complete. ◧

We next give our generalization of [17, Theorem].

THEOREM 4. *Assume that* $k \subset R$ *and that* $I = N^m$ *for some* even *positive integer* m. *Then* R *is Gorenstein if (and only if) there is no strictly larger overring* T *such that* $R \subset T \subset V$ *and* $(T : V) = I$.

Proof. Proposition 2 takes care of the parenthetical assertion. Conversely, suppose that R is maximal as a subring of V having conductor I. We claim that $\widehat{R}$ is maximal as a subring of $\widehat{V}$ having conductor $\hat{I}$.

Indeed, suppose that B is a ring such that $\widehat{R} \subset B \subset \widehat{V}$ and $(B : \widehat{V}) = \hat{I}$. Put $A := B \cap V$. Of course, $R \subset A \subset V$, and so $I \subset (A : V)$. We next establish the reverse inclusion. In fact,

$$(A : V) \subset \widehat{(A : V)} = (\widehat{A} : \widehat{V}) = (\widehat{R}A : \widehat{V}) \subset (B : \widehat{V}) = \hat{I}$$

whence $(A : V) \subset \hat{I} \cap V = I$ (the last step following since $\widehat{V}$ is a faithfully flat V-module). Hence $(A : V) = I$ and so, by the hypothesis on R, we have $A = R$; that is, $B \cap V = R$.

Consider $b \in B$. Let i be a positive integer. Since V is dense in $\widehat{V}$, there exists $v \in (b + \widehat{N}^{i}) \cap V$. (One could just as well take $v \in (b + \widehat{M}^{i}\widehat{V}) \cap V$ since the proof of Lemma 3 showed that the M-adic and the N-adic completions of V may be identified.) As $\widehat{N}$ is the maximal ideal of $\widehat{V}$, $\widehat{V}$ is a DVR and $I = N^m$ (by hypothesis), we have $\hat{I} = \widehat{N}^m$ (cf. also Lemma 3). Taking i = m leads to

$$v \in b + \widehat{N}^m = b + \hat{I};$$

write $v = b + e$, with $e \in \hat{I} \subset \widehat{R} \subset B$. Hence $v \in B \cap V = R$. Thus

$$b = v - e \in R + \widehat{R} = \widehat{R},$$

and we have proved the claim, namely that $B = \widehat{R}$.

The residue field of $\widehat{V}$ is $\widehat{V}/\widehat{N} = \widehat{V/N} = \hat{k} = k$, which is canonically contained in (R, $\widehat{R}$, and) $\widehat{V}$. Since $\widehat{V}$ is a complete DVR, it follows that $\widehat{V} \cong k[[X]]$ (cf. for instance [10, Theorem 6.3]). Since $\hat{I} = \widehat{N}^m$, the above claim allow us to apply [17, Theorem]. The upshot is that (the Noetherian conducive domain) $\widehat{R}$ is Gorenstein. Therefore, by [26, Teorema 4], R is Gorenstein. The proof is complete. ■

REMARK 5. (a) The last part of the proof of Theorem 4, namely that $A = R$ implies that $B = \widehat{R}$, may also be carried out as follows. We have canonical isomorphisms:

$$\widehat{R} \cong A \otimes_R \widehat{R} = (B \cap V) \otimes_R \widehat{R} \cong (B \otimes_R \widehat{R}) \cap (V \otimes_R \widehat{R}),$$

the last one holding via [9, Remark 1, page 18]. Now $V \otimes_R \widehat{R} \cong \widehat{V} \cong \widehat{\widehat{V}}$ canonically contains $B \otimes_R \widehat{R}$ since $\widehat{R}$ is R-flat. Thus, the displayed isomorphisms can continue with $B \otimes_R \widehat{R} \cong B\widehat{R}$ (via [9, Proposition 10, page 34], which applies since $\widehat{R}$ is R-faithfully flat). In other words, $\widehat{R}$ is identified with $B\widehat{R} = B$, as desired.

(b) In view of the role of [23] in the above work, it seems useful to make the following remark. A domain D is Noetherian and conducive if and only if D is (Noetherian) local, $\dim(D) \leq 1$ and D is analytically irreducible.

