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Factoring ideals in almost Dedekind domains

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Abstract. A well-known property of Dedekind domains is that each nonzero ideal can be uniquely factored as a finite product of powers of the maximal ideals that contain the ideal. One of the questions to be addressed in this paper is to what extent this property can be extended to the finitely generated ideals of an almost Dedekind domain. A related question involves a way to measure how far a given almost Dedekind domain is from being a Dedekind domain.

1. Introduction

There are two different but related notions which inspire our work in this paper. Both are derived from elementary properties of Dedekind domains. The first involves factorability of finitely generated ideals and the second is based on work of R. Gilmer [1]. We wish to consider both in relation to almost Dedekind domains—those one-dimensional domains with the property that each maximal ideal is locally principal. An alternate characterization of almost Dedekind domains is that a domain D is almost Dedekind if D_M is a discrete rank one valuation domain for each maximal ideal M.

Recall that in a Dedekind domain, each nonzero ideal can be factored uniquely as a finite product of positive powers of maximal ideals. What we would like to determine is how close can an almost Dedekind domain come to satisfying a similar factorization property. Our exact question is the following: Given an almost Dedekind domain D with maximal ideals $\mathrm{Max}(D) = \{M_\alpha\}$, when can we find a family of finitely generated ideals $\{J_\alpha\}$ such that each finitely generated nonzero ideal of D can be factored as a finite product of powers of ideals from the family $\{J_\alpha\}$ with the family indexed over the set of maximal ideals $\{M_\alpha\}$ in such a way that $J_\alpha D_{M_\alpha} = M_\alpha D_{M_\alpha}$? We refer to such a family of ideals as a *factoring family* for D. Two things we most likely will have to give up in the general case are uniqueness of factorizations and the ability to restrict to using only positive powers (regarding the latter, see the remark following the proof of Theorem 2.5 and Example 3.2). We will find that in some cases, each nonzero finitely generated fractional ideal may factor uniquely over the underlying set of some factoring family, but not factor uniquely with respect to the family. Specifically, we might have a factoring family $\{J_\alpha\}$ with family members J_β and J_γ such that $M_\beta \neq M_\gamma$ (equivalently $\beta \neq \gamma$) but $J_\beta = J_\gamma$. This would mean that while $I = J_\beta = J_\gamma$ may factor uniquely over the underlying set of the family (as itself), it does not factor uniquely over the family. In Theorem 2.10, we give a general scheme for constructing almost Dedekind domains that will have factoring families for which factorization will be unique over the underlying set of ideals making up the family. The technique applies to all of the examples we construct in Section 3. At this time we do not know of an example of an almost Dedekind domain possessing a factoring family such that there is no factoring family for the domain for which factorizations are unique over the underlying set of factors. However, we give an example where uniqueness does fail for a particular family (Example 3.2).

In a Dedekind domain, each nonzero ideal is invertible. The same happens for each nonzero finitely generated ideal in an almost Dedekind domain. Domains for which each nonzero finitely generated ideal is invertible are referred to as Prüfer domains ([2], Theorem 22.1).

In a paper that appeared in 1966, Gilmer introduced the notion of a $\#$-domain (read as "sharp domain") as an integral domain D such that for each pair of subsets $\mathcal{M}$ and $\mathcal{N}$ of $\mathrm{Max}(D)$, having $\bigcap_{M_\alpha \in \mathcal{M}} D_{M_\alpha} = \bigcap_{M_\beta \in \mathcal{N}} D_{M_\beta}$ implies $\mathcal{M} = \mathcal{N}$ ([1]). If D is a Prüfer domain, then it is a $\#$-domain if and only if each maximal ideal contains a finitely generated ideal which is contained in no other maximal ideal ([1], Theorem 2). Thus each Dedekind domain is a $\#$-domain. Moreover, an almost Dedekind domain is a $\#$-domain if and only if it is a Dedekind domain ([1], Theorem 3). On the other hand, an almost Dedekind domain that is not Dedekind does have overrings which are $\#$-domains. A trivial example of such an overring is simply the localization of the domain in question at one of its maximal ideals. In some sense what we will be studying is how far a particular almost Dedekind domain is from an overring that is a Dedekind domain.

With the exception of Theorem 2.6 and Corollary 2.7, D will always represent a one-dimensional Prüfer domain, frequently one which is an almost Dedekind domain. The definitions which follow are restricted to one-dimensional Prüfer domains. First, we say that a maximal ideal M of a one-dimensional Prüfer domain D is a *sharp prime* if it contains a finitely generated ideal which is contained in no other maximal ideal. Since D is one-dimensional, this is equivalent to saying that M is the radical of a finitely generated ideal. Obviously we can split $\mathrm{Max}(D)$ into two disjoint sets, $\mathcal{M}_\#(D)$ containing the sharp primes and $\mathcal{M}_\dagger(D)$ containing the maximal ideals that are not sharp primes, for lack of a better name we shall refer to these ideals as *dull primes*.

Mixing Gilmer's terminology with ours we can say that a one-dimensional Prüfer domain D is a $\#$-domain if (and only if) $\mathcal{M}_\#(D) = \mathrm{Max}(D)$. If D fits the other extreme, namely $\mathcal{M}_\dagger(D) = \mathrm{Max}(D)$, we will say that D is a *dull domain*. The second concern of this paper involves constructing almost Dedekind domains that fit between these two extremes.

For a one-dimensional Prüfer domain D we recursively define domains $D_1 = D$, $D_2 = \bigcap_{M_\beta \in \mathcal{M}_\dagger(D_1)} (D_1)_{M_\beta}$ and $D_n = \bigcap_{M_\beta \in \mathcal{M}_\dagger(D_{n-1})} (D_{n-1})_{M_\beta}$, with $D_n = K$, the quotient field of D, if $\mathcal{M}_\dagger(D_{n-1})$ is empty. In the event $D_{n+1} = K$ and D_n is not K, we say that D has *sharp degree* n. On the other hand we say that D has *dull degree* n if $D_{n+1} = D_n \neq K$ and $D_{n-1} \neq D_n$ (or $n = 1$). In section 3, we will give a fairly elementary way to construct almost Dedekind domains with any prescribed finite dull or sharp degree (the latter for n greater than one). The scheme we employ will also give rise to defining various infinite sharp and dull degrees. Note that a $\#$-domain is the same as a domain with sharp degree 1 and a dull domain is the same as a domain with dull degree 1.

As a convenience, we also define sharp degrees for ideals of D, both integral and fractional. For a fractional ideal I of D, we say that I has sharp degree n if $ID_n \neq D_n$ but $ID_{n+1} = D_{n+1}$. It turns out that the primes of D which generate sharp primes of D_n are exactly the prime ideals of sharp degree n. For any ideal I, of finite sharp degree or not, we let $\mathcal{M}(I)$ denote the set of maximal ideals that contain I and let D_I denote the ring $\bigcap_{M \in \mathcal{M}(I)} D_M$. A property we shall use throughout the paper is that the only primes of D that survive in D_I are those which contain I. The proof is quite elementary, for suppose P is a maximal ideal that does not contain I. Then there is an element $d \in P$ such that $dD + I = D$. It follows that d is not contained in any ideal M from the set $\mathcal{M}(I)$. Hence $1/d \in D_M$ for each $M \in \mathcal{M}(I)$, which in turn implies that $1/d$ is in D_I.

For fractional ideals that are not integral, we will mainly be concerned with those that are finitely generated. In Corollary 2.4, we show that if each prime ideal has finite sharp degree, then not only does there exist a factoring family for D, but there is one for which each finitely generated fractional ideal factors uniquely and the factoring family is actually a set with each member corresponding to a unique maximal ideal of D. Thus we are led to declaring that a factoring family $\{J_\alpha\}$ is a *factoring set* if no member appears more than once.

Throughout the paper we use $\subset$ to denote proper containment.

2. Factoring finitely generated ideals

We start with a lemma which characterizes primes of finite sharp degree in one-dimensional Prüfer domains.

