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Universally Going-Down Homomorphisms of Commutative Rings

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1. INTRODUCTION

Let $f: R \to T$ be a homomorphism of commutative rings. It is known that change of base need not preserve the property that f satisfies going-down (henceforth abbreviated GD, as in [12]). Indeed, [4, Example 3.9] presents a Noetherian local integral domain R of (Krull) dimension 1 and an overring T of R (that is, a ring T contained between R and the quotient field of R) such that, although the inclusion map $R \to T$ necessarily satisfies GD, the induced map $R[X] \to T[X]$ of polynomial rings does not satisfy GD. As a strengthening of the GD property, we are thus led to consider homomorphisms $f: R \to T$ such that the induced homomorphisms f_n: $R[X_1,..., X_n] \to T[X_1,..., X_n]$ satisfy GD for each $n \geqslant 1$. It is easy to see (cf. Corollary 2.3) that such f are "universally going-down," in the sense that $S \to S \otimes_R T$ satisfies GD for each change of base $R \to S$. On the other hand, McAdam has shown that such f are of interest for the following additional reason ([13, Theorems B and C; 14, Theorem 4]). If f is the inclusion of an integral domain R in an integral overring T, then each f_n satisfies GD if and only if each f_n is unibranched; and moreover, in this context, each f_n satisfies GD if (and only if) f_1 satisfies GD or is unibranched. The aim of this paper

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is to identify and study the universal properties underlying such investigations.

Section 2 presents a unified treatment of universally i-, universally unibranched, and universally mated homomorphisms (definitions recalled below). Its main results, Theorems 2.1 and 2.5, show in each of the three cases that the "universal" property is equivalent to the conjunction of the "ordinary" property and the condition that the induced inclusions of residue fields be purely inseparable extensions. Much of the material in Section 2 may be considered folklore. Indeed, the above observation for "universally i-" is essentially due to Grothendieck–Dieudonné, for "universally i-" means precisely *radiciel*, in the terminology of [10]. In addition, in the "universally unibranched" case, the observation has been somewhat anticipated by work of Andreotti–Bombieri [1] on weak normalization and by [13, Theorem 3]. However, it is the "universally mated" property which will play a key role in the analysis of "universally going-down" in Section 3. Section 2 also contains some useful technical results (Lemma 2.4(a)) concerning stability of various properties under direct limit.

The third, and most important, section begins by describing an essential difference between the behaviour of GD and the behaviour of the three "related" properties noted in Section 2. Specifically, Example 3.1 presents an overring T of an integral domain R such that the the inclusion map $R \to T$ satisfies GD and is universally unibranched (and hence satisfies the "purely inseparable" condition for residue field extensions) and such that the induced map $R[X] \to T[X]$ does not satisfy GD. Accordingly, we pause next to apply the criterion from Corollary 2.3 in order to obtain some examples of universally going-down homomorphisms, including all $R \to T$ for which $\dim(R) = 0$ (cf. Proposition 3.3). Then, in an attempt to obtain an "internal" characterization of "universally going-down," we modify some constructions of Andreotti–Bombieri [1] and Traverso [17], and thus introduce the notion of a UGD homomorphism. In case $f: R \to T$ is injective and integral, we find that f is UGD if and only if f satisfies GD and T is the weak normalization of R with respect to f (in the sense of [1]). For arbitrary (not necessarily integral) f, UGD is shown to have several useful consequences, notably "radiciel" and "universally mated" (cf. Corollary 3.12). After observing that UGD implies a weak variant of going-up, we infer our main result, Theorem 3.15, concerning universality of UGD. Its specializations include Theorem 3.17, a characterization of "universally going-down" for (the inclusion map of) an *arbitrary* overring of an integral domain. This entails consequences for certain nonintegral maps, which cannot be handled by the riding hypotheses in [13, 14]. The upshot for an integral overring of an integral domain is that "universally going-down" and UGD are equivalent (cf. Corollary 3.20). In particular, there is but one type of integral overring extension $R \to T$ of the type studied by McAdam [13, 14], viz. for which the induced homomorphisms f_n satisfy GD or are unibranched: T must be the weak normalization of R inside T.

Throughout, all rings are assumed commutative, with unit; and all ring-homomorphisms are assumed unital. If $f: R \to T$ is a ring-homomorphism and p is a prime ideal of R, then T_p denotes $T_{f(R \setminus p)}$ ($\cong T \otimes_R R_p$); and $k(p) = k_R(p)$ denotes R_p/pR_p. In addition, $X = X_1, \ldots, X_n$ denote commuting, algebraically independent indeterminates over the appropriate rings; and R' denotes the integral closure of R. Any unexplained material is standard, as in [9] and [12].

2. Matedness and Universality

This section's goal is to characterize the universally mated ring-homomorphisms. This work will be used in the study of "universally going-down" in Section 3. Our methods will also yield characterizations of the universally i- and the universally unibranched ring-homomorphisms, thereby recovering some facts about "radiciel" homomorphisms from [10] and placing [13, Theorem B] into a more general setting.

The underlying definitions are the following. Let $f: R \to T$ be a ring-homomorphism. We say that f is an *i-homomorphism* in case $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$, its induced function on prime ideals, is an injection. We say that f is *unibranched* (resp. *mated*) if, for each $p \in \operatorname{Spec}(R)$ (resp., for each $p \in \operatorname{Spec}(R)$ such that $f(p)T \neq T$), there exists a unique $q \in \operatorname{Spec}(T)$ such that $f^{-1}(q) = p$. Finally, if **P** is a property of (some) ring-homomorphisms, then f is said to be (have, satisfy) *universally* **P** if, for each change of base $R \to S$, the induced homomorphism $S \to S \otimes_R T$ satisfies **P**.

The first three definitions given above appear in [15, 13, 4], respectively, for the case in which f is an inclusion map of integral domains. For the general situation, it is evident that

$$\text{unibranched} \Rightarrow \text{mated} \Rightarrow i\text{-homomorphism}.$$

Moreover, neither of these implications has a valid converse. To see this in the first case, it is enough to consider $R \to R_p$, where p is a nonmaximal prime of R; and the second is also easy: cf. [15, Example 2.3].