We remind the reader that ***for the rest of the paper, the previous notation*** $(R, M, etc.)$ ***is no longer fixed.***

In the following Remark 9 (b), we shall show that the conclusion of Theorem 4 fails if we omit the hypotheses $k \subset R$ and $I = N^m$ for some *even* integer m. The most natural way to omit these hypotheses is to take R to be a PVD. Recall that PVD's have tractable pullback descriptions [2, Proposition 2.6]. The Noetherian upshot is this (cf. [2, Corollary 3.29], [6, Theorem 6]) : R is a Noetherian PVD if and only if $R \cong V \times_k k_0$, where (V, N) is a DVR, $k := V/N$, and $k_0 \subset k$ is a finite-dimensional field extension. Necessarily in this situation, we have that N is (also) the maximal ideal of R, $V = (N : N) = R'$, and $k_0 = R/N$.

We next use the conductors to develop a characterization of Gorenstein conducive domains that does not have the restrictive hypotheses of Theorem 4. Since any DVR is Gorenstein, we focus on the case $R \neq R'$.

PROPOSITION 6. *Let R be a Noetherian conducive domain with maximal ideal M. Assume that R is not integrally closed. Put $E := (M : M)$ and $k_0 := R/M$. Then the following conditions are equivalent:*

(1) *R is Gorenstein;*

(2) $\ell_R(E/R) = 1$;

(3) $\dim_{k_0}(E/M) = 2$;

(4) either $E/M \cong k_0[X]/(X^2)$ *as a k_0-algebra* or E/M *is a two-dimensional field extension of* k_0.

Proof. Note that $E \cong \text{End}_R(M)$. Hence, by [15, Proposition 4.7], (1) is equivalent to $\ell_R(E/R) \leq 1$. However, if R is Gorenstein (not integrally closed, by hypothesis) then $\ell_R(E/R) \neq 0$, that is, $E \neq R$. Indeed, since $R \neq R'$, we see in this case via [21, final paragraph] that E is a minimal proper overring of R. Thus, (1) $\Leftrightarrow$ (2).

The equivalence of (2) and (3) results from the following calculation of lengths:

$$\ell_R(E/R) = \ell_{k_0}(E/R) = \dim_{k_0}((E/M)/k_0) = \dim_{k_0}(E/M) - 1 .$$

Since (4) $\Longrightarrow$ (3) trivially, it suffices to show that (2) $\Longrightarrow$ (4). Assume (2). Hence there is no R- module contained strictly between R and E. In particular, the inclusion map $R \hookrightarrow E$ is a minimal homomorphism, in the sense of [15]. By [15, Lemme 1.4 (i)] $k_0 \hookrightarrow E/M$ is also a minimal homomorphism. According to [15, Lemme 1.2], (4) will follow once we show that E/M is not isomorphic to $k_0 \times k_0$ as a k_0-algebra. This, in turn, holds since E (and hence E/M) is local. The proof is complete. ∎

We proceed to analyze the dichotomy in condition (4) of Proposition 6.

REMARK 7. (a) Put $R := k + X^2k + X^4k + X^6k[[X]]$, where k is a field. Evidently, $R' = k[[X]]$ and $(R : R') = X^6k[[X]]$. Hence R is a conducive domain. Moreover, R is Noetherian (by Eakin's Theorem, cf. [25, Theorem 3.7 (i)], since $R' = R[X]$). In fact, R is Gorenstein by [23, Theorem], since $v(R)$ is a symmetric semigroup. (This last assertion may be seen via [23, Lemma]: $g(v(R)) = 5$, $\{0, 2, 4\} \subset v(R)$, and $\{1, 3, 5\} \subset \mathbb{N} \setminus v(R)$.) In particular, R is the kind of domain considered in Theorem 4 (and [17, Theorem]).

Moreover, R satisfies the first possibility in condition (4) of Proposition 6. To see this, first note that the maximal ideal of R is $M := X^2k + X^4k + X^6k[[X]]$, so that $E = (M : M) = k + X^2k + X^4k[[X]]$. Then $E/M = k[\varepsilon]$, where $\varepsilon := X^5 + M$. Observe that $\varepsilon^2 = X^{10} + M = 0$, whence $E/M \cong k[X]/(X^2)$, as asserted.

