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Universally going-down integral domains

By

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1. Introduction and background. The present study is based on our work in [5] on the "universally GD" and "UGD" properties of homomorphisms. Recall that a (unital) homomorphism $R \to T$ of (commutative) rings is said to be a *universally* GD-*homomorphism* in case $S \to S \otimes_R T$ is a GD-homomorphism for each commutative R-algebra S. The most natural examples of such are the flat maps (since flatness implies GD and flatness is a universal property). By analogy with [4], we shall say that a (commutative integral) domain R is a *universally* GD-*domain* if, for each overring T of R, the inclusion map $R \subset T$ is a universally GD-homomorphism. The most natural examples of such are the Prüfer domains (since each domain containing a Prüfer domain R is R-flat). It is easy to see (cf. [5, Corollary 2.3]) that, in testing for a universally GD-domain, one may restrict to $S = R_n \equiv R[X_1, \ldots, X_n]$, and then test the induced inclusion maps between polynomial rings, $R_n \subset T_n$, for GD.

Apart from Remark 2.5(a) and part of the proof of Theorem 2.6, the reader will need to know only the following from [5] regarding UGD. An inclusion map of overrings of domains is UGD if it is universally GD, with the converse holding in the integral case [5, Theorem 3.17]; and each UGD map is universally mated [5, Corollary 3.12]. (As in [2], a ring-homomorphism $f\colon A \to B$ is called *mated* if, for each $p \in \operatorname{Spec}(A)$ such that $f(p)B \neq B$, there exists a unique q in $\operatorname{Spec}(B)$ such that $f^{-1}(q) = p$.)

For the two items mentioned above, the reader will need the following technical definition. A ring homomorphism $f\colon R \to T$ is said to be UGD if it is GD and, for each $p \in \operatorname{Spec}(R)$ such that $pT \neq T$ and $\ker(f) \subset p$, T_p coincides with

$$R_p^* = \{v \in T_p\colon \text{for some } n \geqq 1,\ v^{l^n} \in f_p(R_p) + \cap \{q \in \operatorname{Spec}(T_p)\colon f_p^{-1}(q) = pR_p\}\},$$

where $f_p\colon R_p \to T_p$ is the induced map, and l is $\operatorname{char}(R_p/pR_p)$ or 1 according as that characteristic is positive or 0. If T is integral over R, the above intersection is precisely $J(T_p)$, the Jacobson radical of T_p. Recall next that the *weak normalization of* R (in the sense of Andreotti-Bombieri [1]) is the largest integral overring T of R such that $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ is a homeomorphism and for each $q \in \operatorname{Spec}(T)$ and $p = q \cap R$, the field extension $R_p/pR_p \to T_q/qT_q$ is purely inseparable. Thus we see that R' is the weak normalization

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of R if and only if $(R')_p = R_p^*$ for each $p \in \mathrm{Spec}(R)$, that is, if and only if $R \subset R'$ is universally GD.

Our main results, Theorems 2.4 and 2.6, allow Prüfer domains and weak normality to play central roles in studying the class of universally GD-domains. It, unlike the class of GD-domains, is shown to be stable for overrings (Proposition 2.2(a)) and to lead to a pleasant characterization (Corollary 2.3) of Prüfer domains without need of ancillary finiteness conditions. As well, Remark 2.5 sheds new light on some known GD-phenomena.

2. Results. We begin by stating a straightforward, but useful, result. Its first assertion was observed by McAdam [10, Lemma 1 (2)]. Notation is as in [8].

Lemma 2.1. *Let $R \subset S \subset T$ be domains such that $R \subset T$ is* GD. *Then:*

(a) *If $S \subset T$ is LO, then $R \subset S$ is* GD.
(b) *If $R \subset S$ is mated, then $S \subset T$ is* GD.

It was shown in [4, Corollary 4.4(ii)] that an overring of a GD-domain need not be a GD-domain. The first assertion of the next result shows that the class of universally GD-domains enjoys greater stability.

Proposition 2.2. *Let R be a universally* GD-*domain. Then:*

(a) *Each overring of R is a universally* GD-*domain.*
(b) *$R \subset S$ is mated for each overring S of R.*

Proof. Consider overrings $S \subset T$ of R and any integer $n \geqq 0$. Since $R \subset S$ is a universally GD extension of overrings, it is universally mated (and hence mated), by the remarks in Section 1. Thus, $R_n \subset S_n$ is mated. However, $R_n \subset T_n$ is GD since $R \subset T$ is universally GD. By Lemma 2.1(b), $S_n \subset T_n$ is GD. Thus, $S \subset T$ is universally GD.

As explained in [4, p. 447 – p. 448], the theory of Prüfer domains has motivated much of the study of GD-domains. Indeed, with the aid of various finiteness conditions, GD has figured in several characterizations of Prüfer domains (cf. [3, Corollary 4], [4, Proposition 2.7]). The next result avoids any such finiteness hypotheses. It follows directly since it is well known (cf. [7, Theorems 26.1 and 26.2], [4, Proposition 3.6]) that Prüfer domains are just the integrally closed domains R which satisfy the condition in Proposition 2.2(b).

Corollary 2.3. *A domain R is a Prüfer domain if and only if R is an integrally closed universally* GD-*domain.*

Theorem 2.4. *For a domain R, the following are equivalent:*

(i) *R is a universally* GD-*domain;*
(ii) *R' is a Prüfer domain and $R \subset R'$ is universally* GD;
(iii) *R' is a Prüfer domain and R' is the weak normalization of R.*

Proof. (ii) $\Leftrightarrow$ (iii): This follows directly from [5, Corollary 3.19], which is itself the result of combining the main theorem in [5] with [1].