To aid our study of universal properties, we make the following definition. Let **P** be a property of (some) ring-homomorphisms and let $f: R \to T$ be a ring-homomorphism. We say that f is *strongly* **P** if f satisfies **P** and, for each $q \in \operatorname{Spec}(T)$, the induced extension of fields, $k_R(f^{-1}(q)) \to k_T(q)$, is (algebraic) purely inseparable. Note that a strongly i-homomorphism is termed "radiciel" in [10], and a strongly unibranched inclusion map of integral domains is called a U-extension in [13]. It is easy to use the $D + M$

construction to find a unibranched ring-homomorphism which is not strongly unibranched. Indeed, in [13, Example, p. 709], McAdam presents such a unibranched inclusion map $g: R \to T$ of integral domains which is not strongly unibranched, and he notes that the induced map $g_1: R[X] \to T[X]$ is not unibranched. It will follow from Theorem 2.1 that g_1 is not even an i-homomorphism.

An "i-homomorphism" variant of the next result was anticipated by Grothendieck–Dieudonné [10, Proposition 3.7.1, p. 246]. The "unibranched" assertion was obtained by McAdam [13, Theorems B, 3 and 4] in case f is an inclusion map of integral domains. Finally, we note that [4, Proposition 3.1] has anticipated the "mated" case of the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 2.1. We shall only sketch how the techniques of [13] adapt to treat the general situation.

THEOREM 2.1. *For a ring-homomorphism $f: R \to T$, the following are equivalent*:

(i) $f_1: R[X] \to T[X]$ *is an i-homomorphism* (*resp., unibranched*; *resp., mated*).

(ii) f *is a strongly i-homomorphism* (*resp., strongly unibranched*; *resp., strongly mated*).

(iii) *There exists* $n \geqslant 1$ *such that* $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ *is an i-homomorphism* (*resp., unibranched*; *resp., mated*).

(iv) *For each* $n \geqslant 0$, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ *is an i-homomorphism* (*resp., unibranched*; *resp., mated*).

(v) *For each* $n \geqslant 0$, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ *is a strongly i-homomorphism* (*resp., strongly unibranched*; *resp., strongly mated*).

Proof. It is evident that (v) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (iii).

Next, we remark that [13, Theorem 2 and Lemma 4] extend directly from the context of inclusions of integral domains to that of ring-homomorphisms. (Details of such modifications of [13] will be left to the reader.) In particular, the resulting variant of [13, Theorem 2] now describes the preimage of any prime in $R[X]$ under the map $\mathrm{Spec}(T[X]) \to \mathrm{Spec}(R[X])$ induced by f_1.

(iii) $\Rightarrow$ (i). It is enough to show that if f_1 has any of the three properties (i-homomorphism, unibranched, or mated), then so does f. The first and second of these are direct consequences of the following facts. If $p \in \mathrm{Spec}(R)$ and $q \in \mathrm{Spec}(T)$, then $f^{-1}(q) = p$ if and only if $f_1^{-1}(qT[X]) = pR[X]$; and if $Q \in \mathrm{Spec}(T[X])$, then $f_1^{-1}(Q) \cap R = f^{-1}(Q \cap T)$. For the "mated" conclusion, we also need to observe that if $p \in \mathrm{Spec}(R)$, then $f_1(pR[X])\, T[X] \neq T[X]$ if and only if $f(p)T \neq T$.

(i) $\Leftrightarrow$ (ii). Combine the preceding argument with a straightforward modification of the first and second paragraphs of the proof of [13, Theorem 3].

(ii) $\Rightarrow$ (v). By induction on n, we can assume that $n=1$. Since (ii) $\Rightarrow$ (i), it remains only to show that if $Q \in \operatorname{Spec}(T[X])$, then $k_{R[X]}(f_1^{-1}(Q)) \to k_{T[X]}(Q)$ is a purely inseparable extension. In case $Q = qT[X]$ for some $q \in \operatorname{Spec}(T)$, we can argue as in [13, p. 710, lines 25–30]. In the remaining case, Q is an "upper" and, since f_1 is an i-homomorphism, so is $f_1^{-1}(Q)$; thus, the (modified) proof of [13, Lemma 4] may be applied, and the proof is complete.

Before showing that strongly **P** is equivalent to universally **P** for each of **P** $= i$-homomorphism, unibranched and mated, we collect some useful information about universal properties. (As usual, a property **P** is said to be a *universal property* in case **P** is equivalent to universally **P**: cf. [10, pp. 239–240].)

PROPOSITION 2.2. *Let* **P** *be a property of (some) ring-homomorphisms which is preserved by direct limits. Then* **P** *is a universal property if and only if both the following conditions hold:*

(a) *If $f: R \to T$ has* **P**, *then so does $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ for each $n \geqslant 1$.*

(b) *If $f: R \to T$ has* **P** *and if J is an (finitely generated) ideal of R, then the induced homomorphism $R/J \to T/JT$ also has* **P**.

Proof. The "only if" half is clear, even without the hypothesis about direct limits, since $T[X_1,..., X_n] \cong R[X_1,..., X_n] \otimes_R T$ and $T/JT \cong R/J \otimes_R T$.

Conversely, consider any change of base, $g: R \to S$. By [10, Proposition 6.3.8, p. 136], S is a direct limit of finitely presented R-algebras. More precisely there exists a directed system of ring-homomorphisms, $\{g_i: i \in I\}$, such that $g_i: R \to S_i$ is finitely presented for each $i \in I$ and $\varinjlim g_i: R \to \varinjlim S_i$ is R-isomorphic to g. For each i, write $S_i \cong R[X_1,..., X_{n_i}]/J_i$ for a suitable (finitely generated) ideal J_i. Assuming that $f: R \to T$ has **P**, we may use (a) and (b) to show that $S_i \to T[X_1,..., X_{n_i}]/J_i T[X_1,..., X_{n_i}]$ has **P**, for each i; that is, $S_i \to S_i \otimes_R T$ has **P**. Since tensor product commutes with direct limit, the direct limit of $\{S_i \to S_i \otimes_R T\}$ is identified as $S \to S \otimes_R T$, which, by the preservation hypothesis, must have **P**. This completes the proof.

COROLLARY 2.3. *A ring-homomorphism $f: R \to T$ is universally going-down if (and only if) $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ satisfies GD for each $n \geqslant 0$.*

Proof. It was shown in [5, Theorem 2.1] that GD is preserved by direct limits. Moreover, it is easy to see that $\mathbf{P} = \mathrm{GD}$ satisfies condition (b) in the statement of Proposition 2.2. Thus, the required assertion follows from the *proof* of Proposition 2.2.

LEMMA 2.4. *Let* $\mathbf{P}$ *be one of the following six properties of (some) ring-homomorphisms: i-homomorphism, unibranched, mated, strongly i-homomorphism, strongly unibranched, and strongly mated. Then:*

(a) $\mathbf{P}$ *is preserved by direct limits.*

(b) *If a ring-homomorphism* $f: R \to T$ *has* $\mathbf{P}$ *and if* J *is an ideal of* R, *then the induced homomorphism* $\bar{f}: R/J \to T/JT$ *also has* $\mathbf{P}$.