Lemma 2.1. *Let D be a one-dimensional Prüfer domain. Then*:

(a) *If M is a maximal ideal of D_n, then there is a maximal ideal P of D such that $P = MD_n$ and PD_{n-1} is a dull prime of D_{n-1}.*

(b) *If $P \in \mathrm{Max}(D)$ survives in D_n, then* (i) *PD_{n-1} is a dull prime of D_{n-1}, and* (ii) *PD_n is in $\mathcal{M}_\#(D_n)$ if and only if there is a finitely generated ideal I of D which is contained in P and no other maximal ideal which survives in D_n.*

Proof. Let M be a maximal ideal of D_n. Since D is a one-dimensional Prüfer domain, each prime of D_n is extended from a prime of D ([1], Theorem 1). Thus $M = PD_n$ for some $P \in \mathrm{Max}(D)$. To show that PD_{n-1} is a dull prime of D_{n-1}, consider what happens to a sharp prime Q of D_{n-1}. Since D_{n-1} is a Prüfer domain ([2], Theorem 26.1), Q is the radical of a finitely generated ideal J ([1], Theorem 2). Thus J^{-1} is contained in each localization of D_{n-1} at a dull prime. Hence J^{-1} is contained in D_n. But then $JD_n = JJ^{-1}D_n = D_n$ and therefore $QD_n = D_n$. Hence PD_{n-1} must be a dull prime of D_{n-1}.

To prove (b), suppose $P \in \mathrm{Max}(D)$ survives in D_n. Then by the above, PD_{n-1} must be a dull prime of D_{n-1}. Obviously, if there is a finitely generated ideal I of D such that PD_n is the only maximal ideal of D_n that contains ID_n, then $PD_n = \sqrt{ID_n}$ is a sharp prime of D_n. Conversely, if PD_n is a sharp prime of D_n, then there is a finitely generated ideal J_n of D_n for which $PD_n = \sqrt{J_n}$. Since PD_n is generated by the elements of P, there is a finitely generated ideal I of D whose extension to D_n is contained in PD_n and contains J_n. $\square$

Note that if PD_n is a sharp prime of D_n, any ideal I that satisfies the conditions in Lemma 2.1 must be contained in infinitely many primes which do not survive in D_n, for otherwise P will be a sharp prime of D_k for some $k < n$ and thus not survive in D_n.

It is known that if a finitely generated ideal of an almost Dedekind domain is contained in only finitely many maximal ideals, then the ideal is a product of positive powers of these maximal ideals ([2], Theorem 37.5). The converse is trivial. In our next lemma we show that the finitely generated fractional ideals of sharp degree one in an almost Dedekind domain are those that can be factored into finite products of nonzero powers of maximal ideals.

Lemma 2.2 (cf. [2], Theorem 37.5). *Let D be an almost Dedekind domain and let I be a finitely generated fractional ideal of D. Then I is a finite product of nonzero powers of maximal ideals if and only if I has sharp degree one.*

Proof. First, assume $I = M_1^{r_1} M_2^{r_2} \cdots M_n^{r_n}$ with each r_i a nonzero integer and no $M_i^{r_i} = D$. Since I is finitely generated, it is invertible. Thus each M_i is invertible and therefore a sharp prime. As $M_i D_2 = D_2$ for each i. The same happens for M_i^{-1}. Thus $ID_2 = D_2$ and we have that I has sharp degree one.

To complete the proof assume I has sharp degree one. Then $ID_2 = D_2$. Partion $\mathrm{Max}(D)$ into sets $\mathscr{M}^0(I) = \{P \in \mathrm{Max}(D) \mid ID_P = D_P\}$, $\mathscr{M}^+(I) = \{P \in \mathrm{Max}(D) \mid ID_P \subsetneqq PD_P\}$ and $\mathscr{M}^-(I) = \{P \in \mathrm{Max}(D) \mid I^{-1}D_P \subsetneqq PD_P\}$ (note that one or two of these may be empty). Since each dull prime survives in D_2 and $ID_2 = D_2$, each dull prime must be in the set $\mathscr{M}^0(I)$. Therefore $D_I^+ = \bigcap_{P \in \mathscr{M}^+(I)} D_P$ and $D_I^- = \bigcap_{P \in \mathscr{M}^-(I)} D_P$ are both Dedekind domains with nonzero Jacobson radicals. Thus each is semilocal which means that both $\mathscr{M}^+(I)$ and $\mathscr{M}^-(I)$ are finite sets. Note that $\mathscr{M}^-(I)$ is empty if I is an integral ideal of D, but both may be nonempty if I is fractional. Let $\mathscr{M}^+(I) = \{M_1, M_2, \ldots, M_n\}$ and $\mathscr{M}^-(I) = \{N_1, N_2, \ldots, N_m\}$. It follows that $ID_I^+ = M_1^{r_1} \cdots M_n^{r_n} D_I^+$ and $I^{-1}D_I^- = N_1^{s_1} \cdots N_m^{s_m} D_I^-$ for some positive integers r_i and s_j. We also have $ID_I^- = N_1^{-s_1} \cdots N_m^{-s_m} D_I^-$. By checking locally we see that $I = M_1^{r_1} \cdots M_n^{r_n} N_1^{-s_1} \cdots N_m^{-s_m}$. This representation is unique since each M_i and N_j is a maximal ideal. $\square$

If D is an almost Dedekind domain and P is a maximal ideal of sharp degree n, then not only is there a finitely generated ideal I of D such that no other maximal ideal of D_n contains I, but we may assume $ID_P = PD_P$ since PD_P is principal. Thus in D_n, we have $ID_n = PD_n$.

As a consequence, each prime of D_n is extended from a prime of D ([1], Theorem 1). Also if J is a finitely generated ideal whose radical is a maximal ideal M, then J^{-1} is contained in D_P for each prime P different from M. Hence both J and M will blow up in D_2. Thus $\mathrm{Max}(D_2) = \{PD_2 \mid P \in \mathscr{M}_\dagger(D)\}$. As long as $D_n \neq D_{n+1} \neq K$, $\mathrm{Max}(D_{n+1}) = \{PD_{n+1} \mid P \in \mathscr{M}_\dagger(D_n)\}$.

Theorem 2.3. *Let D be an almost Dedekind domain. For each positive integer k and each prime P_α of sharp degree k, let J_α be a finitely generated ideal of D such that $J_\alpha D_{P_\alpha} = P_\alpha D_{P_\alpha}$ and J_α is contained in no other prime of D_k. If I is a finitely generated fractional ideal of finite sharp degree, then I factors uniquely into a finite product of nonzero powers of ideals from the family $\{J_\alpha\}$. In particular, the members of the family $\{J_\alpha\}$ are distinct.*

Proof. First note that if P_α is a sharp prime of D, then by checking locally we see that the corresponding J_α is simply P_α itself. Moreover, by checking locally in D_k we see that if P_α has sharp degree k, then $J_\alpha D_k = P_\alpha D_k$. Let P_α and P_β be distinct maximal ideals of D with P_α of finite sharp degree k. Then in D_k we have $J_\alpha D_k = P_\alpha D_k$ with $P_\alpha D_k$ a maximal ideal of D_k. Thus the only way to have $P_\beta D_k$ contain J_α is to have P_β blow up in D_k. In such a case P_β would have sharp degree $m < k$. While it might be that $J_\alpha D_{P_\beta} = P_\beta D_{P_\beta}$, $J_\alpha D_m$ would be contained in $P_\alpha D_m$ so that $J_\alpha D_m$ cannot equal $P_\beta D_m$. Thus $J_\alpha \neq J_\beta$. It follows that if both P_α and P_β have finite sharp degree, then $J_\alpha \neq J_\beta$. Moreover, no nonzero powers can be equal and $J_\alpha D_n = D_n$ for each $n > k$. We will take care of uniqueness first. For this it suffices to show that there is no nontrivial factorization of D since each of the J_αs is invertible.

Assume $D = \prod J_{m,i}^{e_{m,i}}$ is a finite factorization of D over the set $\{J_\alpha\}$ with each $J_{m,i}$ having sharp degree m and $e_{m,i}$ an integer, perhaps 0. Let n denote the highest sharp degree of any "factor". Then in D_n, we have $D_n = \prod J_{n,i}^{e_{n,i}}$ since $J_{m,i}D_n = D_n$ for $m < n$. As $J_{n,i}D_n = P_{n,i}D_n$ is a maximal ideal of D_n, it must be that each $e_{n,i} = 0$. Thus the factors $J_{n,i}^{e_{n,i}}$ are all superfluous. Continue the process to show all $e_{m,i}$ are 0.