(i) $\Rightarrow$ (ii): Combine Proposition 2.2(a) and Corollary 2.3.

(ii) $\Rightarrow$ (i): Assume (ii). We shall show that $R_n \subset T_n$ is GD for each overring T of R and each integer $n \geqq 0$. Note first that $R_n \subset (R')_n$ is GD (since $R \subset R'$ is universally GD) and $(R')_n \to (R' T)_n$ is also GD (since R', being a Prüfer domain, is a universally GD-domain). Hence, the composite extension, $R_n \subset (R'T)_n$, is GD. However, $T_n \subset (R'T)_n$ is LO, by virtue of integrality, and so an application of Lemma 2.1(a) completes the proof.

Remark 2.5. (a) A universally GD-domain need not be integrally closed. To see this, let F be a field of positive characteristic p, $k = F(Y^p)$, $K = F(Y)$, $T = K[\![X]\!] = K + M$ (where $M = XT$), and $R = k + M$. Then R is a universally GD-domain and $R \neq R' = T$. (By Theorem 2.4 and the comments in Section 1, we need only check that $R \subset T$ is UGD. To do this, one checks easily that $R_M^* = T = T_M$ and $R_{(0)}^* = K((X)) = T_{(0)}$.)

(b) Despite (a) and the situation for GD-domains [6, Corollary], the $k + M$ construction does *not* always produce universally GD-domains. To see this, replace the choice of k in (a) with F, and apply Corollary 2.3.

(c) A domain satisfying the condition in Proposition 2.2(b) need not be a universally GD-domain. We need only consider the extension $R = \mathbb{R} + X\mathbb{C}[\![X]\!] \subset T = \mathbb{C}[\![X]\!]$. As noted by McAdam [9, Example, p. 709], $R_1 \subset T_1$ is (integral but) not mated, and so by the remarks in Section 1, $R \subset T$ is not universally GD.

Our final result is the analogue of "test extension" results for GD-domains [6, Theorem 1]. When combined with Theorem 2.4, it yields Corollary 2.7.

Theorem 2.6. *For a domain R, the following are equivalent:*

(i) *R is a universally* GD*-domain;*
(ii) *$R \subset T$ is universally* GD *for each domain T containing R;*
(iii) *$R \subset T$ is universally* GD *for each valuation overring T of R;*
(iv) *$R \subset R[u]$ is universally* GD *for each element u in L, the quotient field of R.*

Proof. The implications (ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv) are trivial.

(i) $\Rightarrow$ (ii): Assume (i). Consider an integer $n \geqq 0$ and a domain T containing R. By applying Lemma 2.1(a) to $R_n \subset T_n \subset (T')_n$, we may assume that $T = T'$. Then $S = T \cap L$ is integrally closed and, by Proposition 2.2(a), is also a universally GD-domain. Therefore, by Corollary 2.3, S is a Prüfer domain. Then $S \subset T$ is flat, hence universally GD. Hence $R_n \subset T_n$ is the composite of GD-extensions, $R_n \subset S_n$ and $S_n \subset T_n$, and is therefore GD.

(iii) $\Rightarrow$ (i): Assume (iii). We shall show first that $R \subset R'$ is mated (that is, unibranched). It is enough to prove that if $p \in \operatorname{Spec}(R)$ and if W is a valuation overring dominating R_p, then $W = (R')_p$. At any rate, $(R_p)' = (R')_p \subset W$, and so it suffices to show that W is integral over R_p. This is well known to be equivalent to showing that $R_p \subset W$ is universally going-up. Since $R_p \subset W (= W_p)$ is universally GD, it is enough to show that $R_p \subset W$ is universally unibranched. By the remarks in Section 1, $R_p \subset W$ is universally mated. Thus, by [5, Theorem 2.5], it suffices to prove that $R_p \subset W$ is LO. This, however, is clear from dominance since (iii) assures that R (and hence also R_p) is a GD-domain. Thus, $R \subset R'$ is indeed mated.

We shall show that $R_n \subset T_n$ is GD for each overring T of R and each integer $n \geqq 0$. Given prime ideal data to test for GD, we may replace T with a suitable localization at a prime, and select a dominating valuation overring V of (the now quasilocal) T. By the proof of Proposition 2.2 and [11, Proposition 2.18 and Corollary 2.13], the result in the preceding paragraph and (iii) guarantee that T is a GD-domain. Hence $T \subset V$ is GD and, by dominance, also LO. Since [9, Proposition 1] then yields that $T_n \subset V_n$ is LO, the assertion follows from Lemma 2.1 (a) once we notice via (iii) that $R_n \subset V_n$ is GD.

(iv) $\Rightarrow$ (i): Assume (iv). By the remarks in Section 1, $R \subset R[u]$ is mated, for each $u \in L$. Then it follows easily (cf. [11, Proposition 2.14]) that R' is a Prüfer domain. By Theorem 2.4, it now suffices to show that $T = R'$ is the weak normalization of R, that is, that $T_p \subset R_p^*$ for each prime p of R. View any $v \in T_p$ as a fraction u/z, where $u \in T$ and $z \in R \backslash p$. By (iv) and the remarks in Section 1, $R \subset R[u]$ is UGD. Thus, if l is as in Section 1, there exists $n \geqq 1$ so that $v^{l^n} \in R_p + J(R[u]_p)$ which, by integrality, is contained in $R_p + J(T_p)$, as desired.

Corollary 2.7. *Let R be an integral domain. Then the inclusion map $R \to T$ is universally going-down for each integral domain T containing R if and only if R' is a Prüfer domain and R' is the weak normalization of R.*

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