(b) Now assume that R is as in Proposition 6 and satisfies the second possibility in condition (4). We shall show that R is *not* of the kind considered in Theorem 4 (and [17, Theorem]). Indeed, we show that R is a PVD. (Hence, *any* Gorenstein domain of the kind in Theorem 4 must satisfy the first possibility in condition (4).)

Let $V := R'$. It suffices to show that $E = V$. We notice that E is local one-dimensional Noetherian domain, and M is also the maximal ideal of E. Since $(M : M) = E$, then by [4, Theorem 1.8] it is easy to conclude that $E = E' = R' = = V$. The proof is complete.

It is convenient to record the PVD case of Proposition 6 ((1) $\Leftrightarrow$ (4)).

COROLLARY 8. *Let R be a Noetherian PVD with maximal ideal M and canonically associated valuation ring $V := (M : M) = R'$. Then R is Gorenstein if and only if $[V/M : R/M] \leq 2$.* ∎

REMARK 9. (a) It seems worthwhile to record an example of a (non-integrally closed) Gorenstein PVD. In view of Corollary 8, one natural example is $\mathbb{R} + X\mathbb{C}[[X]]$.

(b) Let (R, M) be a Noetherian PVD with $V := R'$ such that $[V/M : R/M] =$ p, an odd prime. (A concrete example with p = 3 is given by $R := \mathbb{Q} + X\mathbb{Q}(\sqrt[3]{2})[[X]]$. It is interesting to note that a similar construction was introduced in [3] for other purposes.) By Corollary 8, R is not Gorenstein. However, R *is* maximal as a subring of V with given conductor to V. Indeed, there is no ring strictly between R and V (since any such ring would induce a field strictly between R/M and V/M). In particular, the "if" assertion of Theorem 4 fails if we omit the hypotheses " $k \subset R$ and $I = N^{m}$ for some *even* integer m ".

It is wellknown that a domain is Gorenstein if and only if it is locally Gorenstein [19, Th. 9.6]. Accordingly, it is of interest to move beyond the conducive case and consider "locally conducive" Gorenstein domains. A natural family of locally conducive domains is given by the LPVD's (and GPVD's) introduced in [12]. We turn to these next.

We assume familiarity with the definitions of LPVD (locally pseudo-valuation domain) and GPVD given in [12]. We also assume the fact that each GPVD is an LPVD [12, page 156]; and that the converse is false, even for one-dimensional Noetherian domains [12, Example 3.4].

Example 11 will exhibit nonlocal Gorenstein domains, some of which are GPVD's (hence, locally conducive), others being non-locally conducive. To do this, we use quadratic orders of algebraic integers. Recall that each nonmaximal such order is unique expressible as $\mathbb{Z}[n\omega_d]$, where d is a squarefree integer, $n \geq 2$ is an integer, and

$$\omega_d := \begin{cases} (1+\sqrt{d})/2, & \text{if } d \equiv 1 \pmod 4; \\ \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Of course, $\mathbb{Z}[\omega_d]$ is the (Dedekind, hence Gorenstein) maximal order of $\mathbb{Q}(\sqrt{d})$.

PROPOSITION 10. *Let* $R := \mathbb{Z}[n\omega_d]$, *where* d *is a squarefree integer and* $n \geq 2$ *is an integer. Then:*

(a) *R is Gorenstein;*

(b) *R is a GPVD if (and only if) R is an LPVD.*

Proof. (a) Each nonzero ideal I of R is a *module of* $\mathbb{Q}(\sqrt{d})$, in the sense of [1]. Indeed, I satisfies the conditions of [1, Theorem 6, page 226]; for if $0 \neq r \in I$, then $\{r, rn\omega_d\}$ is linearly independent over $\mathbb{Q}$. In particular, I is a 2-generated ideal of R. Thus if $P \in \text{Spec}(R)$, each ideal J of R_P is 2-generated (since $J = (J \cap R)R_P$). By a result of Bass and Matlis (cf. [7, Proposition 6.4], [24, Theorem 13.2, page 120]), R_P is Gorenstein. Hence, so is R.