For existence of factorizations we use induction and Lemma 2.2.

By Lemma 2.2, if I has sharp degree one, then I is a product of nonzero powers of finitely many sharp maximal ideals, say $I = M_1^{e_1} M_2^{e_2} \cdots M_n^{e_n}$.

Now assume I has sharp degree two. Then ID_2 is a finitely generated fractional ideal of D_2 whose sharp degree as an ideal of D_2 is one. Thus by Lemma 2.2 there are finitely many maximal ideals $P_1D_2, P_2D_2, \ldots, P_nD_2$ of D_2 which locally contain either ID_2 or $(ID_2)^{-1}$. For each i, we have a finitely generated ideal J_i in the set $\{J_\alpha\}$ such that $J_iD_2 = P_iD_2$. Thus in D_2 we can factor ID_2 uniquely as $P_1^{e_1}P_2^{e_2}\cdots P_n^{e_n}D_2$ for some nonzero integers $e_1, e_2, \ldots, e_n$. This factorization is the same as the factorization $J_1^{e_1}J_2^{e_2}\cdots J_n^{e_n}D_2$ since $P_iD_2 = J_iD_2$ for each i. Let $J = J_1^{e_1}J_2^{e_2}\cdots J_n^{e_n}$. Then $I(D:J)D_2 = D_2$. As both I and $(D:J)$ are finitely generated fractional ideals of D, $I(D:J)$ is a finitely generated fractional ideal of D. It has sharp degree one since $I(D:J)D_2 = D_2$. Thus by Lemma 2.2 there are finitely many maximal ideals $M_1, M_2, \ldots, M_m$ such that $I(D:J) = M_1^{r_1}M_2^{r_2}\cdots M_m^{r_m}$ for some nonzero integers r_i. Thus $I = I(D:J)J = M_1^{r_1}M_2^{r_2}\cdots M_m^{r_m}J_1^{e_1}J_2^{e_2}\cdots J_n^{e_n}$.

Now assume a factorization exists for each finitely generated fractional ideal of sharp degree k or less (in every almost Dedekind domain). Let I be a finitely generated fractional ideal of D which has sharp degree $k+1$. Then ID_2 is a finitely generated fractional ideal of D_2 which has sharp degree k. Thus ID_2 factors into a finite product, say $ID_2 = J_1^{e_1}J_2^{e_2}\cdots J_m^{e_m}D_2$. To complete the proof simply repeat the steps used above for the case of an ideal of sharp degree 2. Namely, set $J = J_1^{e_1}\cdots J_m^{e_m}$ and factor the fractional ideal $I(D:J)$ over the sharp primes of D. This establishes existence of a factorization. $\square$

Corollary 2.4. *Let D be an almost Dedekind domain such that each prime ideal has finite sharp degree. Then there is a factoring set $\{J_\alpha\}$ such that each finitely generated fractional ideal factors uniquely over $\{J_\alpha\}$. In particular, such a factoring set exists for each almost Dedekind domain of finite sharp degree.*

One special case we wish to consider is the one of an almost Dedekind domain with exactly one dull prime.

Theorem 2.5. *Let D be a one-dimensional Prüfer domain. Then D is an almost Dedekind domain with at most one noninvertible maximal ideal if and only if there is an element $d \in D$ such that for each finitely generated nonzero ideal I there is a finite set of maximal ideals $\{M_1, M_2, \ldots, M_m\}$ and integers $e_1, e_2, \ldots, e_m$ and n with $n \geqq 0$ such that $I = M_1^{e_1} M_2^{e_2} \cdots M_m^{e_m}(d)^n$. Moreover, if either (hence both) holds and D is not Dedekind, then the element d must be such that $dD_P = PD_P$ for the noninvertible maximal ideal P and the set $\{dD\} \cup \mathscr{M}_\#(D)$ is a factoring set for D such that each finitely generated fractional ideal factors uniquely.*

Proof. For D Dedekind, we simply set $d = 1$. Thus we may assume D is not Dedekind.

Assume D is an almost Dedekind domain with one noninvertible maximal ideal P. Then $D_2 = D_P$ and therefore there is an element $d \in D$ such $PD_2 = dD_2$ since D_P is a discrete rank one valuation domain. Thus by Theorem 2.3, the set $\{dD\} \cup \mathscr{M}_\#(D)$ is a factoring set for D such that each finitely generated fractional ideal factors uniquely as a finite product of nonzero powers of members of this set.

For the converse, assume there is an element $d \in D$ such that each finitely generated nonzero ideal can be written in the form $M_1^{e_1} M_2^{e_2} \cdots M_m^{e_m}(d)^n$ where each M_i is a maximal ideal, each e_i is a nonzero integer and n is a non-negative integer. Let I be a finitely generated ideal of D and write $I = M_1^{e_1} M_2^{e_2} \cdots M_m^{e_m}(d)^n$ with no $M_i^{e_1} = D$. Since D is a Prüfer domain, I is invertible. Combining this with the assumption that $M_i^{e_i}$ is not equal to D, we have that each M_i is invertible.

As we are not assuming that D is almost Dedekind, we need to show that each sharp prime is invertible. Let $M \in \mathrm{Max}(D)$ be a noninvertible prime ideal of D, such a prime exists since we are assuming D is not Dedekind. Then no (nonzero) power of M can appear as a nontrivial factor (i.e., not D) in a factorization of a finitely generated ideal. Hence d must be contained in M and each finitely generated ideal contained in M must have a positive power of (d) in a factorization. It follows that $MD_M = dD_M$ and D_M is a discrete rank one valuation domain. Such a prime M cannot be sharp since to be sharp it would have to contain a finitely generated ideal J that is contained in no other maximal ideal. By checking locally, we would then find that M is the finitely generated (and therefore invertible) ideal $dD + J$. So all of the sharp primes are invertible and the dull ones are locally principal. Hence D is an almost Dedekind domain.

We next show that D has at most one dull maximal ideal. By way of contradiction assume P_1 and P_2 are distinct dull maximal ideals of D. Let b be an element of P_1 that is not in P_2 and write $(b) = M_1^{e_1} M_2^{e_2} \cdots M_m^{e_m}(d)^n$ with no $M_i^{e_i} = D$. As above, each M_i must be invertible. Thus neither P_1 nor P_2 appears in the factorization. Therefore n must be positive and d must be an element of P_1. By repeating this argument for an element in P_2 that is not in P_1 we find that d is also in P_2. But then we have $(b)D_{P_2} = (d)^n D_{P_2} \subseteqq P_2 D_{P_2}$ which is a contradiction. Hence there must be exactly one dull maximal ideal and the rest is both sharp and invertible. □

Remark. With regard to the situation in Theorem 2.5, let D be an almost Dedekind domain with exactly one dull prime, P, and let J be a finitely generated ideal with the property that $JD_P = PD_P$. By Theorem 2.3, the set $\{J\} \cup \mathscr{M}_{\#}(D)$ forms a factoring family for D where each finitely generated fractional ideal will factor uniquely. As J is not a maximal ideal of D, there is an element $a \in P \backslash J$. Consider the ideal $I = aD + J$. On the one hand, I properly contains J, but on the other we have $PD_P = JD_P \subseteqq ID_P \subseteqq PD_P$. Thus $ID_2 = JD_2$ and thus from the proof of Theorem 2.3, it must be that the factorization of I is of the form $JM_1^{e_1} \cdots M_n^{e_n}$ with each e_i negative since J is properly contained in I.

Also note that it is not possible to deduce that D is one-dimensional from the assumption that there is a fixed element d in D such that each finitely generated ideal factors as in Theorem 2.5. For example, let V be a two dimensional valuation domain with principal maximal ideal M and height one prime Q for which QV_Q is principal. Select an element $d \in Q$ such that $dV_Q = QV_Q$ and let $r \in M$ be such that $rV = M$. Then each nonzero nonunit of V has the form $ud^n r^m$ for some unit u and integers m and $n \geqq 0$ with $m > 0$ whenever $n = 0$. Thus each finitely generated ideal factors as $M^m (d)^n$ as desired.