Proof. (a) The assertion for i-homomorphisms was proved in [5, Proposition 2.3]. To approach the remaining assertions, we fix the following notation. Let $(I, \leqslant)$ be a directed set, and let (A_i, f_{ij}) and (B_i, g_{ij}) each be directed systems of rings indexed by I. For each $i \in I$, let $h_i: A_i \to B_i$ be a ring-homomorphism satisfying the property in question such that, whenever $i \leqslant j$ in I, then $g_{ij} h_i = h_j f_{ij}: A_i \to B_j$. Set $A = \varinjlim A_i$, $B = \varinjlim B_i$, and $h = \varinjlim h_i$. The issue is whether $h: A \to B$ also has the property in question.

For the "unibranched" assertion, it is now enough to show that if $p \in \mathrm{Spec}(A)$, then there exists $q \in \mathrm{Spec}(B)$ such that $h^{-1}(q) = p$. To this end, consider $p_i = f_i^{-1}(p)$, where $f_i: A_i \to A$ is the canonical structure map. By [10, Proposition 6.1.2(ii), p. 128], $\varinjlim p_i \cong p$. By hypothesis, for each i, $p_i = h_i^{-1}(q_i)$ for some uniquely determined $q_i \in \mathrm{Spec}(B_i)$. It can be shown that $g_{jk}^{-1}(q_k) = q_j$ whenever $j \leqslant k$ in I. Indeed, since h_j is assumed to be an i-homomorphism, it is enough to show that $h_j^{-1}(g_{jk}^{-1}(q_k)) = h_j^{-1}(q_j)$, and we leave this calculation for the reader. Now, it follows from [10, Propositions 6.1.2(i) and 6.1.6(i), pp. 128–130] that $\{q_i\}$ is a directed system and that $q = \varinjlim q_i$ is a prime of B. To see that q is as desired, one need only show that $f_i^{-1} h^{-1}(q) = p_i$ for each i. However, if $g_i: B_i \to B$ is the structure map, observe that

$$\begin{aligned} f_i^{-1} h^{-1}(q) &= (hf_i)^{-1}(q) = (g_i h_i)^{-1}(q) \\ &= h_i^{-1} g_i^{-1}(q) = h_i^{-1}(q_i) = p_i. \end{aligned}$$

This establishes the "unibranched" assertion.

The "mated" assertion may be proved by repeating the above argument for the "unibranched" fact, after noticing that $h(p)B \neq B$ implies $h_i(p_i) B_i \neq B_i$ for each i.

For the "strongly..." assertions, consider $q \in \mathrm{Spec}(B)$, and set $p = h^{-1}(q)$ and $l = \mathrm{char}(k_B(q))$. Suppose $l > 0$. (The case $l = 0$ is similar and hence omitted.) We need only to show that for each $u \in k_B(q)$, there exists $n \geqslant 1$

such that $u^{l^n} \in k_A(p)$. Without loss of generality, $u \in B/q$; write $u = b + q$, with $b \in B$. Setting $q_i = g_i^{-1}(q)$ and $p_i = f_i^{-1}(p)$, we can find an index j and an element $b_j \in B_j$ such that $g_j(b_j) = b$. Next a calculation using $g_j h_j = hf_j$ reveals that $h_j^{-1}(q_j) = p_j$, which leads to an inclusion of fields $k_{A_j}(p_j) \to k_{B_j}(q_j)$. By hypothesis, there exists $n \geqslant 1$ such that $(b_j + q_j)^{l^n} \in k_{A_j}(p_j)$. As u may be assumed nonzero, the construction of quotient fields leads to elements c_j, $d_j \in A_j \backslash p_j$ such that

$$h_j(c_j)\, b_j^{l^n} - h_j(d_j) \in q_j.$$

Mapping via $B_j \to B \to B/q$ reveals that $u^{l^n} \in k_A(p)$, the point being that $h_j^{-1}(q_j) = p_j$ and $c_j \notin p_j$ force $h_j(c_j) \in B_j \backslash q_j$, whence $g_j h_j(c_j) \in B \backslash q$.

(b) The "i-homomorphism" assertion follows easily by observing that $\bar{f}^{-1}(q/JT) = f^{-1}(q)/J$ for each (prime) q of T which contains JT; the "unibranched" assertion, by observing that $f^{-1}(q) \supset J$ implies $q \supset JT$; and the "mated" assertion, by observing that if p is a prime of R containing J such that $(p/J)(T/J) \neq T/J$, then $pT \neq T$. For the "strongly..." assertions, note that if $q \in \operatorname{Spec}(T)$ contains JT and if $p = f^{-1}(q)$, then $k_{R/J}(p/J) \to k_{T/JT}(q/JT)$ may be identified with $k_R(p) \to k_T(q)$. This completes the proof.

We next close the section by giving its main result.

THEOREM 2.5. *Let* **P** *be any one of the following three properties: i-homomorphism, unibranched, mated. Then, for each ring-homomorphism* $f: R \to T$, *the following conditions are equivalent*:

(i) f *is universally* **P**;

(ii) f *is strongly* **P**;

(iii) f *is universally strongly* **P**.

Proof. (iii) $\Rightarrow$ (i). Trivial.

(i) $\Rightarrow$ (ii). Apply the implication (i) $\Rightarrow$ (ii) established in Theorem 2.1.

(ii) $\Rightarrow$ (iii). Assume that f is strongly **P**. By the implication (ii) $\Rightarrow$ (v) established in Theorem 2.1, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ is strongly P for each n. Moreover, Lemma 2.4 shows that the strongly **P** property is preserved both under direct limits and under formation of factor-rings. The desired conclusion now follows from the proof of Proposition 2.2.

3. GOING-DOWN AND UNIVERSALITY

This section studies universally going-down ring-homomorphisms (cf. the "morphismes universellement génèrisants" of [10, Définition 3.9.2, p. 253]). Its main result, Theorem 3.15, uses the "weak normalization" notion introduced by Andreotti–Bombieri [1] to obtain, i.a., an "internal" characterization of the universally going-down inclusion maps of integral overrings of integral domains. In this way, new light is shed on the overrings appearing in McAdam [13, Theorem C].

We begin with a sharp contrast to Theorem 2.5: strongly GD does *not* imply universally going-down. Example 3.1 is inspired by work of Doering–Lequain [6, Example C].

