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G-DOMAINS AND SPECTRAL SPACES

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Each G-domain R has a canonically associated overring $R^\square$ such that Spec($R^\square$) is homeomorphic to Spec(R). In general, $R^\square$ is more tractable than R, since $R^\square$ is a pullback of a ring of fractions T of R such that each nonzero prime of T is contained in the union of height 1 primes. Domains with this latter property are dubbed 'essential', we construct several two-dimensional essential G-domains. Often, for instance if R is a seminormal going-down G-domain, $R = R^\square$. Interest in $R^\square$ is justified by establishing a natural bijection between homeomorphism classes of spectral spaces and homeomorphism classes of spectra of G-domains.

1. Introduction, summary, and notation

Let R be a (commutative integral) domain with quotient field K. R is said to be a *G-domain* if K is a finite-type R-algebra. This terminology honors the approach of Goldman [17] to the Hilbert Nullstellensatz. Following [13], we let $p(R)$, or simply p if no confusion results, denote the *pseudo-radical* of R, that is, the intersection of the set of nonzero prime ideals of R. It is well known that R is a G-domain if and only if $p(R) \neq 0$ (cf. [19, Theorem 19]). Thus, if Spec(R) is a finite set, then R is evidently a G-domain. The Artin–Tate theorem [19, Theorem 146] establishes the converse in the Noetherian case; that is, if R is Noetherian, then R is a G-domain if and only if R is semilocal of (Krull) dimension at most 1. Regarding the general situation, Kaplansky [19, p. 13] has stated, "... the facts are more complex, and we seem to lack even a reasonable conjecture concerning the structure of general G-domains".

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In recent years, examples of G-domains with diverse spectra [21, 25] have served to reinforce Kaplansky's assertion. Nevertheless, this article identifies a more tractable type of G-domain, called a 'G-domain of pullback type' (see Section 2 for its definition), sufficiently general so that to every G-domain R, there is canonically associated a G-domain of pullback type, denoted by $R^{\square}$. $R^{\square}$ is an overring of R and the induced map $\mathrm{Spec}(R^{\square}) \to \mathrm{Spec}(R)$ is a homeomorphism. In certain cases, for instance R a Prüfer G-domain, R coincides with $R^{\square}$. (For proofs of these assertions, see Theorems 2.9 and 2.15, Corollaries 2.10 and 2.6, and Proposition 2.19(c).) As usual, if A is a commutative ring with unit, $\mathrm{Spec}(A)$ is equipped with the Zariski topology. Following [18], any topological space homeomorphic to such a $\mathrm{Spec}(A)$ is called a *spectral space*. It is convenient to call a *G-space* any topological space homeomorphic to $\mathrm{Spec}(T)$, for T a suitable G-domain. Now, in order to justify the above attention paid to G-spaces (and hence to $R^{\square}$), we can cite Proposition 4.2: there is a natural one-to-one correspondence between the homeomorphism classes of G-spaces and the homeomorphism classes of spectral spaces.

Section 2 develops the facts about $R^{\square}$ stated above. In order to explicate the construction of $R^{\square}$, it is convenient to fix some more notation. Assume now that R is a G-domain. Let $S = S(R)$ denote $R \setminus \bigcup \{P \in \mathrm{Spec}(R) \mid \mathrm{ht}(P) = 1\}$, let $\bar{R} = R/p$, and let T denote the total quotient ring of $\bar{R}$. Then $R^{\square}$ is just the pullback $\bar{R} \times_T S^{-1}R$, an overring of R contained in $S^{-1}R$. (The structure map $S^{-1}R \to T$ which is implicit in the pullback notation is available because of Lemma 2.5(b): T is canonically isomorphic to $S^{-1}\bar{R}$. We assume familiarity with the construction and universal property of pullback.) Since $R^{\square}$ is an overring of R, $R^{\square}$ is also a G-domain; and, since $R^{\square}$ is a pullback, its ideal-theory and spectrum are tractable (cf. [10]).

It is convenient to say that a G-domain R is *essential* (or that R is a *G-domain of essential type*) if each nonzero prime ideal of R is contained in the union of the height 1 prime ideals of R. This concept is motivated in part by the observation that many G-domains are one-dimensional (cf. Artin–Tate), but its principal motivation is that $S^{-1}R$ is essential, so that $R^{\square}$ is a pullback of an essential G-domain.

For each commutative ring A, we let $\mathrm{Spec}^i(A)$ denote the subspace of $\mathrm{Spec}(A)$ consisting of the height i primes. Section 3 is devoted to the study of G-domains R such that $\mathrm{Spec}^1(R)$ is a finite set. Such R are tractable, for if $\mathrm{Spec}^1(R) = \{P_1, \ldots, P_n\}$, then it follows easily from the prime avoidance lemma that $S^{-1}R = \bigcap R_{P_i}$. Section 3 pays attention to the G-domains, especially the essential ones, admitting such a representation in several categories of domains.

As for Section 4, suffice it to mention here that, besides the above-noted correspondence between G-spaces and spectral spaces, the section also contains a pair of interesting essential two-dimensional G-domains: see Example 4.1 and Remark 4.6(b).

In addition to the above standing notation (R, K, p, S, $\bar{R}$, T and $R^{\square}$), the following will be in force. R^+, R', and R^* denote the seminormalization (in the sense of [30]), the integral closure, and the complete integral closure of R, respectively; the corresponding pseudo-radicals are denoted by p^+, p', and p^*. Finally, $X(R)$ (resp., $X^1(R)$) denotes the set of all (resp., all one-dimensional) valuation overrings of R, and m_V denotes the maximal ideal of any given valuation ring V.

2. Essential G-domains and G-domains of pullback type

Lemma 2.1. *Let R be a G-domain. Then*

(a) *Every valuation overring of R other than K is contained in a maximal valuation overring of R distinct from K;*

(b) *Every $0 \neq Q \in \mathrm{Spec}(R)$ contains a minimal nonzero prime. Therefore, $p(R) = \bigcap\{P \mid P \in \mathrm{Spec}^1(R)\}$;*

(c) $p(R) = \bigcap\{(m_V \cap R) \mid V \in X^1(R)\}$.

Proof. (a) We need only verify that Zorn's lemma applies. Let $\{V_\alpha\}$ be a chain of valuation rings in $X(R)\backslash\{K\}$. Then, $W = \bigcup V_\alpha$ is necessarily a valuation overring of R. The only possible difficulty might be that $W = K$. Let $0 \neq x \in p(R)$. Since x lies in every nonzero prime ideal of R, $1/x \notin V$ for every nontrivial $V \in X(R)$. It follows that $1/x$ cannot lie in W and so $W \neq K$.

(b) Again, we need only verify that Zorn's lemma applies. Let $\{Q_\alpha\}$ be a chain of prime ideals in $\mathrm{Spec}(R)\backslash\{0\}$. Then, $P = \bigcap Q_\alpha$ is a prime ideal. Since $0 \neq p(R) \subset Q_\alpha$ for every α, $p(R) \subset P$, and $P \neq 0$.

(c) Given $P \in \mathrm{Spec}^1(R)$, there exists $V \in X(R)$ such that $m_V \cap R = P$. By (a), there exists $W \in X^1(R)$ such that $V \subset W$. Thus $m_W \cap R$ is a nonzero prime inside P, whence $m_W \cap R = P$. □

Lemma 2.2. *Let R be a G-domain. Then*

(a) *Every overring of R is a G-domain. In particular $S^{-1}R$ is a G-domain;*

(b) $p(S^{-1}R) = S^{-1}(p(R))$;

(c) $S^{-1}R \subset \bigcap\{R_Q \mid Q \in \mathrm{Spec}^1(R)\}$.

Proof. (a) Immediate from the definition of a G-domain.

(b) By Lemma 2.1, $p(R) = \bigcap\{Q \mid Q \in \mathrm{Spec}^1(R)\}$ and

$$\begin{aligned} p(S^{-1}R) &= \bigcap\{S^{-1}Q \mid S^{-1}Q \in \mathrm{Spec}^1(S^{-1}R)\} \\ &= \bigcap\{S^{-1}Q \mid Q \in \mathrm{Spec}^1(R) \text{ and } Q \cap S = \emptyset\}. \end{aligned}$$

But, $S = R\backslash\bigcup\{Q \mid Q \in \mathrm{Spec}^1(R)\}$ implies $Q \cap S = \emptyset$ is true for every $Q \in \mathrm{Spec}^1(R)$. Thus,

$$\begin{aligned} p(S^{-1}R) &= \bigcap\{S^{-1}Q \mid Q \in \mathrm{Spec}^1(R)\} \\ &\supset S^{-1}(\bigcap\{Q \mid Q \in \mathrm{Spec}^1(R)\}) \\ &= S^{-1}(p(R)). \end{aligned}$$

To verify the reverse containment, let

$$x = \frac{x_1}{x_2} \in p(S^{-1}R) \subset S^{-1}R \quad \text{where } x_1 \in R \text{ and } x_2 \in S\,.$$

Since $x \in \bigcap\{S^{-1}Q \mid Q \in \operatorname{Spec}^1(R)\}$, it follows that, for every $Q \in \operatorname{Spec}^1(R)$ there exists $s_Q \in S$ such that $s_Q x \in Q$. Then, since $s_Q x_1 = s_Q x_2 x \in Q$, it follows that $x_1 \in Q$ for every $Q \in \operatorname{Spec}^1(R)$ and, therefore, $x \in S^{-1}(\bigcap\{Q \mid Q \in \operatorname{Spec}^1(R)\})$ as claimed.