(b) We adapt some techniques developed to study GPVD's in [12]. Since R is an LPVD, it is seminormal. In particular, it is "seminormal in" $T := \mathbb{Z}[\omega_d]$, in the sense of [29]. Thus, by [29, Lemma 1.3], $I := (R : T)$ is a radical ideal of T. According to [14, Theorem 2.5], it remains only to show that $\text{Spec}(T) \longrightarrow \text{Spec}(R)$ is an injection. If M is a maximal ideal of R, then R_M is a Noetherian PVD; hence the integral closure of R_M is a DVR (cf. [2, Corollary 3.30]). It follows easily that T cannot have distinct primes contracting to M. The proof is complete. ■

In view of the above "ubiquity" result, it is interesting to note that a cubic order of algebraic integers need *not* be Gorenstein. An explicit example of this has been given by M. Picavet-L'Hermitte [27, pages 28-30]. Also with regard to Proposition 10 (b), it should be noted that the ring R in [12, Example 3.4] can be arranged to be a (one-dimensional) Gorenstein LPVD which is not a GPVD. (Simply take $n = 2$ in the definition of K_i.)

EXAMPLE 11. (a) There exists a Gorenstein nonlocal GPVD (hence locally conducive domain) of the form $R = \mathbb{Z}[n\omega_d]$. Indeed, it suffices to take n to be an odd rational prime p and d a squarefree integer such that the Legendre symbol (d/p) = -1. (For instance, take p = 3, d = 2; or p = 5, d = 3.)

For the verification, observe that R is Gorenstein via Proposition 10; R is nonlocal since $\mathrm{Max}(R) \to \mathrm{Max}(\mathbb{Z})$ is surjective, by integrality; and R is a GPVD by [14, Theorem 2.5].

(b) There exists a Gorenstein nonlocal non-locally conducive domain of the form $R = \mathbb{Z}[n\omega_d]$. Indeed, it suffices to take n = 2 and d a squarefree integer such that $d \equiv 1 \pmod 8$. (Then $R = \mathbb{Z}[2\omega_d] = \mathbb{Z}[\sqrt{d}]$.)

As in the proof of (a), we see that R is Gorenstein and nonlocal. By Proposition 10 (b) and [14, Remark 2.6], R is not LPVD (cf. also [13, Example 4]). However, R is seminormal, by applying [14, Corollary 4.5]. It is known (cf. [11, Corollary 2.9] or [6, Corollary 11]) that a Noetherian seminormal conducive domain must be a PVD. Hence, some localization of R is not conducive, to complete the proof.

For the sake of completeness we give next the analogue of Example 11 for Noetherian nonlocal non-Gorenstein domains.

EXAMPLE 12. (a) There exists a Noetherian nonlocal non-Gorenstein GPVD (hence, locally conducive domain). For instance, consider the polynomials $f_1 := X^4 + X^3 + X^2 + X + 1$ and $f_2 := X^2 + X + 1$, both of which are irreducible over $\mathbb{Q}$. Put $K_i := \mathbb{Q}[X]/(f_i)$. It suffices to take R to be the pullback of the canonical surjection $\mathbb{Q}[X] \to K_1 \times K_2$ and the inclusion $\mathbb{Q} \times \mathbb{Q} \hookrightarrow K_1 \times K_2$.

By a standard "gluing" argument (cf. [16, Theorem 1.4]), the map $\mathrm{Spec}(\mathbb{Q}[X]) \to \mathrm{Spec}(R)$ is a bijection, indeed, a homeomorphism and an order-isomorphism. In particular, R is nonlocal. Note also that R and $\mathbb{Q}[X]$ are one-dimensional domains sharing a nonzero radical ideal $I := f_1 f_2 \mathbb{Q}[X]$. Hence, by definition (cf. [12, Theorem 3.1]), R is a GPVD.

