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Semistar invertibility on integral domains. (English summary)
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The authors have written several interesting papers on the semi-star operation [see, for example, S. El Baghdadi and M. Fontana, Comm. Algebra **32** (2004), no. 3, 1101–1126; MR2063800 (2006a:13003); S. El Baghdadi, M. Fontana and G. Picozza, J. Pure Appl. Algebra **193** (2004), no. 1-3, 27–60; MR2076377 (2005g:13032); M. Fontana and K. A. Loper, Comm. Algebra **31** (2003), no. 10, 4775–4805; MR1998028 (2004e:13034)]. This paper is a continuation of their works on the semistar operation. In this paper, the authors introduce the notions of $*$-invertibility and quasi-$*$-invertibility, and then they extend many results proved for star (e.g., d-, v-, t-, w-) invertibility.

Let D be an integral domain with quotient field K such that $D \neq K$. Let $\overline{F}(D)$ be the set of all nonzero D-submodules of K, $F(D)$ the subset of $\overline{F}(D)$ consisting of fractional ideals of D, and $f(D)$ the set of nonzero finitely generated D-submodules of K; hence $f(D) \subseteq F(D) \subseteq \overline{F}(D)$. A map $*\colon \overline{F}(D) \to \overline{F}(D)$, $E \mapsto E^*$, is called a semistar operation on D if (i) $(xE)^* = xE^*$, (ii) $E \subseteq F$ implies $E^* \subseteq F^*$, and (iii) $E \subseteq E^*$ and $(E^*)^* = E^*$ for each $0 \neq x \in K$ and $E, F \in \overline{F}(D)$.

Let $*$ be a semistar operation on D. A nonzero integral ideal I of D is a $*$-ideal (resp., quasi-$*$-ideal) if $I^* = I$ (resp., $I^* \cap D = I$). A quasi-$*$-ideal maximal among proper quasi-$*$-ideals of D is called a quasi-$*$-maximal ideal. An $I \in \overline{F}(D)$ is said to be $*$-finite if there exists $J \in f(D)$ such that $I^* = J^*$. For each $E \in \overline{F}(D)$, let $E^{*_f} = \bigcup\{F^* |\ F \in f(D)$ and $F \subseteq E\}$. Then $*_f$ is also a semistar operation on D and $(*_f)_f = *_f$. Let $M(*)$ be the set of quasi-$*$-maximal ideals of D; it is known that if $* = *_f$, then $M(*) \neq \varnothing$. Let $N(*) = \{h \in D[X] |\ h \neq 0$ and $(c(h))^* = D^*\}$, where $c(h)$ is the content of the polynomial $h \in D[X]$. Then the ring $\mathrm{Na}(D,*) := D[X]_{N(*)}$ is called the Nagata ring of D with respect to $*$.

Let $*$ be a semistar operation on D. An $I \in F(D)$ is $*$-invertible if $(II^{-1})^* = D^*$. An $I \in \overline{F}(D)$ is quasi-$*$-invertible if $(I(D^*\colon I))^* = D^*$. Note that $I^{-1} = (D\colon I) \subseteq (D^*\colon I)$; so if $I \in F(D)$ is $*$-invertible, then I is also quasi-$*$-invertible. The authors give an example which shows that a quasi-$*$-invertible (integral) ideal need not be $*$-invertible. Among other things, the authors prove the following. (1) An $I \in F(D)$ is $*_f$-invertible if and only if I and I^{-1} are $*_f$-finite and I is $*$-invertible. (2) Each $*$-invertible ideal $I \in F(D)$ is $*_f$-invertible if and only if $M(*_f) = M(*)$. (3) An $I \in \overline{F}(D)$ is quasi-$*_f$-invertible if and only if I and $(D^*\colon I)$ are $*_f$-finite and I is quasi-$*$-invertible. (4) If $D^* = D$, then a quasi-$*$-invertible ideal $I \in F(D)$ is $*$-invertible. (5) An $I \in f(D)$ is $*_f$-invertible if and only if ID_Q is invertible for each $Q \in M(*_f)$, if and only if $I\mathrm{Na}(D,*_f)$ is invertible. (6) For $0 \neq h \in D[X]$, $c(h)$ is $*_f$-invertible if and only if $h\mathrm{Na}(D,*_f) = c(h)\mathrm{Na}(D,*_f)$. (7) If H is an invertible ideal of $\mathrm{Na}(D,*)$, then H is principal in $\mathrm{Na}(D,*)$.

Reviewed by *Gyu Whan Chang*