(c) It suffices to note that $S \subset R \backslash Q$ for every $Q \in \operatorname{Spec}^1(R)$. □

Corollary 2.3. $(S^{-1}R)/(p(s^{-1}R)) \simeq S^{-1}(\bar{R})$ *for every G-domain R.* □

Remark 2.4. If R is a seminormal G-domain, and $S^{-1}R \subset R^*$, one can strengthen Lemma 2.2(b). In that case, $p(S^{-1}R)$ actually equals both $p(R)$ and $p(R^*)$. (See Theorem 2.9 and Corollary 2.11.) A complete characterization of those G-domains for which $p(R) = p(S^{-1}R)$ would be interesting since this condition is necessary and sufficient for R to be isomorphic to a pullback of $S^{-1}R$. (See Theorem 2.15.)

Let W be any ring. We denote by ZD(W) (respectively, NZD(W)) the set of all zero divisors (respectively, all nonzero divisors) of W. The total quotient ring of W, denoted Tot(W), is $\{r/s \mid r \in W \text{ and } s \in \operatorname{NZD}(W)\}$.

Lemma 2.5. *Let R be a G-domain with pseudo-radical p, and let Y be the union of all minimal primes of* $\bar{R} = R/p$. *Then*

(a) $Y = \operatorname{ZD}(\bar{R})$;

(b) $\operatorname{Tot}(\bar{R}) = S^{-1}(\bar{R}) \simeq (S^{-1}R)/(p(S^{-1}R))$.

Proof. (a) That $Y \subset \operatorname{ZD}(\bar{R})$ is well known [19, Theorem 84]. For the reverse inclusion, note that $p = \bigcap\{Q \mid Q \in \operatorname{Spec}^1(R)\}$ (Lemma 2.1). Thus, if $\bar{x}\bar{y} = 0$ and $\bar{y} \neq 0$, then there exists $Q \in \operatorname{Spec}^1(R)$ such that $y \not\in Q$. It follows that $x \in Q$ and so $\bar{x} \in Y$.

(b) Let $\bar{x} \in \bar{R}$. By (a) $\bar{x} \in \operatorname{NZD}(R)$ if and only if $x \not\in \bigcup\{Q \mid Q \in \operatorname{Spec}^1(R)\}$; that is, if and only if $x \in S$. Thus $\operatorname{Tot}(\bar{R}) = S^{-1}(\bar{R})$. That $S^{-1}(\bar{R}) \simeq (S^{-1}R)/(p(S^{-1}R))$ was already noted in Corollary 2.3. □

Remark 2.6. By the proof of Lemma 2.5, a G-domain R has essential type if and only if $\bar{R} = \operatorname{Tot}(\bar{R})$. Thus every one-dimensional G-domain has essential type. Since it is known that all Noetherian and all Krull G-domains satisfy $\dim(R) \leq 1$ (a fresh proof of this relatively easy fact is included in Remark 2.18), it follows that all Noetherian and Krull G-domains have essential type. However, any valuation ring V of finite dimension $n \geq 2$ is a G-domain of nonessential type; indeed, the pseudo-radical of V is the unique height 1 prime P of V and so $S^{-1}V = V_P \neq V$.

Remark 2.7. In many essential G-domains, (for example, if $\text{Spec}^1(R)$ is a finite set), $R = \bigcap\{R_P \mid P \in \text{Spec}^1(R)\}$. By Lemma 2.2(c), $R \subset S^{-1}R \subset \bigcap\{R_P \mid P \in \text{Spec}^1(R)\}$ is always true for G-domains. Hence, if a G-domain $R = \bigcap\{R_P \mid P \in \text{Spec}^1(R)\}$, then $R = S^{-1}R$ is necessarily essential.

Proposition 2.8. *Let R be an integrally closed G-domain. Then*
(a) $R^* = \bigcap\{V \mid V \in X^1(R)\}$;
(b) $p(R) = p(R^*) = \bigcap\{m_V \mid V \in X^1(R)\}$.

Proof. (a) This is a result due to Gilmer and Heinzer [15].
(b) $p(R^*) = \bigcap\{(m_V \cap R^*) \mid V \in X^1(R^*)\}$ by Lemma 2.1(c). But $X^1(R) = X^1(R^*)$ by (a). Therefore $p(R^*) = (\bigcap\{m_V \mid V \in X^1(R)\}) \cap R^*$.
On the other hand, applying Lemma 2.1(c) to R yields $p(R) = \bigcap\{(m_V \cap R) \mid V \in X^1(R)\} = (\bigcap\{w_V \mid V \in X^1(R)\}) \cap R$. An application of Lemma 2.1(a) makes clear that $\bigcap\{m_V \mid V \in X^1(R)\} = \bigcap\{m_V \mid V \in X(R)\} \subset \bigcap\{V \mid V \in X(R)\} = R$ (since R is integrally closed). The assertions now follow easily. □

In Corollary 2.16, we establish, for a certain class of G-domains R, that R is a pullback of its associated essential G-domain $S^{-1}R$. We first characterize that class.

Theorem 2.9. *Let R be a G-domain. Then, the following are equivalent:*
(a) $\{x \in K \mid x^2 \in p \text{ and } x^3 \in p\} \subset R$;
(b) $p = p^+$;
(c) $p = p'$;
(d) $p = p^*$;
(e) $\bigcap\{m_V \mid V \in X^1(R)\} \subset R$.
We say that the G-domain R is saturated *if it satisfies any of these equivalent conditions.*

Proof. If S is any overring of R, then $p(R) \subset p(S)$ (since every nonzero prime of S intersects to a nonzero prime of R). In particular, $p \subset p^+ \subset p' \subset p^*$, from which (d) $\Rightarrow$ (c) $\Rightarrow$ (b) is an immediate consequence.
By Proposition 2.8 and the fact that $X^1(R) = X^1(R')$, $p' = p^* = \bigcap\{m_V \mid V \in X^1(R)\}$. (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) follows immediately. Also, $p^* \cap R = (\bigcap\{m_V \mid V \in X^1(R)\} \cap R) = \bigcap\{m_V \cap R \mid V \in X^1(R)\} = p$ by Lemma 2.1(c). If we assume (e), then $p^* \subset R$ so $p^* = p^* \cap R = p$, which verifies (d).
To prove (b) $\Rightarrow$ (a), it suffices to show that the set $\Omega = \{x \in K \mid x^2 \in p \text{ and } x^3 \in p\}$ is contained in p^+. Since p^+ is a radical ideal in R^+ and every $x \in \Omega$ satisfies $x^2 \in p \subset p^+$, the problem reduces to showing that $\Omega \subset R^+$. This is immediate from a standard characterization of seminormal domains.

The most difficult part of this theorem is the fact that (a)$\Rightarrow$(d). Its proof will proceed through a remark and two lemmas which we number for convenient reference. As Remark 2.9.1 is well known (and easily verified), its proof is omitted.

Remark 2.9.1. Let R be a domain with quotient field K; let S be an overring of R. Then

(a) $(R:S)$ is the largest subset of K which is an ideal in both R and S; (In particular, $(R:S) \subset R$.)

(b) If J is any ideal in S, then $(R:S) \cap J$ is an ideal in both R and S;

(c) If I is an ideal in R, $(I:I)$ is the largest overring of R in which I remains an ideal.

Lemma 2.9.2. *Let R be a G-domain and S an overring of R. Assume that $\{x \in K \mid x^2 \in p(R)$ and $x^3 \in p(R)\} \subset p(R)$ and that $(R:S) \neq 0$. Then, $p(R) = p(S)$ and, consequently, $(R:S) \supset p(S)$.*

Proof. Let $I = (R:S) \cap p(S)$. By Remark 2.9.1(b), I is an ideal in both R and S and therefore we need to distinguish the radical of I in R, $\mathrm{rad}_R I$, from that of I in S, $\mathrm{rad}_S I$. The heart of the proof is to show that $\mathrm{rad}_R I = \mathrm{rad}_S I$.

Clearly, $\{x \in R \mid x^N \in I \text{ some } N \in \mathbb{Z}^+\} \subset \{x \in S \mid x^N \in I \text{ some } N \in \mathbb{Z}^+\}$. So it suffices to verify the reverse containment. Since $(R:S) \neq 0$, the two rings have the same complete integral closure: $R^* = S^*$. Thus, using the facts that $p(S^*) \cap S = p(S)$ and $p(R^*) \cap R = p(R)$ (see proof of (d)$\Leftrightarrow$(e) above), $p(S) \cap R = p(S^*) \cap S \cap R = \bigcap\{m_V \cap S \mid V \in X^1(S^*)\} \cap R = \bigcap \{m_V \cap R \mid V \in X^1(R^*)\} = p(R^*) \cap R = p(R)$.