Consider a nonzero $P \in \mathrm{Spec}(R)$, with N the unique prime of $\mathbb{Q}[X]$ lying over P. As in the proof of Proposition 10 (b), we have $R_P \cong \mathbb{Q}[X]_N$ if $I \not\subset P$. If $I \subset P$, then N is either (f_1) or (f_2). Let P_i denote the corresponding primes of R. By standard isomorphism theorems, $R/P_i \cong \mathbb{Q}$. Since $\mathbb{Q}[X]$ is a

Dedekind domain, [12, Proposition 3.6] now yields that R is Noetherian. Moreover, R_{P_1} is a PVD with canonically associated valuation ring $\mathbb{Q}[X]_{(f_1)}$ (by [12, page 156]) and a four-dimensional induced extension of residue class fields. Hence, by Corollary 8, R_{P_1} is not Gorenstein; thus neither is R.

(b) Arguing as in (a), we may produce also a Gorenstein nonlocal GPVD (cf. also Example 11 (a)). It suffices to take the pullback of the surjection $\mathbb{C}[X] \to \mathbb{C}[X]/(X^2 - X) \cong \mathbb{C}\times\mathbb{C}$ and the inclusion $\mathbb{R}\times\mathbb{R} \hookrightarrow \mathbb{C}\times\mathbb{C}$.

(c) Finally, we exhibit an example of a nonlocal, non locally conducive, non-Gorenstein, Noetherian one-dimensional domain (cf. also Example 11 (b)).

Let $k' \hookrightarrow k$ be a proper inclusion of fields of characteristic zero and let X be an indeterminate over k. Suppose that $3 \leq [k : k'] < \infty$. Set $T := k[X]$, $M_0 := (X)$, $M_1 := (X - 1)$, $N := (X - 2)$. We consider the following rings, viewed as pullbacks in the obvious ways:

$$R := \{ f \in T : \ f(0) = f(1) \text{ and } f(2) \in k' \},$$

$R_1 := \{ \varphi \in T_{M_0} \cap T_{M_1} : \ \varphi(0) = \varphi(1) \}$ and $R_2 := \{ \varphi \in T_N : \ \varphi(2) \in k' \}$. It is easy to see that $R \hookrightarrow T$ is finite, having as conductor the ideal $I := M_0 \cap M_1 \cap N$. By [16, Theorem 1.4 and Proposition 1.8] R is a one-dimensional Noetherian domain. A direct calculation shows that $M_0 \cap R = M_1 \cap R$. Set $m := M_0 \cap R$ and $n := N \cap R$. It is clear, from the universal mapping property of pullbacks, that $R_m \subset R_1$ and $R_n \subset R_2$.

We next prove the reverse inclusions. If $\varphi \in R_1$ then, by applying the Lagrange interpolation formula, it is possible to find $\alpha \in R \setminus m$ such that $\alpha\varphi \in R$. (Indeed write $\varphi = u/v$ with $u, v \in T$ and $v(0) \neq 0$, $v(1) \neq 0$. If $\varphi(0) = \varphi(1) \neq 0$, put $\alpha := av$ where $a \in T$ satisfies $a(0) = u(1)$, $a(1) = u(0)$, $a(2) = 0$; if $\varphi(0) = \varphi(1) = 0$, put $\alpha := bv$ where $b \in T$ satisfies $b(0) = v(1)$, $b(1) = v(0)$, $b(2) = 0$.) Thus $R_m = R_1$.

By a similar argument, one can prove that $R_n = R_2$. (Suppose $\varphi = u/v \in R_2$ with $u, v \in T$, $v(2) \neq 0$ and $\varphi(2) \in k'$. Then $(av)\varphi \in R$, where $a \in T$ satisfies $a(0) = a(1) = 0$ and $a(2) = 1/v(2)$.)

By Remark 5 (b), R_1 is non-conducive, because $R_1' = T_{M_0} \cap T_{M_1}$ is not a valuation domain; and (the Noetherian PVD) R_2 is non-Gorenstein by Corollary 8, because $[k : k'] \geq 3$. Hence R is a domain with the announced properties.

We have already given an example of a nonlocal, locally conducive Gorenstein domain (cf. Example 11 (a)). Such a domain satisfies locally the second possibility in condition (4) of Proposition 6, *i.e.* it is locally a PVD. We close with a class of examples of nonlocal, locally conducive, non necessarily LPVD, Gorenstein domains.