EXAMPLE 3.1. There exist an integral domain R and an overring T of R such that the inclusion map $f: R \to T$ satisfies GD and is strongly (universally) unibranched, although $f_1: R[X] \to T[X]$ does not satisfy GD.

For the construction, let F be a field and $\{T_i\}$ a denumerable set of (independent) indeterminates over F; set $k = F(T_1, T_2, \ldots)$. Let X, Y_1, Y_2 be indeterminates over k, set $K = k(X, Y_1, Y_2)$ and consider the one- (resp., two-) dimensional valuation domain $V_1 = k(Y_1, Y_2)[X]_{(X)}$ (resp., $V_2 = k(X)[Y_1]_{(Y_1)} + Y_2 k(X, Y_1)[Y_2]_{(Y_2)}$), with quotient field K. Let M_1 (resp., M_2) denote the maximal ideal of V_1 (resp., V_2). It is well known (cf. [12, Theorem 107]) that the maximal ideals of the Prüfer domain $A = V_1 \cap V_2$ are $m_1 = M_1 \cap A$ and $m_2 = M_2 \cap A$; and that $A_{m_i} = V_i$ for $i = 1, 2$. Hence the only nonzero nonmaximal prime of A is

$$p = Y_2 k(X, Y_1)[Y_2]_{(Y_2)} \cap A$$

and it is straightforward to verify that $A_p = k(X, Y_1)[Y_2]_{(Y_2)}$ (cf. [2, Theorem 1, (a), p. 376]).

Next, note that there is an F-algebra isomorphism $g_1: k_A(m_1) \to k$ since $k_A(m_1) = V_1/M_1 \cong k(Y_1, Y_2)$; similarly, $k_A(m_2) \cong k(X)$, leading to an isomorphism $g_2: k_A(m_2) \to k$. Let $u: k(m_2) \to k(m_1) \times k(m_2)$ be the composite of g_2, the diagonal map $k \to k \times k$, and $g_1^{-1} \times g_2^{-1}: k \times k \to k(m_1) \times k(m_2)$. Let $v: A \to k(m_1) \times k(m_2)$ be the product map, which is surjective (by the Chinese remainder theorem). Define R to be the pullback of u and v; view $R \subset A$ in the usual way. The topological description of Spec(R) in [8, Theorem 1.4, especially (c), p. 335] yields that R has but three distinct primes $0 \subset q = p \cap R \subset m = m_1 \cap m_2$, and that $R_q = A_p$. In particular, $k_R(q) = k_A(p)$.

Set $T = V_2$ and consider the inclusion map $f: R \to T$. The above information about spectra easily reveals that f is unibranched and satisfies GD (and going-up). Moreover, for each prime $w \in \operatorname{Spec}(T)$, the canonical injection $k_R(f^{-1}(w)) \to k_T(w)$ is the identity map (and, hence, a purely inseparable extension). To see this for w the maximal ideal of T, recall from [8, Theorem 1.4(a),(b)] that R/m is canonically $k(m_2)$, viz., $k_T(M_2)$; for w the nonzero nonmaximal prime of T, note that $k_T(w) = k_A(p)$, which we have seen coincides with $k_R(q)$; and for $w=0$, the injection in question is just the identity map $K \to K$. Thus f is strongly unibranched.

Returning to our construction, we infer that $f_1: R[X] \to T[X]$ is unibranched by Theorem 2.1 (or [13, Theorem 3]). Hence, to show that f_1 does not satisfy GD, it is enough to show that f_1 does not satisfy going-up. This may be done in two ways: either modify the argument of Doering–Lequain [6, p. 592, lines 28–35] or use the folklore result that an inclusion $h: B \to D$ is integral if (and only if) $h_1: B[X] \to D[X]$ satisfies going-up. (Of course, f is not integral: cf. [16, Proposition 2].) This completes the proof of Example 3.1.

We pause to emphasize that [14, Theorem 4] shows that *no* example $R \subset T$ with the properties asserted in Example 3.1 can be integral. Put differently, Example 3.1 shows that one cannot remove the integrality hypothesis for the (i) $\Rightarrow$ (iv) part of [13, Theorem C].

Example 3.1 has indicated that the universally going-down ring-homomorphisms *cannot* be characterized as the strongly GD maps. Before trying to see how they *can* be characterized, it seems prudent to collect some examples. A first source of examples has a homological origin: if a ring-homomorphism $f: R \to T$ makes T a flat R-module, then f is universally going-down. The reason is simply that flat implies GD (cf. [12, Exercise 37, p. 44]) and flatness is a universal property [2, Corollary 2, p. 19]. A second family of examples is topological: each universally open ring-homomorphism is universally going-down. (Cf. [10, Corollaire 3.9.4, p. 254]. As usual, a ring-homomorphism $R \to T$ is said to be *open* if its induced function $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is open in the Zariski topology.) In his study of "universally open," Ferrand [7] has observed that any (GD) ring-homomorphism satisfying the hypotheses of the classical "going-down theorem" of Cohen–Seidenberg [3, Theorem 5] actually fits into this second family of examples. It is interesting to note that "universally open" does not imply "flat" (cf. [10, Remarques 7.3.12(ii), p. 341]); "flat" does not imply "open" (consider $R \to R_p$, for a nonmaximal prime p); and "open" does not imply "universally going-down" (consider, for instance, [4, Example 3.9]). A third family of examples, the homomorphisms defined on zero-dimensional rings, will be treated in Proposition 3.3. For motivation, recall the case of discrete schemes treated in [10, Proposition 7.3.13, p. 341].

As usual, $r(A)$ will denote the radical of a ring A; A_{red} will denote $A/r(A)$; and if $f: R \to T$ is a ring-homomorphism, then f_{red} will denote the induced map $R_{\mathrm{red}} \to T_{\mathrm{red}}$.

LEMMA 3.2. (a) *A ring-homomorphism $f: R \to T$ satisfies GD if and only if f_{red} satisfies GD.*

(b) *If I is an ideal of a ring R such that $I \subset r(R)$, then the canonical surjection $R \to R/I$ satisfies GD.*

Proof. (a) This follows easily from the observation that if $q \in \mathrm{Spec}(T)$, then $f_{\mathrm{red}}^{-1}(q/r(T)) = f^{-1}(q)/r(R)$.

(b) One need only recall that $r(R)$ is the intersection of the prime ideals of R. (Much more could be said. For instance, $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is a homeomorphism. Cf. [10, Proposition 1.1.6, p. 195].)