Now, we claim that $\mathrm{rad}_S I \subset R$ (from which it follows that $\mathrm{rad}_S I \subset \mathrm{rad}_R I$). To prove this claim, let $x \in \mathrm{rad}_s I$ be arbitrary. For some integer $N \geq 1$, $x^N \in I$. Of course, then $x^n \in I = (R:S) \cap p(S) \subset R \cap p(S) = p(R)$ whenever $n \geq N$. The set $\{d \in Z \mid x^d \not\in p(R)\}$ is therefore bounded above by N. Let t be the largest nonnegative integer such that $x^t \not\in p(R)$. If $t > 0$, then $3t > 2t > t$ from which it follows that $(x^t)^2$ and $(x^t)^3$ lie in $p(R)$. By our assumption, this forces x^t into $p(R)$ which is a contradiction. Hence, $t = 0$ and $x \in p(R) \subset R$. Thus, $\mathrm{rad}_S I \subset R$. Hence, $\mathrm{rad}_S I = \mathrm{rad}_R I \subset p(R) \subset p(S)$. However, $p(S)$, being the intersection of all nonzero primes, is contained in every nonzero radical ideal in S. Therefore, $p(S) = \mathrm{rad}_S I$, forcing $p(R) = p(S)$. $\square$

Lemma 2.9.3. *Let R be a G-domain, with pseudo-radical p, satisfying $\{x \in K \mid x^2 \in p$ and $x^3 \in p\} \subset p$. Then*

(a) *For every $x \in R^*$, $(R:R[x]) \supset p$ and $p = p(R[x])$;*

(b) $(p:p) = R^*$.

Proof. (a) By Lemma 2.9.2 with $S = R[x]$, it suffices to show that $(R:R[x]) \neq 0$. But this is immediate from the definition of complete integral closure.

(b) Let $x \in R^*$. By (a), $p \subset (R:R[x])$; in fact, p is an ideal common to both rings. Given any $y \in p$, $yx \in pR[x] = p$; whence, $x \in (p:p)$. Thus, $R^* \subset (p:p)$. For the reverse inclusion, set $S = (p:p)$ and note via Remark 2.9.1 that $(R:S) \supset p \neq 0$. Therefore, $S \subset R^*$. □

We now *complete the proof of Theorem* 2.9. Assume (a), namely that $\{x \in K \mid x^2 \in p \text{ and } x^3 \in p\} \subset R$. Then, $\{x \in K \mid x^2 \in p \text{ and } x^3 \in p\} \subset \operatorname{rad}_R(p) = p$. By Lemma 2.9.3(b), $(p:p) = R^*$ and, therefore, p is a common ideal of both R and R^*. This implies that $(R:R^*) \neq 0$ from which it follows by Lemma 2.9.2 (with $S = R^*$) that $p = p^*$.□

Corollary 2.10. *If R is a seminormal G-domain, then*
(a) *R is saturated*; *and*
(b) $p(R) = p(R') = p(R^*)$. □

Corollary 2.11. *For a saturated G-domain R,* $p(R) = p(S^{-1}R)$ *if and only if* $S^{-1}R \subset R^*$.

Proof. If $S^{-1}R \subset R^*$, then $p(R) \subset p(S^{-1}R) \subset p(R^*)$. But, for a saturated G-domain, $p(R) = p(R^*)$, so $p(R) = p(S^{-1}R)$ is forced.
Conversely, $p = p(S^{-1}R) = p(R)$ implies $S^{-1}R \subset (p:p)$. But when R is saturated, $(p:p) = R^*$. □

As a consequence of Theorem 2.9, we can generalize the result of Gilmer and Heinzer quoted in Proposition 2.8 as follows:

Corollary 2.12. *If R is a saturated G-domain, then* $R^* = \bigcap \{V \mid V \in X^1(R)\}$ *and is completely integrally closed.*

Proof. Obviously $R \subset R' \Rightarrow R^* \subset (R')^*$. By Proposition 2.8, $(R')^* = \bigcap \{V \mid V \in X^1(R')\} = \bigcap \{V \mid V \in X^1(R)\}$ and is completely integrally closed. We need only show that $R^* \supset (R')^*$. By Theorem 2.9 applied to R', $p' = p(R'^*)$. By Theorem 2.9 applied to R, $p = p'$. Hence $p = p(R'^*)$; so $(R:R'^*) \supset p \neq 0$, whence $R'^* \subset R^*$. □

Remark 2.13. A different characterization of the saturated condition was pointed out by Campanella [3, Teorema 1.5] who proved that a G-domain R is saturated if and only if every height one maximal ideal of $R[X]$ is principal. Campanella also showed that a Noetherian G-domain is saturated if and only if it is seminormal. However, the saturation condition does not, in general, imply seminormality.

For details of the connection between pullback diagrams of commutative rings and the resulting pushout diagrams of Spec, we refer the reader to [10, Theorem 1.4]. For future references, we list in Lemma 2.14 some facts that we will need later on.

Lemma 2.14. *A pullback diagram of commutative rings*

$$\begin{array}{ccc} A \times_T R & \xrightarrow{\pi_1} & A \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \phi_2} \\ R & \xrightarrow{\phi_1} & T \end{array}$$

where ϕ_1 is surjective, naturally gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A \times_T R) & \xleftarrow{\mu_1} & \mathrm{Spec}(A) \\ \uparrow{\scriptstyle \mu_2} & & \uparrow \\ \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(T) \end{array}$$

in such a way that $\mathrm{Spec}(A \times_T R)$ *is identified with the topological space* $\mathrm{Spec}(A) \cup_{\mathrm{Spec}(T)} \mathrm{Spec}(R)$ *via the maps μ_1 and μ_2. Moreover, π_1 is a surjective map and, thus, μ_1 gives a closed embedding of* $\mathrm{Spec}(A)$ *into* $\mathrm{Spec}(A \times_T R)$. □

We have already seen, for a G-domain R, that $p(S^{-1}R) = S^{-1}(p(R))$ (Lemma 2.2(b)) and that $S^{-1}R/p(S^{-1}R) = S^{-1}(R/p(R))$, the total quotient ring T of $R/p(R)$ (Corollary 2.3 and Lemma 2.5(b)). Thus, using the canonical surjection from $S^{-1}R$ to T and the canonical injection from $\bar{R}$ to T, we can construct the pullback $R^{\square} = \bar{R} \times_T S^{-1}R$ which is also a G-domain that we will identify (canonically) with an overring of R (contained in $S^{-1}R$). Recall that for every G-domain R, its overring $S^{-1}R$ is called the *essential G-domain associated to R*, and that R is an essential G-domain if and only if $S^{-1}R = R$. In the same spirit, we say that R is a G-domain of *pullback type* if $R = R^{\square}$ and we call $R^{\square}$ *the G-domain of pullback type associated to R*. This terminology is justified by Theorem 2.15(c) below.

Theorem 2.15. (a) *If R is a G-domain, then* $\mathrm{Spec}(R^{\square})$ *is homeomorphic to* $\mathrm{Spec}(R)$ (*via the map induced by the natural inclusion of R in $R^{\square}$*).

(b) *If R is a saturated* (*e.g. seminormal*) *G-domain and $S^{-1}R \subset R^*$, then $R^{\square} = R$ is of pullback type.*

(c) *Let R be any ring of the form $A \times_T B$ such that T is the total quotient ring of A, B is a G-domain, and $B/p(B) \simeq T$. Then R is a G-domain, B is naturally identified with $S^{-1}R$, and $R = R^{\square}$. In particular, for any G-domain R, $(R^{\square})^{\square} = R^{\square}$.*