The following result may be known but we have been unable to locate a reference.

Theorem 2.6. *Let M be a height one maximal ideal of an integral domain D with nonzero Jacobson radical. Then M is invertible if and only if it is principal.*

Proof. Assume M is invertible and let d be a nonzero element in the Jacobson radical of D. Since M is invertible, MD_M is principal. Let $b \in M$ be such that $bD_M = MD_M$. Since M is height one, there are elements $r, s \in D \backslash M$ and a positive integer n, $rd = sb^n$. Since d is in the Jacobson radical, s must be in each maximal ideal that does not contain b. Since s is not in M, $sbD_M = MD_M$ with sb in the Jacobson radical.

Now consider the ideal sbM^{-1}. This is a finitely generated ideal of D which is not contained in M. But since sb is contained in the Jacobson radical, it is contained in every other maximal ideal. Thus there is an element $a \in M$ such that $a + sbM^{-1} = D$. Since sbM^{-1} is contained in every maximal ideal except M, a must be contained in M and no other maximal ideal of D. If $aD_M = MD_M$, we have $(a) = M$, otherwise we have $(a + sb) = M$. □

We have several corollaries.

Corollary 2.7. *Let D be an integral domain with $\mathscr{J}(D) \neq (0)$. If M is a height one maximal ideal of D which is locally principal and the radical of a principal ideal, then M is principal.*

Proof. Let M be a height one maximal ideal of D which is locally principal and the radical of the principal ideal (a). Let $b \in M$ be such that $bD_M = MD_M$. By checking locally we see that $M = (a, b)$. As M is finitely generated and locally principal, it must be invertible. Thus M is principal by Theorem 2.6. □

Corollary 2.8. *Let M be a maximal ideal of D, an almost Dedekind domain with nonzero Jacobson radical. If M has finite sharp degree k, then MD_k is principal. The converse holds provided MD_k is a proper ideal of D_k, otherwise M has sharp degree less than k.*

Proof. Since D is a Prüfer domain, each overring, other than the quotient field, has a nonzero Jacobson radical. In particular, the Jacobson radical of D_k is not zero as long as $D_k \neq K$. Thus if $M \in \mathrm{Max}(D)$ has finite sharp degree k, then MD_k is finitely generated and thus principal. □

Theorem 2.9. *Let D be an almost Dedekind domain where each finitely generated ideal has finite sharp degree. If $\mathscr{J}(D) \neq (0)$, then D is Bezout.*

Proof. By Corollary 2.8, each maximal ideal M of sharp degree k is a principal ideal of D_k. Thus the ideals J_α of Lemma 2.2 can be assumed to be principal. The result follows. □

Next we give a general construction scheme for producing an almost Dedekind domain which will have a factoring family for finitely generated ideals. By carefully selecting the members we can produce a family such that each nonzero finitely generated fractional ideal will factor uniquely over the underlying set of allowable factors.

Theorem 2.10. *Let $R_1 \subset R_2 \subset \cdots$ be a chain of Dedekind domains which satisfy all of the following*:

(i) *For $i < j$, each maximal ideal of R_i survives in R_j.*

(ii) *Each maximal ideal of R_j contracts to a maximal ideal of R_1.*

(iii) *If M' is a maximal ideal of R_j and $M = M' \cap R_1$, then $MR_{jM'} = M'R_{jM'}$.*

Let $D = \bigcup R_n$. Then:

(a) *D is an almost Dedekind domain.*

(b) *For $i < j$, each maximal ideal of R_i is contained in only finitely many maximal ideals of R_j. Moreover, if M_i is a maximal ideal of R_i and $M_{j,1}, M_{j,2}, \ldots, M_{j,r}$ are the maximal ideals of R_j that contain M_i, then $M_iR_j = \prod M_{j,k}$.*

(c) *For each finitely generated ideal I of D, there is a finitely generated ideal I_i of some R_i such that $I = I_iD$.*

(d) *A maximal ideal M is a sharp prime of D if and only if $M = M_nD$ for some $M_n = M \cap R_n$.*

(e) *There is a family $\{J_\alpha\}$ that is a factoring family for D for which each nonzero finitely generated fractional ideal can be factored uniquely over the underlying set of the family.*

(f) *D is a Dedekind domain if and only if each maximal ideal of D_1 is contained in only finitely many maximal ideals of D.*

Proof. For each n, we let K_n denote the quotient field of R_n.

Proof of (a). Let M be a maximal ideal of D and let $M_i = M \cap D_i$. Obviously, some M_i is not zero. But then no M_i is zero. Let $r/s \in MD_M$ with $s \in D \backslash M$. For some i, both r and s are in D_i. So $r \in M_i$. But then there is an element $b \in M_1$ and an element $t \in D_i \backslash M_i$ such that $b/t = r/s$. It follows that $MD_M = M_i D_M$ for each i. Since each D_i is Dedekind, $M_i D_{iM_i}$ is principal. Thus MD_M is principal and height one. Hence D is an almost Dedekind domain.

Proof of (b). The first statement is a simple consequence of the fact that each ideal of a Dedekind domain is contained in only finitely many maximal ideals. For the second let M_i be a maximal ideal of R_i and let $M_{j,1}, M_{j,2}, \dots, M_{j,r}$ be the maximal ideals of R_j that contain M_i. Since the $M_{j,k}$s are maximal ideals of R_j, their intersection is the same as their product. Thus $M_i R_j$ is contained in $\prod M_{j,k}$. Equality comes from our assumption that $M_i R_{jM_{j,k}} = M_{j,k} R_{jM_{j,k}}$.

Proof of (c). Since the set $\{R_i\}$ forms a chain, each finitely generated ideal of D can be generated by some finite subset of some R_i.

Proof of (d). Since D is an almost Dedekind domain, a maximal ideal is sharp if and only if it is finitely generated. Hence by (c), M is sharp if and only if some R_n contains a generating set for M. As $M \cap R_n = M_n$ is a maximal ideal of R_n, $M = M_n D$.

Proof of (e). For each maximal ideal M of D and each positive integer i, let $M_i = M \cap R_i$. It is easy to see that $M = \bigcup M_i$. Hence the chain $\{M_i\}$ is uniquely determined by M. Moreover, if $N_1 \subseteqq N_2 \subseteqq \cdots$ is a chain with each N_k a maximal ideal of R_k, then $N = \bigcup N_i$ is a maximal ideal of D. We say that $\{M_i\}$ is the chain determined by M, and that N is the maximal ideal determined by the chain $\{N_i\}$. Each member N_j of the chain $\{N_i\}$ uniquely determines the members of the chain below it since we have $N_i = N_j \cap R_i$ for each $i < j$. Thus for each j, N is determined by the truncated chain $\{N_i\}_{i=j}^{\infty}$.

Since each R_n is a Dedekind domain, the primes of any ring between R_n and its quotient field, K_n, are all extended from primes of R_n. With the restrictions we have placed on the maximal ideals, the quotient field of R_n properly contains the quotient field of R_{n-1} with $R_{n-1} = R_n \cap K_{n-1}$.

Let I be a fractional ideal of R_{n-1}. We will show that $I = IR_n \cap K_{n-1}$. We at least have $I \subseteqq IR_n \cap K_{n-1}$. Since R_{n-1} is a Dedekind domain, each of its fractional ideals is invertible and therefore divisorial. Thus it suffices to show that each element of $(R_{n-1} : I)$ multiplies $IR_n \cap K_{n-1}$ into R_{n-1}. Since both $(R_{n-1} : I)$ and $IR_n \cap K_{n-1}$ are contained in K_{n-1}, the product is there as well. Now use the fact that both I and $IR_n \cap K_{n-1}$ will generate IR_n together with the fact that each element of $(R_{n-1} : I)$ is in $(R_n : IR_n)$ to verify that $(R_{n-1} : I)(IR_n \cap K_{n-1})$ is contained in R_{n-1}. Thus $IR_n \cap K_{n-1} = I$.