PROPOSITION 3.3. *If $f: R \to T$ is a ring-homomorphism and if* $\dim(R) = 0$, *then f is universally going-down.*

Proof. Consider any change of base, $R \to S$. Since R_{red} is zero-dimensional and reduced, it is von Neumann regular (cf. [12, Exercise 12, p. 63]), that is, "absolutely flat" in the terminology of [2]. Accordingly, by [2, Exercise 16(d), p. 143], f_{red} is flat and thus, by the above remarks, universally going-down. Therefore the map $g: S_{\mathrm{red}} \to S_{\mathrm{red}} \otimes_{R_{\mathrm{red}}} T_{\mathrm{red}}$ satisfies GD.

Notice that the induced map $f_S: S \to S \otimes_R T$ leads to a commutative diagram

$$\begin{array}{ccc} S_{\mathrm{red}} & \longrightarrow & (S \otimes_R T)_{\mathrm{red}} \\ \downarrow & & \downarrow \\ S_{\mathrm{red}} \otimes_{R_{\mathrm{red}}} T_{\mathrm{red}} & \longrightarrow & (S_{\mathrm{red}} \otimes_{R_{\mathrm{red}}} T_{\mathrm{red}})_{\mathrm{red}} \end{array}$$

in which the top arrow is $(f_S)_{\mathrm{red}}$, the bottom is the canonical surjection π, the left vertical map is g, and the right is the canonical isomorphism [10, Corollaire 4.5.12, p. 271]. Since Lemma 3.2(b) impies that π satisfies GD, so do πg and $(f_S)_{\mathrm{red}}$. An application of Lemma 3.2(a) completes the proof.

It is interesting to note that the notation in the preceding proof becomes more compact if one uses the criterion from Corollary 2.3, for then $S = R[X_1, ..., X_n]$ and $g: R_{\mathrm{red}}[X_1, ..., X_n] \to T_{\mathrm{red}}[X_1, ..., X_n]$ is just $(f_{\mathrm{red}})_n$.

We next introduce a notion which is general enough to encompass both the integral contexts of [1, 17] and such quintessentially nonintegral GD contexts as localizations. To wit, we say that a ring-homomorphism $f: R \to T$ is *quasi-lying-over* (in short, QLO) if, for each $p \in \mathrm{Spec}(R)$ such that $f(p)T \neq T$ and $\ker(f) \subset p$, there exists at least one $q \in \mathrm{Spec}(T)$ such that $f^{-1}(q) = p$. Evidently, as the terminology suggests, lying-over implies QLO. Moreover, $f: R \to T$ is QLO if and only if the inclusion map $f(R) \to T$ is QLO. Thus, integral maps and the canonical inclusions $A \to A[X]$ all satisfy QLO (cf. [12, Theorem 44]). Moreover, it is essentially well known (and easy to see) that GD implies QLO (cf. [12, Exercise 38, p. 45]). In particular, flat maps (and, a fortiori, localizations) all satisfy QLO. We next record a useful example.

Remark 3.4. (a) (Quasi-)lying-over does not imply GD. One way to see this is to appeal to the famous examples of Cohen–Seidenberg [3, Section 3]. For another way, let R be a quasilocal one-dimensional integral domain with maximal ideal m, quotient field K, and residue field $k = R/m$. One verifies easily that the product map $f: R \to K \times k$ is QLO (indeed, strongly unibranched) but does not satisfy GD.

(b) QLO does not imply LO. To see this, it is enough to consider the inclusion map $R \to K$ of an integral domain R which is properly contained in its quotient field K.

(c) It seems worthwhile to observe that not every ring-homomorphism satisfies QLO. For example, consider the inclusion map of any local (Noetherian) integral domain of (Krull) dimension $n \geqslant 2$ into a dominating DVR overring.

We next put the QLO concept to work. Let $f: R \to T$ be a ring-homomorphism which is QLO. Let $p \in \operatorname{Spec}(R)$ be such that $f(p)T \neq T$, and consider the canonical map $f_p: R_p \to T_p$. If $\ker(f) \subset p$, it is easy to see that some prime of T_p lies over pR_p, and so

$$F_p = \bigcap \{q \in \operatorname{Spec}(T_p): f_p^{-1}(q) = pR_p\}$$

is a proper ideal of T_p. If $\ker(f) \not\subset p$, $F_p = T_p$. Set

$$l = \begin{cases} \operatorname{char}(k_R(p)), & \text{if} \quad \operatorname{char}(k_R(p)) \neq 0 \\ 1, & \text{if} \quad \operatorname{char}(k_R(p)) = 0. \end{cases}$$

Inspired by Andreotti–Bombieri [1, p. 433] and Traverso [17, p. 585], we introduce the *sets*

$$R_p^+ = f_p(R_p) + F_p$$

and

$$R_p^* = \{x \in T_p: \text{there exists } n \geqslant 1 \text{ such that } x^{l^n} \in R_p^+\}.$$

(Thus, if $\operatorname{char}(k_R(p)) = 0$, then $R_p^+ = R_p^*$.) In general, we have

$$f_p(R_p) \subset R_p^+ \subset R_p^* \subset T_p.$$

Proposition 3.7(a) will establish that R_p^+ and R_p^* are rings: it will then follow that R_p^* is integral over R_p^+.

It is now possible to make the key definition in this paper. If $f: R \to T$ is a ring-homomorphism (satisfying QLO), we say that f satisfies the UGD property in case f satisfies GD and, for each $p \in \operatorname{Spec}(R)$ such that $f(p)T \neq T$, one has $R_p^* = T_p$. (Of course, one need only consider such p which also satisfy $\ker(f) \subset p$.) We turn immediately to some examples.

Remark 3.5. (a) If p is a prime ideal of a ring R, then the canonical ring-homomorphism $R \to R_p$ satisfies UGD. To see this, it remains only to verify that $R_q^* = (R_p)_q$ for each prime q of R which is contained in p. This, in turn, is evident since $(R_p)_q$ may be identified with R_q and $R_q^+ = R_q + qR_q = R_q$, whence $R_q^* = R_q$ as well.

(b) If T is a flat overring of an integral domain R, then the inclusion map $R \to T$ satisfies UGD. Indeed, if $p \in \operatorname{Spec}(R)$ is such that $pT \neq T$, then flatness assures that $T \subset R_p$ (cf. [16, Theorem 1]), whence $T_p = R_p$ and R_p^+, R_p^* both coincide with T_p.