Proof. (a) From the pullback diagram of canonical homomorphisms,

$$\begin{array}{ccc} R^{\square} & \xrightarrow{\pi_1} & \bar{R} \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \Phi_2} \\ S^{-1}R & \xrightarrow{\Phi_1} & T \end{array}$$

we obtain a commutative diagram (Lemma 2.14)

$$\begin{array}{ccc} \mathrm{Spec}(R^{\square}) & \xleftarrow{\mu_1} & \mathrm{Spec}(\bar{R}) \\ \uparrow{\scriptstyle \mu_2} & & \uparrow{\scriptstyle \alpha_2} \\ \mathrm{Spec}(S^{-1}R) & \xleftarrow{\alpha_1} & \mathrm{Spec}(T) \end{array}$$

such that $\mathrm{Spec}(R^{\square})$ is identified with $\mathrm{Spec}(\bar{R}) \cup_{\mathrm{Spec}(T)} \mathrm{Spec}(S^{-1}R)$ and μ_1 is a closed embedding. The map α_1, being induced by the surjection ϕ_1, is just the standard correspondence between prime ideals in $T = (S^{-1}R)/(p(S^{-1}R))$ and prime ideals in $S^{-1}R$ that contain $p(S^{-1}R)$. Since every nonzero prime contains the pseudo-radical, the image of α_1 is $\mathrm{Spec}(S^{-1}R) \backslash \{0\}$. But every $\alpha_1(P)$ in this image is identified with the corresponding $\alpha_2(P)$ in $\mathrm{Spec}(\bar{R})$. Thus, up to homeomorphism, $\mathrm{Spec}(\bar{R}) \cup_{\mathrm{Spec}(T)} \mathrm{Spec}(S^{-1}R) = \mathrm{Spec}(\bar{R}) \cup \{0\}$ (the second union being disjoint). Moreover, since μ_1 is a closed embedding, $\mathrm{Spec}(\bar{R})$ is a closed set in $\mathrm{Spec}(\bar{R}) \cup \{0\}$ and the proper closed sets of $\mathrm{Spec}(\bar{R}) \cup \{0\}$ are in 1–1 correspondence with all the closed sets of $\mathrm{Spec}(R)$. Thus, we have a bijection $\mathrm{Spec}\, \bar{R} \cup_{\mathrm{Spec}\, T} \mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$ which is both continuous and closed; therefore, it is a homeomorphism.

(b) By the universal property of pullback diagrams, R is always identified with a subring of $R^{\square}$ via the injection given by $\phi(r) = (\bar{r}, r/1)$. If R is saturated and $S^{-1}R \subset R^*$, we claim that ϕ must be surjective as well. To see this, let $(\bar{r}, a/t) \in \bar{R} \times_T S^{-1}R$ be arbitrary. By definition, $\bar{r} = \overline{(a/t)}$ in T; whence, $b = r - a/t \in p(S^{-1}R)$. By Lemma 2.2(b) and Corollary 2.11, $p(S^{-1}R) = S^{-1}(p(R)) = p(R)$. Thus, $b \in p(R) \subset R$. But, then, $a/t = r - b \in R$, and $(\bar{r}, a/t) = (\overline{(a/t)}, a/t) = \phi(a/t) \in \phi(R)$.

(c) We are given a canonical surjection $B \xrightarrow{\phi_1} T$ and a canonical injection $A \xrightarrow{\phi_2} T$. By the definition of pullback, we have 'coordinate' maps π_1 and π_2 from R to A and B respectively. We may identify R with the subring $\pi_2(R)$ inside B. Now, just as in part (a), $\mathrm{Spec}(R) = \mathrm{Spec}(A) \cup \{0\}$, and $\mathrm{Spec}(R) \backslash \{0\} = \{\pi_1^{-1}(P) \mid P \in \mathrm{Spec}(A)\}$. If $P \in \mathrm{Spec}(A)$, $\pi_1^{-1}(P) = \{(x, y) \in R \mid x \in P\}$. Thus, we can compute $p(R)$ more explicitly: $p(R) = \bigcap \{\pi_1^{-1}(P) \mid P \in \mathrm{Spec}(A)\} = \{(x, y) \in R \mid x \in P$ for all $P \in \mathrm{Spec}(A)\}$. By hypothesis, $\Gamma^{-1}A \simeq B/p(B)$ where $\Gamma = \mathrm{NZD}(A)$. This implies that $\Gamma^{-1}A$, and therefore A, are both reduced. Thus, $\bigcap \{P \mid P \in \mathrm{Spec}(A)\} = 0$.

Note, too, that for $y \in B$, $(0, y) \in R$ if and only if $\bar{y} = 0$ in $B/p(B)$; that is, if and only if $y \in p(B)$. Consequently,

$$\begin{aligned} p(R) &= \{(x, y) \in R \mid x \in \bigcap \{P : P \in \operatorname{Spec}(A)\} \\ &= \{(0, y) \mid (0, y) \in R\} \\ &= \{(0, y) \mid y \in p(B)\} . \end{aligned}$$

It follows that, identifying via the injection $\pi_2 : R \to B$, $p(R) = p(B) \neq 0$, proving that R is indeed a G-domain. Moreover, the kernel of π_1 is $\{(0, y) \mid (0, y) \in R\} = p(R)$; so, $A \simeq R/p(R) = \bar{R}$.

To prove that $R \simeq \bar{R} \times_T S^{-1}R$, it remains to show that $B = S^{-1}R$. Since $A = \bar{R}$, $\operatorname{Spec}^1(R) = \operatorname{Spec}^0(A)$. Since the set of zero-divisors of A is $\bigcup \{P \mid P \in \operatorname{Spec}^0(A)\}$, it is easy to see that

$$S = \{(x, y) \in R \mid x \in \operatorname{NZD}(A)\} .$$

Hence, $(x, y) \in S \Leftrightarrow x \in \operatorname{NZD}(A)$ and $x/1 = \bar{y}$ (working in $B/p(B) \simeq T) \Leftrightarrow x/1 = \bar{y}$ is a unit in $B/p(B) \Leftrightarrow x/1 = \bar{y}$ in T and y is a unit in B. Thus, $S = \{(x, y) \in R \mid y$ is a unit in $B\}$, and $\pi_2(S)$ is contained in the units of B. It follows that

$$S^{-1}R \simeq \pi_2(S)^{-1}(\pi_2(R)) \subset B .$$

For the reverse containment, choose any $y \in B$. Then $\bar{y} = a/s$ in T for some choice of $a \in A$ and $s \in \operatorname{NZD}(A)$. As $B \to T$ is surjective, there exist m and b in B such that $\bar{m} = s$ and $\bar{b} = a$ in T. It follows that (s, m) and (a, b) are elements of R and that $(\bar{y})(\overline{\pi_2(s, m)}) = \bar{y}\bar{m} = \bar{y}s = a = \bar{b}$ in $T = B/p(B)$. Therefore, $x = (y)(\pi_2(s, m)) - b \in p(B)$. But, we have seen above that $p(B) = \pi_2(p(R))$ and that, since $s \in \operatorname{NZD}(A)$, $(s, m) \in S$. Hence, $y = (x + b)/m \in \pi_2(S)^{-1}(\pi_2(R)) \simeq S^{-1}R$ as desired.

Now, the fact that $(R^{\square})^{\square} \simeq R^{\square}$ is immediate since $R^{\square}$ has the form $A \times_T B$ prescribed in the hypothesis of (c). □

Corollary 2.16. *A saturated G-domain R has pullback type if and only if* $S^{-1}R \subset R^*$.

Proof. The 'if' assertion is a restatement of Theorem 2.15(b).

Conversely, if $R = A \times_T B$ has pullback type, then $B = S^{-1}R$ by Theorem 2.15(c). Moreover, we showed in the proof of Theorem 2.15(c) that $p(R) = p(B)$; whence, $B \subset R^*$. □

The condition $S^{-1}R \subset R^*$ may seem, at first, quite mysterious and restrictive. However, if a G-domain R is Prüfer, Noetherian, or Krull, then $S^{-1}R \subset R^*$. (See

Remark 2.18 and Corollary 2.20.) Thus, the condition is satisfied by G-domains which belong to the most commonly studied classes of commutative rings. In fact, we do not know an example where the condition $S^{-1}R \subset R^*$ fails. Of course, even when it does fail, we still have $(S^{\square})^{-1}(R^{\square}) \subset (R^{\square})^*$ and $\operatorname{Spec} R^{\square} \approx \operatorname{Spec}(R)$ by Theorem 2.15 and Corollary 2.16. Consequently, the condition $S^{-1}R \subset R^*$ often may be assumed when considering questions about R which are topological in nature.

The *essential spectrum* of a G-domain R, denoted $\operatorname{EssSpec}(R)$, is the set $\{Q \in \operatorname{Spec}(R) \mid Q \cap S = \emptyset\}$. Obviously, $\operatorname{EssSpec}(R) \supset \operatorname{Spec}^1(R)$.

Remark 2.17. R is an essential G-domain if and only if $R = \bigcap \{R_Q \mid Q \in \operatorname{EssSpec}(R)\}$; indeed,

$$\begin{aligned} S^{-1}R &= \bigcap \{(S^{-1}R)_{S^{-1}Q} \mid S^{-1}Q \in \operatorname{Spec}(S^{-1}R)\} \\ &= \bigcap \{R_Q \mid Q \in \operatorname{EssSpec}(R)\}\,. \end{aligned}$$

Remark 2.18. If R is a Krull G-domain, then $R = \bigcap \{R_P \mid P \in \operatorname{Spec}^1(R)\}$ and this intersection is locally finite. Therefore, by Remark 2.17, every Krull G-domain R is essential, with $\operatorname{Spec}^1(R)$ a finite set. (If $0 \neq x \in p$, then $x \in P$ for every $P \in \operatorname{Spec}^1(R)$.) In our terminology, then, the Artin–Tate theorem can be 'generalized' by the following simple fact. If $R \neq K$ is an essential G-domain such that $\operatorname{Spec}^1(R)$ is finite, then R is one-dimensional and semiquasilocal. To apply this fact to an arbitrary Noetherian G-domain $R \neq K$, note that R' is a Krull G-domain; and therefore, as already pointed out, R' is one-dimensional and semiquasilocal. Hence, by integrality (essentially, the going-up and incomparability properties), R must be one-dimensional and semi-local. By Remark 2.17, it then follows that every Noetherian G-domain is also essential.