For each n and each maximal ideal M_n of R_n, let $\mathscr{C}(M_n)$ denote the set of maximal ideals of R_{n+1} that contract to M_n. The set $\mathscr{C}(M_n)$ is finite since R_{n+1} is a Dedekind domain. Now select a member M_{n+1} of $\mathscr{C}(M_n)$ and then set $\mathscr{F}(M_n) = \mathscr{C}(M_n)\backslash\{M_{n+1}\}$. We will refer to M_{n+1} as a (or the) *discarded prime* sometimes including the phrase of "R_{n+1}" for emphasis. We refer to the members of $\mathscr{C}(M_n)$ as *conjugates* or *conjugate factors* of M_n. If $\mathscr{C}(M_n)$ is a singleton set, then M_nR_{n+1} is a maximal ideal of R_{n+1} and $\mathscr{F}(M_n)$ will be the empty set. Note that in this case we will refer to M_nR_{n+1} as a discarded prime even if M_n is not a discarded prime of R_n. For $n \geqq 1$, let $\mathscr{F}(R_n) = \bigcup\{\mathscr{F}(M_n) \mid M_n \in \mathrm{Max}(R_n)\}$, then set $\mathscr{F}(D) = \bigcup \mathscr{F}(R_n)$. Next, let $\mathscr{G}(D) = \{MD \mid M \in \mathrm{Max}(R_1) \cup \mathscr{F}(D)\}$. We will show that each finitely generated ideal of D can be factored uniquely as a finite product of integer powers of ideals from the set $\mathscr{G}(D)$. Then we will show how to build a factoring family for D using only the members of $\mathscr{G}(D)$.

For each integer n, let $\mathscr{G}(R_n)$ denote the set $\{PR_n \mid P \in \mathrm{Max}(R_1)$ or $P \in \mathscr{F}(R_k)$ for some $k < n\}$. We use induction to show that each nonzero fractional ideal of R_n can be factored uniquely as a finite product of nonzero integer powers of members of $\mathscr{G}(R_n)$. Since $ID \cap K_n = I$ for each fractional ideal I of R_n, each finitely generated fractional ideal of D will factor uniquely over $\mathscr{G}(D)$.

Let I_n be a nonzero fractional ideal of R_n. The result is trivial if $n = 1$ since $\mathscr{G}(R_1) = \mathrm{Max}(R_1)$, so we move on to the case $n = 2$. Since R_2 is a Dedekind domain, each nonzero fractional ideal has sharp degree one. Thus Lemma 2.2 guarantees that the fractional ideal I_2 factors uniquely as a finite product of nonzero integer powers of maximal ideals of R_2, say $I_2 = \prod_{i=1}^{k} P_i^{r_i}$. If each P_i is in $\mathscr{F}(R_1)$, then we at least have existence of a factorization. If not, then some P_i must be a discarded prime. In such a case there is a maximal ideal M_i of R_1 that has P_i as a factor in R_2. If P_i is the only maximal ideal of R_2 that is a factor of M_i, then we have $M_iR_2 = P_i$, and we simply "substitute" M_iR_2 for P_i—they are in fact equal. On the other hand, if M_i has more than one prime factor in R_2, then the other factors are in the set $\mathscr{G}(R_2)$ as only one prime factor is discarded from a set of conjugates. In this case, $M_iR_2 = P_iQ_1 \cdots Q_m$ where the Q_is are the conjugates of P_i each of which is in $\mathscr{G}(R_2)$. Thus $P_i = M_iR_2 \prod_{s=1}^{m} Q_s^{-1}$ and therefore $P_i^{r_i}$ can be replaced by the product $M_i^{r_i}R_2 \prod_{s=1}^{m} Q_s^{-r_i}$. By doing this for each of the discarded primes in the product $\prod P_i^{r_i}$ we obtain a finite factorization of I_2 using ideals in the set $\mathscr{G}(R_2)$.

Now assume that for each $k < n$, each finitely generated ideal of R_k can be factored into a finite product of nonzero integer powers of members of the set $\mathscr{G}(R_k)$. Let I_n be a nonzero fractional ideal of R_n. As above, R_n is a Dedekind domain so I_n factors uniquely as finite product of nonzero powers of maximal ideals of R_n. If each P_i is in $\mathscr{F}(R_{n-1})$, then we have a factorization of I_n over $\mathscr{G}(R_n)$. If not, then some P_i must be a discarded prime. Let $Q_i = P_i \cap R_{n-1}$ and let $\prod_{a=1}^{b} N_a^{s_a}R_{n-1}$ be a factorization over the set $\mathscr{G}(R_{n-1})$ for Q_i. If $Q_iR_n = P_i$, we simply take the factorization of Q_i in R_{n-1} and extend each factor to R_n to get a replacement for P_i. If $Q_iR_n \neq P_i$, then $Q_iR_n = P_i \prod_{c=1}^{s} M_c$ where the M_cs are the conjugates to P_i. Thus each is in the set $\mathscr{G}(R_n)$. As in the case $n = 2$, $P_i = Q_iR_n \prod_{c=1}^{s} M_c^{-1}$. Now replace Q_iR_n by $\prod_{a=1}^{b} N_a^{s_a}R_n$ to get $P_i = \prod_{a=1}^{b} N_a^{s_a}R_n \cdot \prod_{c=1}^{s} M_c^{-1}$. Do this for each discarded prime in the original factorization of I_n. This will yield a finite factorization of I_n over the set $\mathscr{G}(R_n)$. Extending both I_n and each factor to D will yield a finite factorization of I_nD over the set $\mathscr{G}(D)$. As each finitely generated ideal of D is the extension of some ideal I_n in some R_n, we have that each finitely generated ideal of D has a finite factorization over the set $\mathscr{G}(D)$.

Since R_1 is a Dedekind domain, Lemma 2.2 implies each fractional ideal of R_1 can be factored uniquely over the set $\mathrm{Max}(R_1)$. This forms the base for a proof by induction. Assume that for each integer $k < n$, each fractional ideal of R_k can be factored uniquely over the set $\mathscr{G}(R_k)$. Since each member of $\mathscr{G}(R_k)$ extends to a member of $\mathscr{G}(R_m)$ for each $m > k$, our assumption is equivalent to simply saying that each fractional ideal of R_{n-1} factors uniquely over $\mathscr{G}(R_{n-1})$.

Let J be a nonzero fractional ideal of R_n and let $J = \prod_{i=1}^{m} Q_i^{r_i} \cdot \prod_{a=1}^{n} (P_aR_n)^{s_a}$ with the Q_is in $\mathscr{F}(R_{n-1})$ and the P_as in $\mathscr{G}(R_n)\backslash\mathscr{F}(R_{n-1})$. Suppose $\prod_{c=1}^{k} N_c^{t_c} \cdot \prod_{e=1}^{q} (M_eR_n)^{u_e}$ with the N_cs in $\mathscr{F}(R_{n-1})$ and the M_es in $\mathscr{G}(R_n)\backslash\mathscr{F}(R_{n-1})$ is a potentially different factorization of J over $\mathscr{G}(R_n)$. By multiplying by inverses we may obtain $\prod_{i=1}^{m} Q_i^{r_i} \cdot \prod_{c=1}^{k} N_c^{-t_c} = \prod_{e=1}^{q} (M_eR_n)^{u_e} \cdot \prod_{a=1}^{n} (P_aR_n)^{-s_a}$. Since the left hand side of the equation is a product of integer powers of maximal ideals of R_n, its form is unique once common factors are combined. Moreover, the primes on the left hand side are all nontrivial factors of primes from R_{n-1} and for each N_c and Q_i exactly one conjugate factor cannot appear in this product. On the other hand, each M_e and each P_a is a prime of some smaller R_k that either factors nontrivially in R_n or generates a maximal ideal of R_n. Those that generate maximal ideals of R_n can have no factor on the left hand side of the equation and those that have a nontrivial factorization must be missing the corresponding discarded prime on the left hand side. Thus the left hand side must reduce to R_n. This can occur only if the factors in $\prod Q_i^{r_i}$ are simply a rearrangement of the factors in $\prod N_c^{t_c}$. As each factor is an invertible fractional ideal of R_n, we may cancel the products $\prod Q_i^{r_i}$ and $\prod N_c^{t_c}$ and obtain $\prod (P_aR_n)^{s_a} = \prod (M_eR_n)^{u_e}$. Since $IR_n \cap K_{n-1} = I$ for each fractional ideal of R_{n-1}, we have $\prod (P_aR_{n-1})^{s_a} = \prod (M_eR_{n-1})^{u_e}$. Now simply invoke the induction hypothesis to get uniqueness of factorizations.