(c) It is easy to show that QLO does not imply UGD. Indeed, if R is a field k, then the inclusion map $k \to k[X]$ does not satisfy UGD since the prime $p = 0$ is such that $T_p = k[X]$ properly contains $k = k + 0 = R_p^+$ $(=R_p^*)$.

Matters simplify in the integral case, essentially for "going-up" reasons. Indeed, if a ring-homomorphism $f: R \to T$ is integral (that is, makes T integral over $f(R)$), then for each $p \in \operatorname{Spec}(R)$, which contains $\ker(f)$, F_p is just the Jacobson radical of T_p, since the prime ideals of T_p which lie over pR_p are precisely the maximal ideals of T_p (cf. [12, Theorem 47]).

Inspired by [1, Definizione 1, p. 437] and [17], we define the *weak* (resp., *semi-*) *normalization of* R (*inside* T) *with respect to* a given quasi-lying-over ring-homomorphism $f: R \to T$ to be $\{x \in T$: for each $p \in \operatorname{Spec}(R)$ such that $f(p)T \neq T$, R_p^* (resp., R_p^+) contains the canonical image of x in $T_p\}$. As a direct consequence of the definitions, we have

Remark 3.6. Let an injective ring-homomorphism $f: R \to T$ be integral. Then f satisfies UGD if and only if f satisfies GD and T is the weak normalization of R with respect to f.

PROPOSITION 3.7. *Let $f: R \to T$ be a ring-homomorphism which is QLO and let $p \in \operatorname{Spec}(R)$ be such that $f(p)T \neq T$ and $\ker(f) \subset p$. Then*:

(a) *R_p^+ and R_p^* are subrings of T_p.*

(b) *F_p is a maximal ideal of both R_p^+ and R_p^*.*

(c) *The canonical map $k_R(p) \to R_p^+/F_p$ is an isomorphism of fields. The canonical inclusion $k_R(p) \to R_p^*/F_p$ is a purely inseparable field extension.*

Proof. (a) Since F_p is an ideal of T_p, it is evident from the definition that R_p^+ is a ring. As for R_p^*, we may now assume that $l \geqslant 2$. To see that R_p^*

is closed under products, consider x, y in R_p^*. Then $x^{l^m}, y^{l^n} \in R_p^+$ for some $m, n \geqslant 1$ and, since R_p^+ is closed under products, $(xy)^{l^{m+n}} \in R_p^+$; that is, $xy \in R_p^*$. As for differences, let x, y, m, and n be as above, and observe that the canonical ring-homomorphism

$$k(p) = R_p/pR_p \to T_p/F_p$$

induced by f_p is an injection. Accordingly $\operatorname{char}(T_p/F_p) = l$, and so $lT_p \subset F_p$. However, by the binomial theorem,

$$(x-y)^{l^{m+n}} - x^{l^{m+n}} + y^{l^{m+n}} \in lT_p$$

so that $(x-y)^{l^{m+n}} \in x^{l^{m+n}} - y^{l^{m+n}} + F_p \subset R_p^+$, as desired.

(b) and (c). The canonical ring-homomorphism from the field R_p/pR_p to R_p^+/F_p has image $(f_p(R_p) + F_p)/F_p = R_p^+/F_p$. Thus $R_p^+/F_p \cong k(p)$, a field, and so F_p is a maximal ideal of R_p^+.

We shall show next that F_p is also a maximal ideal of R_p^*. By integrality of R_p^* over R_p^+ (cf. [12, Theorem 47]), it is enough to prove that F_p is prime in R_p^*. Without loss of generality, $l \geqslant 2$. Suppose $x, y \in R_p^*$ are such that $xy \in F_p$. Since $x^{l^m}, y^{l^n} \in R_p^+$ for suitable m and n, we compute as above that

$$(x^{l^m})^{l^n} \cdot (y^{l^n})^{l^m} = (xy)^{l^{m+n}} \in F_p.$$

As F_p is a prime (because maximal) ideal of R_p^+, we may assume, without loss of generality, that $(x^{l^m})^{l^n} = x^{l^{m+n}} \in F_p$. However, by its very construction, F_p is a radical ideal of T_p, and so $x \in F_p$, as desired.

The definition of R_p^* now assures that R_p^*/F_p is a purely inseparable field extension of R_p^+/F_p, that is, of $k(p)$. This completes the proof.

If a ring-homomorphism $f: R \to T$ is integral and if $p \in \operatorname{Spec}(R)$ contains $\ker(f)$, then R_p^+ and R_p^* are each quasilocal (with maximal ideal F_p). Indeed, we noted following Remark 3.5 that F_p is the Jacobson radical of T_p in this case; by using integrality of T_p over R_p^* in the same way, we see that the Jacobson radical of R_p^* is $F_p \cap R_p^* = F_p$, which we know, by Proposition 3.7(b), is a maximal ideal of R_p^* and, hence, must be the only maximal ideal of R_p^*. The argument for R_p^+ is similar.

Proposition 3.9 will give a modest generalization of the preceding observation. First, we shall show that some weak version of "integrality" is necessary for the conclusion.

EXAMPLE 3.8. There exist an injective ring-homomorphism $f: R \to T$ which is QLO and a prime p of R for which $f(p)T \neq T$ and R_p^* is not quasilocal.

Indeed, one need only consider the second example mentioned in

Remark 3.4(a), taking care to arrange also that $\text{char}(k) = 0$. Then, with $p = m$ (and $T = K \times k$), we find that

$$R_p^* = R_p^+ = f(R) + (K \times \{0\}) = K \times k \ (= T = T_p)$$

which, as desired, is not quasilocal. (Note also that $R_0^* \cong K \times \{0\} \cong T_0$ canonically, but f is not UGD since f does not satisfy GD.)

The above f does not satisfy going-up, and thus gives additional motivation for

PROPOSITION 3.9. *Let the ring-homomorphism $f: R \to T$ satisfy the QLO, going-up, and incomparability properties. Then for each $p \in \text{Spec}(R)$ such that $f(p)T \neq T$ and $\ker(f) \subset p$, the rings R_p^* and R_p^+ are each quasilocal.*

Proof. The rings in question are well-defined. As R_p^* is integral over R_p^+, it is enough to consider R_p^+. By hypotheses, the maximal ideals of T_p are precisely the primes of T_p which lie over pR_p, and so F_p is just the Jacobson radical of T_p. Consider the commutative diagram of ring-homomorphisms

$$\begin{array}{ccc} R_p^+ & \longrightarrow & k_R(p) \\ \downarrow & & \downarrow \\ T_p & \longrightarrow & T_p/F_p \end{array}$$

in which the top surjection is obtained from Proposition 3.7(c). The diagram is evidently Cartesian (that is, a pullback), and so [8, Theorem 1.4(b),(c)] assures that if m is a maximal ideal of R_p^+ other than F_p, then there exists a unique $M \in \text{Spec}(T_p)$ such that $M \cap R_p^+ = m$. Since the inclusion map $R_p^+ \to T_p$ inherits incomparability from (f_p and) f, it follows that M is maximal in T_p. By the above observation about the Jacobson radical, $F_p \subset M$, and so $M \cap R_p^+ = F_p$, contradicting the supposition about m. Thus F_p is the only maximal ideal of R_p^+, completing the proof.