We investigate the condition $S^{-1}R \subset R^*$ further in order to prove that Prüfer G-domains have this property. Let $\Omega(R)$ denote $\{V \in X(R) \mid (m_V \cap R) \in \operatorname{EssSpec}(R)\}$, and let $\Omega^1(R) = \Omega(R) \cap X^1(R)$. By a straightforward argument, $\Omega(R) = X(S^{-1}R) \subset X(R)$. The reader should be cautioned, however, that $\Omega(R)$ is not necessarily the same as $\Omega(R')$. Recall from [5] that R is a going-down (GD) domain if, for every overring T of R, the inclusion map $R \to T$ satisfies the going-down property.

Proposition 2.19. *Let R be a G-domain. Then*

(a) *If W is a valuation overring of R and $W \supset \bigcap \{V \mid V \in \Omega(R)\}$, then $W \in \Omega(R)$;*

(b) *If R is saturated, then $S^{-1}R$ is saturated, and $S^{-1}R \subset R^* \Leftrightarrow \Omega^1(R) = X^1(R) \Leftrightarrow R^* = \bigcap \{V \mid V \in \Omega^1(R)\}$;*

(c) *If R is a saturated* GD-*domain, then R has pullback type.*

Proof. (a) For every $0 \neq x \in m_W$, $1/x \notin W$. Thus, $W \supset \bigcap \{V \mid V \in \Omega(R)\}$ implies $1/x \notin \bigcap \{V \mid V \in \Omega(R)\}$ which implies $x \in m_V$ for some $V \in \Omega(R)$. It follows that $m_W \subset \bigcup \{m_V \mid V \in \Omega(R)\}$. Hence, $m_W \cap R \subset \bigcup \{m_V \cap R \mid V \in \Omega(R)\} = \bigcup \{Q \mid Q \in \text{EssSpec}(R)\}$. Since $Q \in \text{EssSpec}(R) \Leftrightarrow Q \subset \bigcup \{P \mid P \in \text{Spec}^1(R)\}$, (a) follows.

(b) Assume that R is saturated. Let $x \in K$ satisfy $x^2 \in p(S^{-1}R)$ and $x^3 \in p(S^{-1}R)$. Then, since $p(S^{-1}R) = S^{-1}(p(R))$, $x^2 = a_1/b_1$ and $x^3 = a_2/b_2$ where a_1, $a_2 \in p(R)$ and b_1, $b_2 \in S$. Therefore, $(b_1b_2x)^2$ and $(b_1b_2x)^3$ are both in $p(R)$. As R is saturated, it follows that $b_1b_2x \in p(R)$. Hence, $x \in S^{-1}(p(R)) = p(S^{-1}R)$ which means that $S^{-1}R$ is saturated.

Next, we prove the equivalence of the three conditions in (b). Obviously, $\Omega^1(R) = X^1(R)$ implies $R^* = \bigcap\{V \mid V \in \Omega^1(R)\}$, by Proposition 2.8 and Corollary 2.12.

Now, assume that $R^* = \bigcap \{V \mid V \in \Omega^1(R)\}$. The fact that $X(S^{-1}R) = \Omega(R)$ implies that $X^1(S^{-1}R) = \Omega^1(R)$. Since $S^{-1}R$ is saturated, $(S^{-1}R)^* = \bigcap \{V \mid V \in \Omega^1(R)\} = R^*$ (Corollary 2.12). Thus $S^{-1}R \subset R^*$.

Finally, assume $S^{-1}R \subset R^*$. Then $R^* \subset (S^{-1}R)^* \subset (R^*)^*$. But $(R^*)^* = R^*$ by Corollary 2.12. Hence, $R^* = (S^{-1}R)^*$. As we just noted above, this forces $R^* = \bigcap \{V \mid V \in \Omega^1(R)\}$. If $W \in X^1(R) \setminus \Omega^1(R)$, then $W = W^* \supset R^* = \bigcap \{V \mid V \in \Omega^1(R)\} \supset \bigcap \{V \mid V \in \Omega(R)\}$. It follows, by part (a), that $W \in \Omega(R) \cap X^1(R) = \Omega^1(R)$, a contradiction.

(c) By Theorem 2.15(b) and (b) of this proposition, it suffices to show that $X^1(R) \subset \Omega(R)$. If $K \neq V \in X(R) \setminus \Omega(R)$, then $Q_1 = m_V \cap R \not\in \text{Spec}^1(R)$. Hence, $Q_1 \supset Q_2$ for some $Q_2 \in \text{Spec}^1(R)$ (Lemma 2.1(b)). Since $R \to V$ satisfies the going-down property, $Q_1 \supset Q_2$ lifts to a chain $m_V \supset P$ in $\text{Spec}(V)$. Hence V_P properly contains V, and so $V \not\in X^1(R)$. □

The interested reader should note that an alternative proof to Proposition 2.19(c) can be obtained using Corollaries 2.12 and 2.16, thus avoiding the use of Proposition 2.19(b).

Corollary 2.20. *If R is a Prüfer G-domain, then $S^{-1}R \subset R^*$, R has pullback type, and $R^* = \bigcap \{R_P \mid P \in \text{Spec}^1(R)\}$ has essential type. If in addition, R is a Bézout domain, then $S^{-1}R = R^*$.*

Proof. Prüfer domains are necessarily GD-domains [5, p. 448]. Hence $S^{-1}R \subset R^*$ and R has pullback type, by Proposition 2.19(c) and Corollary 2.16. Furthermore, for a Prüfer domain, $\Omega^1(R) = X^1(R) = \{R_P \mid P \in \text{Spec}^1(R)\}$. Thus, $R^* = \bigcap \{R_P \mid P \in \text{Spec}^1(R)\}$ by Proposition 2.19(b). Applying the same reasoning to R^* in place of R leads to $R^* \subset (S^*)^{-1}R^* \subset R^{**} = R^*$. Hence, R^* has essential type. If, in addition, R is a Bézout domain, then every overring of R is a ring of fractions of R. In particular $R^* = T^{-1}R$ for some saturated multiplicatively closed set T. By the result above, $S^{-1}R \subset T^{-1}R = R^* = \bigcap \{R_P \mid P \in \text{Spec}^1(R)\}$. Therefore, $T \supset S$ and $R_P \supset T^{-1}R$ for every $P \in \text{Spec}^1(R)$. It follows that $T \cap P = 0$ for every $P \in \text{Spec}^1(R)$, and so $S \supset T$. Thus, $S^{-1}R = T^{-1}R = R^*$. □

It is noteworthy that for R either a Prüfer G-domain or a Noetherian G-domain, R^* is just the intersection of the one-dimensional valuation overrings of R which lie over the height 1 primes of R. Unfortunately, this intersection is not generally locally finite in the Prüfer case; so, it does not necessarily follow that R^* is one-dimensional. It is easy to find examples of G-domains which are not one-dimensional. In Example 4.1 we point out an example of an essential, Bézout (hence, completely integrally closed) G-domain of dimension 2.

3. *G*-domains with only finitely many height 1 prime ideals

If R is a G-domain such that $\mathrm{Spec}^1(R)$ is a finite set, then $S^{-1}R = \bigcap \{R_P \mid P \in \mathrm{Spec}^1(R)\}$, a finite intersection of one-dimensional quasilocal rings. The condition that $\mathrm{Spec}^1(R)$ be finite is characterized by $S^{-1}R$ being semiquasilocal of dimension at most 1. In this section, we note certain conditions which guarantee that $\mathrm{Spec}^1(R)$ is finite and deduce some consequences of $\mathrm{Spec}^1(R)$ being finite which enable us to give a more precise description of certain types of rings where this occurs.

Remark 3.1. It follows readily from [22, Theorem 33.3] that a G-domain R satisfies $\mathrm{Spec}^1(R)$ is finite and R has essential type if and only if every nonzero principal ideal of R is a finite intersection of (height 1) primary ideals.

Remark 3.2. In the structure theory for G-domains given by Theorem 2.15, the case where $S^{-1}R$ is one-dimensional and semiquasilocal precisely corresponds to the requirement that T be a finite direct product of fields. To see this, note first that $T = S^{-1}R/p(S^{-1}R)$. Thus, if $\mathrm{Spec}^1(R) = \{P_1, \ldots, P_t\}$, then $T \simeq \prod_{i=1}^{t} S^{-1}R/S^{-1}P_i$ and each $S^{-1}R/S^{-1}P_i$ is isomorphic to the field $R_{P_i}/P_iR_{P_i}$. Conversely, if $T \simeq \prod_{i=1}^{t} K_i$, a finite direct product of fields, then T is zero-dimensional with t maximal ideals and, consequently, $S^{-1}R$ is one-dimensional with t maximal ideals.