It remains to show that we can build a factoring family using only the members of the set $\mathscr{G}(D)$. This is actually relatively easy because given any ideal J in $\mathscr{G}(D)$, there is some unique integer n such that $J = P_nD$ for some maximal ideal P_n of R_n that is not a discarded prime of R_n. This places P_n in $\mathscr{G}(R_n)$. While there may be primes above P_n that are not discarded primes, there is a unique chain of primes $P_{n+1} \subset P_{n+2} \subset \cdots$ with each P_k a discarded prime of R_k and $P_k \cap R_n = P_n$. Let P_α be the prime of D determined by this particular chain through P_n and set $J_\alpha = J = P_nD$. Since $P_n = P_\alpha \cap R_n$, $J_\alpha D_{P_\alpha} = P_\alpha D_{P_\alpha}$. Note that this means there is a natural one-to-one correspondence between the set $\mathscr{G}(D)$ and the subset of $\mathrm{Max}(D)$ consisting of those maximal ideals M_β for which there is a largest integer n such that $M_\beta \cap R_n$ is not a discarded prime. There may be a (or even infinitely many) maximal ideal M_σ of D for which there is no largest integer n such that $M_\sigma \cap R_n$ is not a discarded prime. For such a prime, simply set J_σ equal to any member $J = M_nD$ of $\mathscr{G}(D)$

such that $M_\sigma \cap R_n = M_n$. With this we have a factoring family for D such that the underlying set allows for unique factorization of nonzero finitely generated fractional ideals.

Proof of (f). By the proof of (e), we see that if each maximal ideal of R_1 is contained in only finitely many maximal ideals of D, then each maximal ideal of D is finitely generated. Thus D is a Dedekind domain. Conversely, if D is a Dedekind domain, each maximal ideal of D is finitely generated. Thus for $M \in \mathrm{Max}(D)$, there is a maximal ideal M_n of some D_n such that $M = M_n D$. Assume $M_1 \in \mathrm{Max}(R_1)$ is contained in infinitely many maximal ideals of D. Then there must be a chain of maximal ideals $\{M_n\}$ with each M_n a maximal ideal of R_n such that each M_n is contained in infinitely many maximal ideals of D. Thus none of these ideals can generate a maximal ideal of D. Hence, $M = \bigcup M_n$ must be a maximal ideal of D which is not finitely generated, a contradiction of the Dedekind assumption. Therefore, each maximal ideal of R_1 is contained in only finitely many maximal ideals of D. $\square$

3. Constructing almost Dedekind domains

Let $\mathscr{P}_0 = \{\mathbb{N}\}$ and let $\mathscr{P}_1 = \{A_{1,1}, A_{1,2}, \dots, A_{1,n_1}\}$ be a partition of $\mathbb{N}$ into finitely many disjoint nonempty sets with $n_1 > 1$. Recursively for each positive integer $m > 1$, let $\mathscr{P}_m = \{A_{m,1}, A_{m,2}, \dots, A_{m,n_m}\}$ be a refinement of the partition $\mathscr{P}_{m-1}$ with $n_m > n_{m-1}$ but allowing some $A_{m_1,k}$ to survive intact in P_m. Let $Y = \prod_{i \in \mathbb{N}} X_i$. For each set $A_{m,k}$, let $Y_{m,k} = \prod_{i \in A_{m,k}} X_i$. For ease of notation, we let $Y_{0,1} = Y$. Let $R_m = \bigcap V_{m,k}$ where $V_{m,k} = K[Y_{m,1}, Y_{m,2}, \dots, Y_{m,n_m}]_{(Y_{m,k})}$. Set $D = \bigcup R_m$. From the construction it is obvious that $R_0 \subset R_1 \subset R_2 \cdots$ is an ascending chain of semilocal Dedekind domains. Moreover, each maximal ideal of R_m contracts to a maximal ideal of R_{m-1}. In particular, each contracts to $YK[Y]_{(Y)}$ in $R_0 = K[Y]_{(Y)}$. We say that a family of sets $\mathscr{A} = \{A_{m,k_m}\}_{m=0}^{\infty}$ is a *chain through the series of partitions* $\mathscr{P} = \{\mathscr{P}_m\}_{m=0}^{\infty}$ if for each m, $A_{m,k_m} \supseteqq A_{m+1,k_{m+1}}$. Depending on the choice of refinements $\mathscr{P}_m$, there may be chains through $\mathscr{P}$ which are eventually constant. As we will see, such a chain corresponds to a sharp prime of D.

Theorem 3.1. *Let D be as above. Then*:

(a) *If P is a nonzero maximal ideal of D, then $P \cap R_0 = YR_0$. Moreover, $PD_P = YD_P$.*

(b) *D is almost Dedekind domain with nonzero Jacobson radical.*

(c) *Each finitely generated ideal of D is principal.*

(d) *There is a natural one-to-one correspondence between the set of maximal ideals of D and the set of chains through the family of partitions $\mathscr{P}$. Moreover, if M is a maximal ideal of D, then the corresponding chain of sets $\mathscr{A}$ is such that $Y_{m,k_m} D_M = MD_M$ for each A_{m,k_m} in $\mathscr{A}$.*

(e) *The set $\{Y_{m,k} \mid 0 \leqq m, 1 \leqq k \leqq m_k\}$ contains the base set for a factoring family for D. Moreover, the set can be selected in such a way that each nonzero finitely generated fractional ideal will factor uniquely.*

(f) *A maximal ideal M of D is sharp if and only if the corresponding chain of sets $\mathscr{A}$ in statement* (d) *stabilizes at some $A_{m,k}$.*

Proof. Statements (a), (b) and (c) follow from Theorem 2.10. In particular, (c) is a result of Theorem 2.10(c) and the fact that each D_i is a PID. Statement (d) follows from the proof of Theorem 2.10(d) and the fact that each $Y_{m,k}$ generates a maximal ideal of R_m. The statement in (e) follows from the proof of Theorem 2.10(e). Since each member of the factoring family is principal, each finitely generated ideal of D must be principal. Statement (f) is simply a combination of statement (d) and Theorem 2.10(d). □

This construction can be used to form almost Dedekind domains with various sharp degrees. Note that the domain D will have finite sharp degree if and only if there is an integer n such that D_n is semilocal.

We first show how to construct an almost Dedekind domain of sharp degree 2. This domain satisfies the hypothesis of Theorem 2.5, so it gives an example of an almost Dedekind domain with a single noninvertible maximal ideal.

Example 3.2. For each $m \geqq 1$, let $\mathscr{P}_m = \{\{1\}, \{2\}, \ldots, \{m\}, \{k \in \mathbb{N} \mid k > m\}\}$. Let D be almost Dedekind domain determined by this chain of partitions of $\mathbb{N}$. Then:

(a) D has exactly one maximal ideal M which is not sharp.

(b) D has sharp degree 2.

(c) D is a Bezout domain.

(d) $\mathscr{M}_{\#}(D) = \{X_n D \mid n \geqq 1\}$ and the set $\{X_n D \mid n \geqq 1\} \cup \{YD\}$ is a factoring set for D such that each finitely generated ideal factors uniquely.

(e) There is a factoring family for D such that no nonzero finitely generated fractional ideal has a unique factorization over the underlying set of ideals.

Proof. Let $Y_n = \prod_{k=n+1}^{\infty} X_k$. The maximal ideals of R_n consist of the ideal $Y_n R_n$ and the ideals of the form $X_k R_n$ for $1 \leqq k \leqq n$. Thus for each integer $n \geqq 1$, $X_n D$ is a maximal ideal of D. Obviously each of these is a sharp prime of D. The only other maximal ideal of D corresponds to the chain $\{Y_n R_n\}$. Thus $D_2 = D_M$ where M is the maximal ideal of D determined by the chain $\{Y_n R_n\}$.

Since M is the only dull prime of D and $YD_M = MD_M$ we have $YD_2 = MD_2$. By Theorem 2.5, the set $\{YD\} \cup \{X_n D \mid n \geqq 1\}$ is a factoring set for D such that each finitely generated fractional ideal factors uniquely over this set.