We next introduce a useful technical concept. A ring-homomorphism $f: R \to T$ will be said to be *quasi-going-up* (in short, QGU) if, for each pair of primes $p_1 \subset p_2$ of R such that $f(p_2)T \neq T$ and each $q_1 \in \text{Spec}(T)$ such that $f^{-1}(q_1) = p_1$, there exists $q_2 \in \text{Spec}(T)$ such that $q_1 \subset q_2$ and $f^{-1}(q_2) = p_2$. It is easy to see that $f: R \to T$ is QGU if and only if the inclusion map $f(R) \to T$ is QGU. The next remark collects some relevant material.

Remark 3.10. (a) It is easy to see that a ring-homomorphism $f: R \to T$ is QGU if and only if $f_p: R_p \to T_p$ satisfies going-up for each $p \in \text{Spec}(R)$ such that $f(p)T \neq T$. (Only such p which also contain $\ker(f)$ need be considered.)

(b) One readily infers from (a) that QGU implies QLO.

(c) Any localization $R \to R_S$ is QGU, indeed universally QGU.

(d) Integral homomorphisms satisfy QGU. Indeed, going-up implies QGU. However, the converse fails: consider $R \to R_p$ for any nonmaximal $p \in \operatorname{Spec}(R)$.

(e) By using the information in Remark 3.5(b), we see that QGU is satisfied by the inclusion map $R \to T$ of each flat overring T of an integral domain R. A generalization of this fact will be given in Corollary 3.12(c).

We turn now to some of the consequences of UGD.

PROPOSITION 3.11. *If a ring-homomorphism $f: R \to T$ is UGD, then f_p: $R_p \to T_p$ is unibranched for each $p \in \operatorname{Spec}(R)$ such that $f(p)T \neq T$ and* $\ker(f) \subset p$.

Proof. Consider primes $p_1 \subset p$ of R such that $f(p)T \neq T$. If distinct primes of T_p each lie over $p_1 R_p$, then (the induced) distinct primes of T_{p_1} each lie over $p_1 R_{p_1}$. Accordingly, suppressing the subscript "1," it is enough to show that F_p is the only prime of T_p lying over pR_p. (Of course, F_p is a maximal ideal of T_p by Proposition 3.7(b) since the UGD hypothesis gives $T_p = R_p^*$.) By its very definition, F_p lies over pR_p and is contained in any $q \in \operatorname{Spec}(T_p)$ which lies over pR_p. Then, by maximality of F_p, any such q must be F_p, completing the proof.

Part (a) of the next result is reminiscent of Proposition 3.9, and is to be contrasted with Example 3.8.

COROLLARY 3.12. *Let $f: R \to T$ be a ring-homomorphism satisfying UGD. Then*:

(a) *For each $p \in \operatorname{Spec}(R)$ such that $f(p)T \neq T$ and* $\ker(f) \subset p$, *the rings R_p^* $(=T_p)$ and R_p^+ are each quasilocal.*

(b) *f is universally (strongly) mated.*

(c) *f is QGU.*

Proof. (a) By integrality, we need only consider R_p^*, that is, T_p. It is enough to show that T_p cannot have a maximal ideal M other than F_p. Setting $m = f_p^{-1}(M)$, we infer via Proposition 3.11 that $m \neq pR_p$. Then, since f_p inherits GD from f, there exists $Q \in \operatorname{Spec}(T_p)$ such that $Q \subset F_p$ and $f_p^{-1}(Q) = m$. Since f_p is unibranched, $Q = M$, although Q is evidently not maximal in T_p. Hence no M of the above kind exists.

(b) If $q \in \operatorname{Spec}(T)$ and $f^{-1}(q) = p$, then $T_p \cong T_q$ by [9, Cor. 5.2]. Consider the inclusions

$$k_R(p) \to k_T(q) \to T_p/qT_p = R_p^*/qT_p.$$

Since we showed in the proof of Proposition 3.11 that F_p is the only prime of T_p which lies over pR_p, it follows that $qT_p = F_p$, and so Proposition 3.7(c) now implies that $k_T(q)$ is purely inseparable over $k_R(p)$. By Theorem 2.5, it remains only to prove that f is mated. However, this is evident since f satisfies GD (by virtue of UGD) and is an i-homomorphism (as a consequence of Proposition 3.11).

(c) If $p \in \operatorname{Spec}(R)$ is such that $\ker(f) \subset p$ and $f(p)T \neq T$, then f_p is unibranched (by Proposition 3.11) and inherits GD from f. Thus f_p satisfies going-up, and an application of the parenthetical part of Remark 3.10(a) completes the proof.

COROLLARY 3.13. *Let $(I, \leqslant)$ be a directed set, and let (A_i, f_{ij}) and (B_i, g_{ij}) each be directed systems of rings indexed by I. For each $i \in I$, let h_i: $A_i \to B_i$ be a ring-homomorphism satisfying UGD such that, whenever $i \leqslant j$ in I, then $g_{ij}h_i = h_j f_{ij}: A_i \to B_j$. Set $A = \varinjlim A_i$, $B = \varinjlim B_i$, and $h = \varinjlim h_i$. Then $h: A \to B$ is UGD.*

Proof. We shall first verify the criterion in Remark 3.10(a), to show that f is QGU. Consider $p \in \operatorname{Spec}(A)$ such that $h(p)B \neq B$, as well as primes $p_1 \subset p_2 \subset p$ of A and $q_1 \in \operatorname{Spec}(B)$ such that $h_p^{-1}(q_1 B_p) = p_1 A_p$. Our task is to produce $q_2 \in \operatorname{Spec}(B)$ such that $q_1 \subset q_2$ and $h_p^{-1}(q_2 B_p) = p_2 A_p$.