Proposition 3.3. *For a G-domain R which is not a field, each of the following* equivalent *conditions implies that R is one-dimensional* (*hence, essential type*) *and semilocal*:

(a) *R is Noetherian and integrally closed*;
(b) *R is a Noetherian strong G-domain*;
(c) *R is a Euclidean domain*;
(d) *R is a principal ideal domain*;
(e) *R is a unique factorization domain*;
(f) *R is a Krull domain*;
(g) *R is a Dedekind domain*;
(h) $R = \bigcap_{i=1}^{n} V_i$, *each* V_i *being a discrete valuation ring* (*DVR*).

Proof. Recall that R is said to be a strong G-domain if every overring of R (including K) has the form $R[1/t]$ for some $t \in R$ (see [25]). All the implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f), (h) $\Rightarrow$ (g) $\Rightarrow$ (f) and (b) $\Rightarrow$ (a) $\Rightarrow$ (f) should be evident to the reader. Note also (h) $\Rightarrow$ (c) [27, Proposition 5] and (h) $\Rightarrow$ (b) [25, Theorem 3.4]. That (f) $\Rightarrow$ (h) hinges upon the fact that R, being a Krull G-domain, is one-dimensional and semiquasilocal (Remark 2.18). □

Proposition 3.3 focused on the uncomplicated case of Noetherian G-domains where, in particular, $\mathrm{Spec}^1(R)$ is finite. In the next results, we 'generalize' by assuming that $\mathrm{Spec}^1(R)$ is finite but not that R is Noetherian.

First recall [19, Theorem 107] that if $R = \bigcap_{i=1}^n V_i$ where each $V_i \in X^1(R)$, then R is a Bézout domain. If, in addition, no two V_j's are comparable, and we let P_i denote $m_{V_i} \cap R$, then the set of maximal ideals of R is precisely $\{P_1, \ldots, P_n\}$ and $R_{P_i} = V_i$.

Now, several important classes of (not necessarily Noetherian) integrally closed essential G-domains coincide when $\mathrm{Spec}^1(R)$ is finite.

Proposition 3.4. *Let $R \neq K$ be an essential G-domain such that* $\mathrm{Spec}^1(R)$ *is finite. Then, the following are equivalent*:

(a) *R is a Prüfer domain*;
(b) *R is a Bézout domain*;
(c) *R is a GCD-domain*;
(d) *R is a strong G-domain*;
(e) *R is integrally closed and every overring is a finitely generated R-algebra*;
(f) *R is integrally closed and every overring is a locally* pqr-*domain*;
(g) *$R = \bigcap_{i=1}^n V_i$ where each V_i is a one-dimensional valuation overring of R.*

Proof. Recall that a domain R is locally pqr if, for every $P \in \mathrm{Spec}(R)$, there exists $t \in R$ such that $R_P = R[1/t]$. It is well known that (g) $\Rightarrow$ (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c). Also, in general, (f) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) (see [21, Proposition 3.7] and [12, Theorem 14]) and (d) $\Rightarrow$ (b) (see [21, Proposition 1.5]). Since $R = \bigcap \{R_P \mid P \in \mathrm{Spec}^1(R)\}$ for any essential G-domain such that $\mathrm{Spec}^1(R)$ is finite, (a) $\Rightarrow$ (g) is evident. Also, R being one-dimensional and semiquasilocal [19, Theorem 107], (g) $\Rightarrow$ (d) [25, Theorem 3.4] and (c) $\Rightarrow$ (b) [4, Corollary 4.4; 28, Theorem 3.7]. □

Remark 3.5. When R is an essential G-domain and $\mathrm{Spec}^1(R)$ is finite, we have seen that $R = \bigcap \{R_P \mid P \in \mathrm{Spec}\, R\}$. It is natural to ask whether, conversely, any finite intersection, $R = \bigcap_{i=1}^n R_i$, of quasilocal G-domains gives rise to a G-domain. It is not surprising that, if each R_i is *one-dimensional* having the same quotient field as R, then R is a G-domain and $\mathrm{Spec}^1(R) = \{m_1 \cap R, \ldots, m_n \cap R\}$ where m_i is the maximal ideal of R_i (apply [19, Theorem 110]). As far as we know, however, R need not be essential unless we make still further assumptions (e.g. if also each R_i is a one-dimensional valuation domain, then R is a one-dimensional semiquasilocal G-domain [19, Theorem 107]).

To pursue Remark 3.5, we consider a specific type of domain for R_i – a finite-dimensional conducive domain (not necessarily quasi-local) – which is a generalization of the finite-dimensional valuation (or pseudovaluation) case. Recall that a conducive domain is a domain T such that the conductor $(T:W) \neq 0$ for every overring W other than the quotient field of T (see [6] or [2]). The standard $D + M$ construction, for example, always gives rise to a conducive domain. Corresponding to any finite-dimensional conducive domain $R \neq K$, there is a *unique* one-dimensional valuation overring V satisfying $(R:V) \neq 0$. (See [6, Propositions 4.5, 4.3].) For this $V = R^*$, $m_V \cap R$ is the *unique* height 1 prime of R. Furthermore, since each such R is a G-domain, Theorem 2.9 implies that if R is saturated and $R \neq R^*$, then $(R:V) = p(R) = p(V) = m_V = m_V \cap R$.

We include the following easy lemma for the sake of completeness:

Lemma 3.6. *Let R be a domain and T be an overring of R such that $(R:T) \neq 0$. Then R is a G-domain if and only if T is a G-domain.*

Proof. That R being a G-domain implies T is a G-domain is patent. Conversely, pick $0 \neq t \in T$ such that $K = T[1/t]$ (possible because T is a G-domain) and pick $0 \neq r \in (R:T)$. Since, for each n, $Tt^{-n} = (Tr)r^n r^{-1}(rt)^{-n} \subset R[r^{-1}, (rt)^{-1}]$, it follows that $K = R[r^{-1}, (rt)^{-1}]$ and, so, R is a G-domain. □

Theorem 3.7. *Let $R_1, \ldots, R_n$ be finite-dimensional conducive domains which are not fields, with Q_i being the (unique) height 1 prime of R_i. Let $R = \bigcap_{i=1}^n R_i$ and $q_i = Q_i \cap R$. Let (V_i, M_i) be the (unique) one-dimensional valuation overring of R_i and let $b_i = (R_i : V_i) \cap M_i$ (which is nonzero by the conducive property). Let $W = \bigcap_{i=1}^n V_i$ and $m_i = M_i \cap W$. Assume further that R and each of the R_i's have a common quotient field K and that $q_i \not\subset q_j$ whenever $i \neq j$. Then*

(a) *R is a G-domain and $R^* = W$, a one-dimensional semiquasilocal Bézout domain*;

(b) $\mathrm{Spec}^1(R) = \{q_1, \ldots, q_n\}$;

(c) $S^{-1}R = \bigcap_{i=1}^n R_{q_i} \subset R^*$;

(d) *R can be represented as a pullback of $S^{-1}R$ and $S^{-1}R$ can be represented as a pullback of R^* via the diagram*

$$\begin{array}{ccc}
R & \longrightarrow & \dfrac{R}{\cap b_i} \\
\downarrow & & \downarrow \\
S^{-1}R & \longrightarrow & \dfrac{S^{-1}R}{\cap b_i} \cong \prod_{i=1}^{n} \dfrac{S^{-1}R}{b_i \cap S^{-1}R} \\
\downarrow & & \downarrow \\
R^* & \longrightarrow & \dfrac{R^*}{\cap b_i} \cong \prod_{i=1}^{n} \dfrac{R^*}{b_i \cap R^*}
\end{array}$$

(e) *If each* R_i *is one-dimensional, then* $R = S^{-1}R$ *is one-dimensional and semiquasilocal*;

(f) *If each* R_i *is saturated, then* R *is saturated and* $R = R^{\square} \simeq \bar{R} \times_{T^*} R^*$ *where* T^* *is the total quotient ring of* R^*/p^*.

Proof. Note that $m_i \cap R = M_i \cap W \cap R = M_i \cap R = M_i \cap R_i \cap R = q_i$ and, so, the condition $q_i \not\subset q_j$ for $i \neq j$ implies $m_i \not\subset m_j$, $M_i \not\subset M_j$, and $V_i \not\subset V_j$ for $i \neq j$.

(a) For each $1 \leq i \leq n$, choose $0 \neq x_i \in b_i \cap R$. Then $0 \neq \prod_{i=1}^n x_i \in ((\bigcap_{i=1}^n R_i):(\bigcap_{i=1}^n V_i)) = (R:W)$; so $R^* = W^* = W$. Also, since $(R:W) \neq 0$ and W is a G-domain, R must be a G-domain (Lemma 3.6). The other facts about W follow from [19, Theorem 107].