Proof of (e). For each n, let $P_n = X_n D$ and write $n = 4k - i$ where $k \geqq 1$ and $0 \leqq i \leqq 3$. Build a factoring family for D as follows: (i) for M again use $J_0 = YD$, (ii) if $i = 0$, let $J_n = X_{2k-1}^3 YD$, (iii) if $i = 1$, let $J_n = X_{2k}^3 YD$, (iv) if $i = 2$, let $J_n = X_{2k-1}^2 YD$, and (v) if $i = 3$, let $J_n = X_{2k}^2 YD$. Since $3 - 2 = 1$, $X_m D$ is the product of $(X_m^3 YD)(X_m^2 YD)^{-1}$. Hence the set $\{J_n\}_{n=0}^{\infty}$ is a factoring family for D. But factorizations are not unique. For example, X_mD can also be factored as $(X_m^2YD)^2(X_m^3YD)^{-1}(YD)^{-1}$. There are in fact infinitely many different ways to factor each nonzero finitely generated fractional ideal of D. By the construction of the family it is clear that each factorization of X_mD must contain nonzero powers of both X_m^2YD and X_m^3YD. On the other hand, YD is redundant, as it can be factored as $(X_m^2YD)^2(X_m^3YD)^{-1}$. $\square$

Next we construct an almost Dedekind domain for which each maximal ideal is dull and where at least some finitely generated ideals will fail to factor uniquely over whatever factoring family we might use—but not necessarily fail to factor uniquely over the underlying set of potential factors.

Example 3.3. For each positive integer n, let $\mathscr{P}_n = \{A_{n,1}, A_{n,2}, \ldots, A_{n,2^n}\}$ where $A_{n,k} = \{m2^n + k \mid m \in \mathbb{Z}, m \geqq 0\}$ for each integer $1 \leqq k \leqq 2^n$.

(a) D is an almost Dedekind domain which is dull.

(b) There exists a factoring family $\{J_\alpha\}$ such that each nonzero finitely generated ideal factors uniquely over the underlying set of ideals making up the family.

(c) Given any factoring family $\{J_\alpha\}$ for D, there exists a nonzero finitely generated ideal I which does not factor uniquely over the family.

Proof. As no chain of sets through $\mathscr{P}$ stabilizes, D has no sharp primes. Hence D is a dull domain. By the proof of Theorem 2.10(e) (or Theorem 3.1(e)), some subset of $\{Y_{m,k}\}$ contains a set such that (i) each nonzero finitely generated fractional ideal factors uniquely, and (ii) this set is the underlying set for a factoring family for D. The nonuniqueness is simply a consequence of the fact that D has only countably many nonzero finitely generated fractional ideals but an uncountable number of maximal ideals. Thus for each factoring family, at least two members are the same ideal of D. $\square$

It is actually rather easy to modify the construction in Example 3.3 to obtain an almost Dedekind domain of dull degree two. One quite trivial way is to simply replace each set $A_{r,1}$, with $r \geqq 1$, by the sets $\{1\}$ and $\{m2^r + 1 \mid m \in \mathbb{N}\}$. This will yield exactly one sharp prime, with the rest dull, and therefore destined to stay that way in D_2. For a more elaborate example with infinitely many sharp primes, we modify the $\mathscr{P}_r$s a bit more.

Example 3.4. Start with the partitions $\mathscr{P}_n$ of Example 3.3. Then for each n and each $0 \leqq r \leqq n$, split each set $A_{n,2^r}$ into the singleton set $\{2^r\}$ and the set $A'_{n,2^r} = \{m2^n + 2^r \mid m \in \mathbb{N}\}$. Then D is an almost Dedekind domain with infinitely many sharp primes and dull degree 2.

Proof. Obviously each singleton set $\{2^r\}$ corresponds to a sharp prime $M_rD = X_{2^r}D$. Each of these primes blows up in D_2, the effect is the same as beginning the construction by partitioning the set $\mathbb{N}\backslash\{2^r \mid r \geqq 0\}$ as in Example 3.3. Thus D_2 is a dull domain. $\square$

Before we construct almost Dedekind domains of larger sharp and dull degrees, we add a little useful terminology. Given a set $A_{m,k}$, we consider the family of sets $\{A_{n,j} \mid A_{n,j} \subseteqq A_{m,k}, n \geqq m\}$ and call this the *branch of the partition from* $A_{m,k}$. Such a branch is said to have *sharp degree p*, if each maximal ideal which has $A_{m,k}$ in its corresponding family of sets has sharp degree less than or equal to p and at least one such maximal ideal has sharp degree p. On the other hand, a branch is said to have *dull degree p*, if there is a maximal ideal which has $A_{m,k}$ in its corresponding family of sets that does not have finite sharp degree, and each maximal ideal of finite sharp degree which has $A_{m,k}$ in its defining family of sets, has sharp degree less than or equal to $p-1$, with at least one such maximal ideal having sharp degree $p-1$.

To build almost Dedekind domains of prescribed sharp and dull degrees, we need a systematic way to build branches of the various sharp and dull degrees. We start with branches of sharp degree two. Essentially these are built not differently than the entire partition used in Example 3.2. Let $\{\mathscr{P}_m\}$ be a series of refinements. For ease of notation assume that for each pair of integers $m < n$, the set $A_{m,1}$ is infinite and $A_{m,1}$ contains $A_{n,1}$. Fix m and order the elements of $A_{m,1}$ as $a_1 < a_2 < a_3 < \cdots$. Then, as in Example 3.2, for each integer $n > m$, let $A'_{n,1} = \{a_1\}$, $A'_{n,2} = \{a_2\}, \dots, A'_{n,n-m} = \{a_{n-m}\}$ and let $A'_{n,n-m+1}$ be the rest of $A_{m,1}$. In each $\mathscr{P}_n$, replace the sets which contain $A_{m,1}$ by the $A'_{n,j}$ sets and leave the rest of $\mathscr{P}_n$ as it is. Then there is exactly one maximal ideal M whose corresponding chain contains $A_{m,1}$ and is not sharp, the one associated with the sets $A'_{n,n-m+1}$. All other maximal ideals associated with $A_{m,1}$ have chains which stabilize at some singleton set $\{a_r\}$. We refer to this technique as building a *standard branch of sharp degree two*. In our next example we utilize this basic construction to build an almost Dedekind domain of sharp degree 3. The construction of the partitions is more complicated, so we will give the details of the construction in the proof rather than the statement of what we are going to build.

Example 3.5. There is a series of partitions $\mathscr{P} = \{\mathscr{P}_m\}_{m=0}^{\infty}$ such that the resulting almost Dedekind domain D has a unique maximal ideal M with sharp degree 3, so $D_3 = D_M$ and D has sharp degree 3.

Proof. Let $\mathscr{P}_1 = \{E, O\}$ where E denotes the positive even integers and O denotes the positive odd integers. From O, build the standard branch of sharp degree two. But for E we proceed a little differently. First split E into the sets $E_{4,0} = \{4m \mid m \geqq 1\}$ and $E_{4,2} = \{4m+2 \mid m \geqq 0\}$. From $E_{4,2}$ build the standard branch of sharp degree two but split $E_{4,0}$ into sets $E_{8,0} = \{8m \mid m \geqq 1\}$ and $E_{8,4} = \{8m+4 \mid m \geqq 0\}$. Then, as with $E_{4,2}$, build the standard branch of sharp degree two from $E_{8,4}$, and, as with $E_{4,0}$, split $E_{8,0}$ into sets $E_{16,0} = \{16m \mid m \geqq 1\}$ and $E_{16,8} = \{16m+8 \mid m \geqq 0\}$. Continue this scheme for each power of 2. Let D be the resulting almost Dedekind domain and let M be the maximal ideal corresponding to the chain $\{E_{2^n,0}\}$.

We will show that there is one prime of sharp degree two associated with O and that each set $E_{2^n,2^{n-1}}$ is associated to exactly one prime of sharp degree two.