EXAMPLE 13. (a) Let C be a semilocal principal ideal domain, P a nonzero prime ideal of C and $V := C_P$. Let R be a Gorenstein conducive domain with integral closure equal to V and suppose that R satisfies the first possibility in condition (4) of Proposition 6. Set $A := R \cap C$. We claim that the canonical map $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is a homeomorphism and moreover that, for every $Q \in \mathrm{Spec}(C)$, if $Q \neq P$, then $A_q = C_Q$, where $q := Q \cap A$, and if $Q = P$, then $A_p = R$, where $p := P \cap A$.

By construction, $I := (R : V) = (PC_P)^n$, where $n \geq 2$, because R is not a PVD. Since $P^n \subset I \subset R$ and $P^n \subset C$, we have $P^n \subset R \cap C = A$, thus $P^n \subset (A : C)$. Consider $(0) \neq q \in \mathrm{Spec}(A)$. If $P^n \not\subset q$ then $(A : C) \not\subset q$, hence by [16, Theorem 1.4, (c)] there exists exactly one prime $Q \in \mathrm{Spec}(C)$ above q and, moreover, $A_q = C_Q$. If $q \supset P^n \supset p^n$, then $q \supset p$.

If $q = p$, first notice that there is only the prime P of C over p. (In fact, if $P' \in \mathrm{Spec}\, C$ and $P' \cap A = p$, then $P' \supset P^n$, so $P' \supset P$ hence $P' = P$.) We want to show now that $A_p = R$. Trivially $A_p \subset R$, because R is a local overring with the maximal ideal over p. For the opposite inclusion, set $S := A \setminus p$ and notice first that, since P is the only prime of C disjoint from S, then $S^{-1}C$ coincides with V. Now, let x be an element of R. Since $R \subset V = S^{-1}C$, there exists $s \in S$ such that $xs \in C$. Since $x \in R$ and $s \in S \subset A \subset R$, we have $xs \in R$. Thus $xs \in C \cap R = A$.

To prove our claim it suffices to show that p is a maximal ideal of A. Suppose $p \subset m \in \mathrm{Spec}(A)$, and set $T := A \setminus m$. Notice that $T^{-1}C = \cap \{C_Q : Q \in \mathrm{Spec}(C),\ Q \text{ disjoint from } T\}$ and that $T^{-1}R = T^{-1}A_p = A_p = R$. Thus if $Q_1, \ldots, Q_r$ are the primes of C disjoint from T and different from P, we have $A_m = T^{-1}A = T^{-1}C \cap T^{-1}R = C_{Q_1} \cap \ldots \cap C_{Q_r} \cap R$. Now the set of non-units in A_m is:

$$mA_m = (Q_1C_{Q_1} \cap A_m) \cup \ldots \cup (Q_r C_{Q_r} \cap A_m) \cup (pA_p \cap A_m).$$

If $q_i := Q_i \cap A$ (for $i = 1, \ldots, r$) then, after intersecting the previous equation with A, we get $m = q_1 \cup \ldots \cup q_r \cup p$. Since $m \neq q_i$ (in fact, $p \not\subset q_i$), for $i = 1, \ldots, r$, we have $m = p$.

Since A is semilocal and locally Noetherian, we deduce easily that A is Noetherian. In conclusion, if C is not local, A is a (one-dimensional) nonlocal locally conducive Gorenstein domain, but it is *not* an LPVD because $A_p = R$ is not a PVD.

(b) The construction in (a) can be generalized in the following way. Let C be a semilocal principal ideal domain. Set $C = V_1 \cap ... \cap V_s$, where (V_i, N_i) is a DVR, for $i = 1,..., s$. Fix $j \leq s$. For each $k = 1,..., j$, let A_k be a Gorenstein conducive domain with integral closure equal to V_k. Arguing as in (a), one can show that $A := A_1 \cap ... \cap A_j \cap V_{j+1} \cap ... \cap V_s$ is a semilocal one-dimensional Noetherian domain with maximal ideals $P_i := N_i \cap A$ $(i = 1,..., s)$, such that $A_{P_k} = A_k$, for $k = 1,..., j$ and $A_{P_i} = V_i$, for $i = j+1,..., s$. Such a domain A is a nonlocal, locally conducive Gorenstein domain; and it is an LPVD or a non-LPVD depending on the choice of the Gorenstein conducive domains A_i.

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