(b) Since R is a G-domain by (a), every nontrivial $V \in X(R)$ is contained in a maximal valuation overring (Lemma 2.1(a)). To show that $\mathrm{Spec}^1(R) \subset \{q_1, \ldots q_n\}$, let $P \in \mathrm{Spec}^1(R)$ and $V \in X^1(R)$ such that $m_V \cap R = P$. But, then, $V = V^* \supset R^* = \bigcap_{i=1}^n V_i$ by (a). Hence, $V = V_i$ for some $1 \leq i \leq n$; so $P = q_i$ for some $1 \leq i \leq n$. Conversely, assume that some $q_i \not\in \mathrm{Spec}^1(R)$. Then q_i properly contains some $P \in \mathrm{Spec}^1(R)$ (Lemma 2.1(b)). As above, $P = q_j$ for some $1 \leq j \leq n$ and so $q_i \supset q_j$ which is a contradiction.

(c) Since $\mathrm{Spec}^1(R) = \{q_1, \ldots, q_n\}$ (by part (b)), $S^{-1}R = \bigcap_{i=1}^n R_{q_i}$. But $R_{q_i} \subset (R_i)_{Q_i} \subset V_i$; therefore, $S^{-1}R \subset \bigcap_{i=1}^n V_i = R^*$.

(d) Keeping in mind that $R^* = \bigcap_{i=1}^n V_i = W$, the pullback descriptions follow easily from the fact that $\bigcap_{i=1}^n b_i$ is a common ideal of R, $S^{-1}R$, and R^*. It remains only to prove that the canonical inclusion maps

$$\frac{S^{-1}R}{\bigcap b_i} \to \pi \frac{S^{-1}R}{(b_i \cap S^{-1}R)} \quad \text{and} \quad \frac{R^*}{\bigcap b_i} \to \pi \frac{R^*}{(b_i \cap R^*)}$$

are surjective. By the Chinese Remainder Theorem, this reduces to showing that $b_i \cap W$ and $b_j \cap W$ are coprime in W and that $b_i \cap S^{-1}R$ and $b_j \cap S^{-1}R$ are coprime in $S^{-1}R$. For the first of these, it is enough to note that $\mathrm{rad}_W(b_i \cap W) = \mathrm{rad}_{V_i}(b_i) \cap W = M_i \cap W = m_i$, and m_i and m_j are obviously coprime. For the second, a similar calculation yields $\mathrm{rad}_{S^{-1}R}(b_i \cap S^{-1}R) = S^{-1}q_i$. But, by (b), $\mathrm{maxSpec}(S^{-1}R) = \{S^{-1}q_1, \ldots, S^{-1}q_n\}$; thus, $S^{-1}q_i$ and $S^{-1}q_j$ are coprime in $S^{-1}R$.

(e) We have already seen in the proof of part (c) that $S^{-1}R = \bigcap_{i=1}^n R_{q_i} \subset \bigcap_{i=1}^n (R_i)_{Q_i}$. But if each R_i is one-dimensional, then $(R_i)_{Q_i} = R_i$ and, so, $\bigcap_{i=1}^n (R_i)_{Q_i} = \bigcap_{i=1}^n R_i = R$. Thus, $R = S^{-1}R = \bigcap_{i=1}^n R_{q_i}$ which is one-dimensional and semiquasilocal.

(f) Since R_i is a G-domain and $R_i^* = V_i$, the fact that R_i is saturated implies that $p(R_i) = p(V_i)$; and, so, $b_i = Q_i = M_i$. By part (b), $p(R) = \bigcap_{i=1}^n q_i = (\bigcap_{i=1}^n Q_i) \cap R = \bigcap_{i=1}^n Q_i = \bigcap_{i=1}^n M_i = p(R^*)$. Thus, R is saturated. Moreover, in the pullback diagrams of part (d), $R/\bigcap b_i = \bar{R}$, $S^{-1}R/\bigcap (b_i \cap S^{-1}R) = \overline{S^{-1}R}$, and $R^*/\bigcap (b_i \cap R^*) = \overline{R^*}$. By Lemma 2.5(b) (noting that R^* must be essential because it is one-dimensional), $\overline{R^*} = \mathrm{Tot}(\overline{R^*})$ and $\overline{S^{-1}R} = \mathrm{Tot}(\overline{S^{-1}R})$. It follows that $R \simeq \bar{R} \times_{T^*} R^*$ as desired. Finally, $R = R^{\square}$, by (c) and Theorem 2.15(b). $\square$

Remark 3.8. Therem 3.7 generalizes several standard results (cf. [19, Theorems 107 and 109]) on the finite intersection of valuation rings and the finite intersection of one-dimensional quasilocal rings. It shows that a common type of finite intersection of conducive domains (not necessarily one-dimensional or quasilocal or valuation) gives rise to a G-domain R such that both $S^{-1}R$ and R^* are one-dimensional and semiquasilocal. Far from being exotic, Theorem 3.7 provides the kind of concrete generalization of the Artin–Tate theorem suggested by Remark 2.18. In fact, the case where R ($\neq K$) is a Krull G-domain is a trivial case of Theorem 3.7 since, then, R is a finite intersection of discrete valuation rings (which are certainly one-dimensional and conducive).

In light of Theorem 2.9, Remark 2.18, and Remark 3.8, we ask questions about R^* when R is a G-domain with $\mathrm{Spec}^1(R)$ finite. Most importantly, we do not know conditions implying $\dim(R^*) = 1$. (Of course, this problem would be trivial if we also assumed that $X^1(R)$ is finite.) One cannot apply directly the results of this section because $\mathrm{Spec}^1(R)$ being finite does not imply that either $X^1(R)$ or $\mathrm{Spec}^1(R^*)$ is finite. This is illustrated below.

Example 3.9. We exhibit a one-dimensional quasilocal domain R such that R^* is a one-dimensional (therefore, essential) Prüfer G-domain, but not semiquasilocal. Let V be a one-dimensional valuation domain with quotient field K such that there exists an algebraic field extension L of K having infinitely many valuation subrings extending V. (For instance, take $V = \mathbb{Z}_{p\mathbb{Z}}$ and L the field of algebraic numbers.) Let T be the integral closure of V in L. Then T is one-dimensional and Prüfer, but not semiquasilocal (cf. [13, Example 1]). Moreover, $T = \bigcap \{T_m \mid m \in \mathrm{Spec}^1(T)\}$ is completely integrally closed. Since every nonzero prime in T lies over m_V in V, $V/m_V \to T/p(T)$ is well defined (and injective). We can define R to be the pullback

$$\begin{array}{ccc} R & \longrightarrow & V/m_V \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/p(T) \end{array}$$

We leave to the reader the routine verifications that $R \to T$ is injective, that $\mathrm{Spec}(R)$ has only 2 elements [10, Theorem 1.4], and that, viewing R as a subring of T, $p(T)$ is the maximal ideal of R. Hence, $(R : T) \neq 0$ so that $R^* = T^* = T$. As asserted, R is quasilocal whereas T has infinitely many height 1 primes.

We close this section by stating some more specialized conditions which imply that $\mathrm{Spec}^1(R)$ is finite.

Example 3.10. $\mathrm{Spec}^1(R)$ is finite for every G-domain R such that R' is a strong G-domain (see, [11, Proposition 1; 21, Proposition 3.7]).

Example 3.11. Spec(R) is finite and $\dim(R) \leq 1$ if (and only if) R is a compactly packed G-domain of essential type. (R is *compactly packed* if, for any subset Ω of Spec(R) and any ideal I in R, the condition $I \subset \bigcup \{P \mid P \in \Omega\}$ implies $I \subset P$ for some $P \in \Omega$. See [24, 26].) In a compactly packed ring, every prime ideal P is the radical of a principal ideal [23, Theorem 1; 29]. By essentiality, $\dim(R) \leq 1$. Thus, for every P, Spec$(R)\backslash\{P\}$ is a quasi-compact Zariski-open set; and therefore it is closed when Spec(R) is endowed with the patch topology. It follows that in the patch topology Spec(R) is discrete. Since the patch topology is compact, Spec(R) must be finite.

4. Describing G-domains via spectral spaces

In this section we describe some different kinds of G-domains with infinitely many height 1 primes. First, relying on the Jaffard–Ohm Theorem [14, Theorem 18.6], we construct a specific example of a two-dimensional Bézout G-domain of essential type. Then, based on Hochster's characterization of spectral spaces [18], we describe a general 'one-point adjunction' construction for G-spaces. Naturally, both techniques are topological, prescribing first the desired prime spectrum and then using the appropriate theorem to establish the existence of a ring with such a spectrum. The diversity of behavior in our examples, using the one-point adjunction, seems to justify our focus on a topological rather than ring-theoretical classification scheme.

Gilmer gave the first example of a one-dimensional Bézout G-domain with infinitely many height 1 primes; namely, the ring T in Example 3.9. To exhibit a completely integrally closed G-domain of essential type whose dimension is greater than 1, we examine further a construction due to Fischer [9].