The only sharp primes of D are those associated with some singleton set $\{a\}$. For each positive integer n, there is exactly one prime of sharp degree two that contains $\prod_{r=0}^{\infty} X_{2^n r+2^{n-1}}$, the one associated with the chain $\{B^{m,n}\}_{m=1}^{\infty}$ where $B^{m,n} = \{2^n r + 2^{n-1} \mid r \geqq m\}$. On the other hand the chain associated with M consists of the sets of the form $\{2^n r \mid n \geqq 0, r \geqq 1\}$, so $\mathbb{N}$, E, $E_{4,0}$, $E_{8,0}$, etc. For each n, there are infinitely many primes of sharp degree two which are associated with $E_{2^n,0}$. Hence M cannot have sharp degree two. As it is the only dull prime which does not have sharp degree two, it must have sharp degree three. Thus D has sharp degree three and $D_3 = D_M$. □

Theorem 3.6. *For each positive integer $k \geqq 2$, there is a series of refinements $\{\mathscr{P}_m\}$ of $\mathscr{P}_0 = \{\mathbb{N}\}$ such that the resulting domain D is an almost Dedekind domain of sharp degree k.*

Proof. The proof is by induction on k. Assume the result holds for k. The partitioning scheme is somewhat a combination of those used in Examples 3.3 and 3.5. As in Example 3.3, we let $\mathscr{P}_1 = \{O, E\}$ and $\mathscr{P}_2 = \{A_{2,1}, A_{2,2}, A_{2,3}, A_{2,4}\}$ with each $A_{2,r} = \{m2^2 + r \mid m \geqq 0\}$. The subsequent partitions will be different. Specifically, from $A_{2,2}$ and $A_{2,3}$ build branches of sharp degree k. On the other hand we split $A_{2,1}$ into $A_{3,1}$ and $A_{3,5}$ and split $A_{2,4}$ into $A_{3,4}$ and $A_{3,8}$ as in the third stage of the process in Example 3.3. Now continue the pattern of splitting the sets $A_{n,2^n}$ and $A_{n,1}$ as in Example 3.3, but split the sets $A_{n,2^{n-1}}$ and $A_{n,2^{n-1}+1}$ into branches of sharp degree k. Each branch of the infinitely many branches of sharp degree k, corresponds to maximal ideals of sharp degree k. But the prime associated with the chain $\{A_{n,2^n}\}$ will not have sharp degree k since each of the sets $A_{n,2^n}$ is in infinitely many chains associated with primes of sharp degree k. The same is true for the prime associated with the chain $\{A_{n,1}\}$. As these are the only chains which do not lead to primes of sharp degree less than or equal to k, each has sharp degree $k+1$ and therefore D is an almost Dedekind domain of sharp degree $k+1$. □

Things are only a slight bit more complicated in building an almost Dedekind domain with arbitrary finite dull degree. The basic underlying notion is to split sets into "thirds" rather than "halves". Unlike in the constructions above, it is convenient to allow infinite sets to stabilize in the series of refinements. We start with an example illustrating how to use thirds to build an almost Dedekind domain of dull degree two with infinitely many sharp primes. The "convenience" is that our construction parallels the "excluded middle" construction of a Cantor set. This makes it rather easy to increase the dull degree.

Example 3.7. For each pair of integers $n \geqq 1$ and $1 \leqq r \leqq 3^n$, let

$$A_{n,r} = \{m3^n + r \mid m \geqq 0\}$$

and let $r = r_m r_{m-1} \cdots r_1$ be the trinary expansion of r. For each integer $n \geqq 1$, let $\mathscr{P}_n = \{A_{n,r} \mid \text{no } r_i \text{ is a } 2\} \cup \{A_{k,s} \mid 1 \leqq k \leqq n \text{ is the smallest integer such that } s_k = 2\}$. The resulting domain D has dull degree two with infinitely many sharp primes.

Proof. We start with an explicit construction for the first few $\mathscr{P}_n$s. First $\mathscr{P}_1 = \{A_{1,1}, A_{1,2}, A_{1,3}\}$. Then for $\mathscr{P}_2$, we leave the set $A_{1,2}$ as is but split $A_{1,1}$ into $A_{2,1}$, $A_{2,4}$ and $A_{2,7}$, and split $A_{1,3}$ into $A_{2,3}$, $A_{2,6}$ and $A_{2,9}$. The set $A_{1,2}$ will appear in each $\mathscr{P}_n$ from here on as will the sets $A_{2,4}$ and $A_{2,6}$. On the other hand, we split $A_{2,1}$ into $A_{3,1}$, $A_{3,10}$, and $A_{3,19}$, $A_{2,3}$ into $A_{3,3}$, $A_{3,12}$ and $A_{3,21}$, $A_{2,4}$ into $A_{3,4}$, $A_{3,13}$ and $A_{3,22}$, and $A_{2,9}$ into $A_{3,9}$, $A_{3,18}$ and $A_{3,27}$. In $\mathscr{P}_4$, we simply keep each "middle third" as it is and split each pair of outer thirds based on the remainders on division by 3^4. Continue this process to build the partitions $\mathscr{P}_n$. As each middle third set is stable once it appears in some $\mathscr{P}_n$, each leads to a sharp prime of D. On the other hand, if the chain of sets corresponding with M contains no middle third set, then each set in the chain is associated with many infinitely many maximal ideals, including infinitely many which are not associated with a middle third set. Thus D has dull degree 2 with infinitely many sharp primes. □

In the proof for the next theorem, we show how the construction in the previous example can be used to construct an almost Dedekind domain of arbitrary (finite) dull degree $k \geqq 2$.

Theorem 3.8. *For each integer $k \geqq 1$, there exists an almost Dedekind domain of dull degree k.*

Proof. Examples 3.3 and 3.4 provide almost Dedekind domains of dull degree one and two, respectively. As in Theorem 3.6, we modify a previous construction by taking out sets which have stabilized and replacing them with branches of the appropriate sharp degree. Our construction is based on that in Example 3.7.

Fix $k \geqq 3$. The outer third sets are left as they are in Example 3.7, but each middle third set is replaced by a branch of sharp degree $k-1$. Each of the new chains will lead to a maximal ideal of sharp degree $k-1$ or less, with infinitely many of sharp degree $k-1$. This is the maximal sharp degree of any maximal ideal of D. Each prime resulting from a chain of outer third sets remains dull in D_k. Thus D_k is a dull domain, with D_{k-1} a proper subring. Hence D has dull degree k. □

Theorem 3.9. *There exists an almost Dedekind domain D such that D_n is a proper subring of D_{n+1} for each positive integer n. Moreover, the ring $D_\infty = \bigcup D_n$ may be a sharp domain, a dull domain or have some other sharp or dull degree.*

Proof. We start with constructing a domain D such that D_∞ has sharp degree one with $D_n \neq D_{n+1}$ for each n. Start with the basic Odd/Even partitioning scheme used to construct branches of sharp degree k, but instead of changing each branch to one of sharp degree $k-1$, allow each new branch to have larger and larger sharp degree. By doing so, once we hit a set high enough up in the branch of sharp degree n, we find a single prime of sharp degree n and all others with smaller sharp degree. But now, the chain corresponding to the powers of 2 sets will not lead to a prime of finite sharp degree. However, once we take the union of the D_ns, we will obtain a domain of sharp degree one as the only prime which does not have finite sharp degree is the one corresponding to the chain $\{E_{n,2^n}\}$.

We use a similar scheme to build a domain D such that D_∞ is a dull domain with primes of each finite sharp degree. Start with the basic scheme used in the proof of Theorem 3.8, but now instead of replacing each middle third set with a branch of the same sharp degree, replace them with branches of larger and larger sharp degree. We may leave the first middle third set, $A_{1,2}$, alone. Then replace $A_{2,4}$ and $A_{2,6}$ by branches of sharp degree two. Continue by replacing each middle third set $A_{k,r}$ by a branch of sharp degree k. The result will be that each branch through a middle third set leads only to primes of finite sharp degree, but there is no uniform bound on the degree that holds for all branches through all middle third sets. As in the proof of Theorem 3.8, the primes whose chains involve only outer third sets will remain dull throughout each D_n and remain dull in D_∞. Thus D_∞ is a dull domain.

For sharp and dull degree two for D_∞, replace branches of finite sharp degree with ones which mimic the construction of a D_∞ with sharp degree one. Continue this fractal like approach to get larger and larger sharp and dull degrees for D_∞. □

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