Example 4.1. Let G be the group of all sequences of integers, $a = \{a_n \mid n \geq 0\}$, that are eventually in arithmetic progression. Addition is given by $(a+b)_n = a_n + b_n$. Partially order G by defining $G^+ = \{a \in G \mid a_n \geq 0 \text{ for } n \geq 0\}$. The greatest lower bound for a and b in G is $a \wedge b = c$ where $c_n = \min(a_n, b_n)$. Evidently, G is a lattice-ordered group. The 'prime' subsets of G are those sets S such that $S + G^+ = S$, $G^+\backslash S$ is closed under addition, and $a \wedge b \in S$ for every a and b in S. Fischer [9, Example 1.4] has shown that, as a partially ordered set, Spec(G) looks like Fig. 1 where

$$P_j = \{a \in G^+ \mid a_j > 0\},$$

$$P' = \{a \in G^+ \mid a_n > 0 \text{ for all large } n\},$$

$$P'' = \{a \in G^+ \mid a_{n+1} > a_n \text{ for all large } n\},$$

$$0 = \emptyset.$$

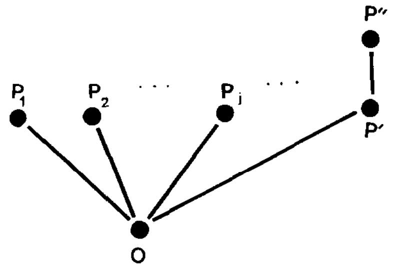

Fig. 1.

Moreover, $\{0\}$ is an open set. By the Jaffard–Ohm Theorem [14, Theorem 18.6], there exists a Bézout domain R with G as its group of divisibility and with Spec(R) homeomorphic to Spec(G). As $\{0\}$ is open in Spec(G), $\{0\}$ must be open in Spec(R), meaning that R is a G-domain. Obviously, $\dim(R) = 2$. From the definitions of G and P'' the reader can easily verify essentiality, namely, $P'' \subset \bigcup_{j=1}^{\infty} P_j$. (Alternatively, it will be established in Remark 4.6(b) that any G-domain with a prime spectrum 'looking like' Spec(G) must be essential.) Hence, by Corollary 2.20, $R = S^{-1}R = R^*$.

Example 4.1 inspires an approach to classifying G-domains. We shall seek to construct all G-spaces and then identify properties of the underlying G-domains. Note that every G-space is an irreducible spectral space.

The key definition of this section can now be introduced. Let X be a topological space and 0 a point disjoint from X. The *one-point adjunction* to X is the topological space $X_0 = X \cup \{0\}$ whose closed sets are X_0 and all the closed sets of X. To motivate this definition, consider an arbitrary G-space Y. The unique minimal point 0 (corresponding to the prime ideal (0)) is an open set. Thus, $Y \setminus \{0\}$ is a closed subspace of Y and therefore a spectral space having Y as its one-point adjunction. Thus, we shall adopt the convention of denoting any such Y by X_0 and $Y \setminus \{0\}$ by X.

Proposition 4.2. *The correspondences* $X \mapsto X \cup \{0\}$ (*via the one-point adjunction*) *and* $Y \setminus \{0\} \leftarrow Y$ *give rise to a bijection between the homeomorphism classes of spectral spaces* X *and the homeomorphism classes of* G*-spaces* Y.

Proof. By the remarks above, it suffices to show that if X is a spectral space, then $X_0 = X \cup \{0\}$ is a G-space. First, note that X_0 is a spectral space because it inherits from X the three criteria of Hochster [18]: It is a T_0 quasi-compact space, its quasi-compact open sets are closed under finite intersection, and every irreducible closed set is the closure of one of its points. Since $\overline{\{0\}} = X_0$ (whence, $0 < x$ for every $x \in X$) and since $\{0\}$ is open in X_0, it follows that X_0 is a G-space. $\square$

Since a nonempty spectral space is zero-dimensional if and only if it is Hausdorff and since $\dim(X_0) = \dim(X) + 1$, we obtain

Corollary 4.3. *The homeomorphism classes of one-dimensional G-spaces are in 1–1 correspondence via the one-point adjunction with the homeomorphism classes of nonempty Hausdorff spectral spaces.* □

Remark 4.4. The nonempty Hausdorff spectral spaces (of dimension 0) are the same as the nonempty Boolean spaces (i.e. spaces homeomorphic to the prime spectrum of a nonzero Boolean ring) (cf. [20, p. 833 and Theorem 6.1]). Thus, the homeomorphism classes of nonempty Boolean spaces are in 1–1 correspondence with the homeomorphism classes of one-dimensional G-spaces.

Recall that a spectral space X is a T_{D}-(respectively, *discrete Alexandroff*) space if for every $x \in X$, $\{x\}$ is open in $\overline{\{x\}}$ (respectively, for every $F \subset X$, $\bar{F} = \bigcup \{\overline{\{x\}} \mid x \in F\}$). Every discrete Alexandroff space is T_{D}.

Corollary 4.5. *Let X_0 be a G-space. Then*

(a) *X_0 is a T_{D}-space if and only if X is a T_{D}-space.*

(b) *X_0 is discrete Alexandroff if and only if X is discrete Alexandroff.*

(c) *Every G-space with only finitely many primes of height greater than* 1 *is a T_{D}-space.*

Proof. (a) and (b) may be verified by the reader. To prove (c), consider $P \in X$. Then, by hypothesis, $\{\bar{P}\} = \{P, P_1, \ldots, P_n\}$ consists of finitely many elements where each $P_i \supsetneq P$. Thus, we can pick $f \in \bigcap_{i=1}^{n} P_i \setminus P$. Thus, $\{P\} = D(f) \cap \{\bar{P}\}$ is open in $\{\bar{P}\}$, as desired. (If $n = 0$, $f = 1$ suffices.) □

Remark 4.6. (a) Y is a Noetherian space if and only if Y^{H} (by which we mean Y endowed with Hochster's opposite order topology [18, Proposition 8]) is a discrete Alexandroff space [7, Corollary 3.4]. Thus, by Proposition 4.2 and Corollary 4.5, $Y \rightarrow Y^{\mathrm{H}} \cup \{0\}$ gives a 1–1 correspondence between homeomorphism classes of Noetherian irreducible spaces and homeomorphism classes of discrete Alexandroff G-spaces.

(b) More concretely, we now construct a two-dimensional essential discrete Alexandroff G-space with infinitely many height 1 elements. Let $\mathbb{Z}$ denote the integers and $Y = \mathrm{Spec}(\mathbb{Z})$. Then, as a partially ordered set, $X_0 = Y^{\mathrm{H}} \cup \{0\}$ looks like Fig. 2. By (a), it remains only to prove that any G-domain R having X_0 as its prime spectrum must be essential. But, if R were not essential, we could pick $f \in M \setminus \bigcup_{i=1}^{\infty} P_i$. Then, $X_0 \setminus \{M\} = D(f)$ would be a G-space (homeomorphic to $\mathrm{Spec}(R[1/f])$); whence, $X_0 \setminus \{M, 0\}$ would also be a spectral space. But $X_0 \setminus \{M, 0\}$, being an infinite discrete topological space, is not quasi-compact and, therefore, not spectral. Thus R is indeed essential.

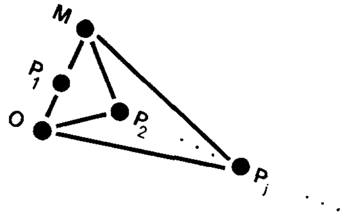

Fig. 2.

(c) By a construction similar to (b), we exhibit a two-dimensional G-space which is T_D but not discrete Alexandroff. Let $W = \text{Spec}(A)$ where A is the one-dimensional Bézout domain of algebraic integers localized at the complement of the union of all maximal ideals lying over a fixed nonzero prime number. Then, by (a), the two-dimensional G-space $X_0 = W^H \cup \{0\}$ is not discrete Alexandroff because W is not Noetherian (cf. [11, Corollaire, p. 6].) Of course, X_0 is T_D by Corollary 4.5(c).

Remark 4.7. It is non-trivial to construct spectral spaces X such that the corresponding G-space X_0 will have a particular prescribed property. For instance, we would like to be able to build essentiality into our space X_0. One tempting approach would be to consider the ring A of real-valued continuous (bounded) functions on a topological space Y. (The question of which information is imparted to A from the topological properties of Y has been discussed often in the literature. See [1, 8, 16] for example.) There are conditions which guarantee that every $Q \in \text{Spec}(A)$ is contained in $\bigcup \{P \mid P \in \text{Spec}^0(A)\}$. Intuitively, this means that A has the right ring-theoretic condition for essentiality. We have been unable to deduce from this that $\text{Spec}(A) \cup \{0\}$ must have an underlying G-domain of essential type. The remaining question is whether, in some 'broad' category, being of essential type is really a topological property.

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