Use the structure maps $f_i: A_i \to A$ and $g_i: B_i \to B$ to yield $p_i = f_i^{-1}(p)$, $p_{1i} = f_i^{-1}(p_1)$, $p_{2i} = f_i^{-1}(p_2)$, and $q_{1i} = g_i^{-1}(q_1)$. Evidently, q_{1i} lies over p_{1i} for each i (since $g_i h_i = h f_i$). Moreover, Corollary 3.12(c) shows that h_i satisfies QGU; and $h_i(p_{2i}) B_i \neq B_i$. Thus we obtain $q_{2i} \in \operatorname{Spec}(B_i)$ such that $q_{1i} \subset q_{2i}$ and $h_i^{-1}(q_{2i}) = p_{2i}$. However, $\{q_{2i}\}$ forms a directed system, indeed $g_{jk}^{-1}(q_{2k}) = q_{2j}$ whenever $j \leqslant k$ in I, since $(h_j)_{p_{2j}}$ is unibranched (and, hence, an i-homomorphism) by virtue of Proposition 3.11. It remains to verify that $q = \varinjlim q_{2i}$ is a satisfactory q_2, and this follows easily from the fact that $A = \bigcup f_i(A_i)$. Thus h is QGU and, a fortiori, QLO.

Moreover, h satisfies GD, since direct limits preserve GD [5, Theorem 2.1]. Hence, it remains only to show that $A_p^* = B_p$ for each $p \in \operatorname{Spec}(A)$ such that $h(p)B \neq B$ and $\ker(h) \subset p$. Consider $p_i = f_i^{-1}(p)$ for each i. As usual, there are canonical isomorphisms $A_p \cong \varinjlim (A_i)_{p_i}$ and $B_p \cong \varinjlim (B_i)_{p_i}$ (cf. [10, Propositions 6.1.5 and 6.1.6(ii), p. 129–130]). Let F_{p_i} (resp., F_p) be the ideal arising in the construction of $(A_i)_{p_i}^+$ (resp., A_p^+). Then there is a canonical map $m: \varinjlim F_{p_i} \to F_p$ which is compatible with the second canonical isomorphism mentioned above; thus m is injective. It suffices to prove that m is surjective, for then $A_p^+ \cong \varinjlim (A_i)_{p_i}^+$, and it readily follows that $A_p^* \cong \varinjlim (A_i)_{p_i}^* \cong B_p$.

As for the surjectivity of m, consider $v \in F_p$. Viewing v as the canonical image of $v_i \in (B_i)_{p_i}$ for some i, we need only show that $v_i \in F_{p_i}$. As $\ker(h) \subset p_i$ and $h_i(p_i) B_i \neq B_i$, Proposition 3.11 gives a unique $q_i \in \operatorname{Spec}(B_i)$

such that $h_i^{-1}(q_i) = p_i$; thus, $F_{p_i} = q_i(B_i)_{p_i}$. Now, since we have seen that h is QLO, there exists $q \in \operatorname{Spec}(B)$ such that $h^{-1}(q) = p$; of course, $v \in qB_p$. Necessarily, $g_i^{-1}(q) = q_i$, whence the inverse image of qB_p under the map $(B_i)_{p_i} \to B_p$ is $q_i(B_i)_{p_i} = F_{p_i}$. As v_i is in this inverse image, the proof is complete.

We come now to a fundamental step in this section's program.

PROPOSITION 3.14. *Let $f: R \to T$ be a ring-homomorphism. If $f_1: R[X] \to T[X]$ is mated and f satisfies GD, then f is UGD.*

Proof. By Theorem 2.1, f is strongly mated. Now, let $p \in \operatorname{Spec}(R)$, such that $f(p)T \neq T$. As f is an i-homomorphism (because mated) and satisfies GD, it is easy to see that $f_p: R_p \to T_p$ is unibranched.

We claim that T_p is quasilocal, with maximal ideal F_p. Indeed, consider any maximal ideal qT_p of T_p (where $q \in \operatorname{Spec}(T)$ is maximal with respect to the property that $p_1 = f^{-1}(q) \subset p$). To check that each element $u \in qT_p$ lies in F_p, we shall show that $u \in QT_p$ for each $Q \in \operatorname{Spec}(T)$ such that $f_p^{-1}(QT_p) = pR_p$. To this end, use the fact that f satisfies GD in order to find $Q_1 \in \operatorname{Spec}(T)$ such that $Q_1 \subset Q$ and $f^{-1}(Q_1) = p_1$. Since f_p is unibranched, it follows that $Q_1T_p = qT_p$, whence $u \in Q_1T_p \subset QT_p$. This proves the claim.

Now let q denote the prime of T such that $qT_p = F_p$. A useful fact about localizations [9, Corollary 5.2] guarantees that $T_p \cong T_q$. It therefore remains only to prove that $R_p^* = T_q$.

It is enough to show that each element $x \in T_q$ lies in R_p^*. Let $\bar{x}$ denote $x + qT_q \in k_T(q)$. By the proof's initial observation, $k_T(q)$ is purely inseparable over $k_R(p)$; thus, $\bar{x}^{l^m} \in k_R(p)$ for some $m \geqslant 1$. Hence there exists $y \in R_p$ such that $x^{l^m} - f_p(y) \in qT_q$. However, $qT_q = F_p$, by equating the maximal ideals of T_q and T_p. Thus $x^{l^m} \in R_p^+$, completing the proof.

The next result is in the spirit of Theorem 2.1 and [13].

THEOREM 3.15. *Let $f: R \to T$ be a ring-homomorphism. Then the following five conditions are equivalent:*

(i) *f is UGD and, for each $m \geqslant 1$, $f_m: R[X_1,..., X_m] \to T[X_1,..., X_m]$ satisfies QGU.*

(ii) *$f_1: R[X] \to T[X]$ is UGD and, for each $m \geqslant 1$, $f_m: [X_1,..., X_m] \to T[X_1,..., X_m]$ satisfies QGU.*

(iii) *There exists $n \geqslant 1$ such that $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ is UGD, and, for each $m \geqslant 1$, $f_m: R[X_1,..., X_m] \to T[X_1,..., X_m]$ satisfies QGU.*

(iv) *For each $n \geqslant 0$, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ is UGD.*

(v) *f is universally UGD.*

Proof. It is trivial that (v) implies (iv). To see the converse, note first by direct calculation that (GD and) UGD each satisfy condition (b) in the statement of Proposition 2.2. (The key point is that if $q \in \operatorname{Spec}(T)$ and $p = f^{-1}(q) \supset J$, then $F_{p/J} = F_p/JT_q$. We omit the details.) In view of Corollary 3.13 and (iv), one may now apply the proof of Proposition 2.2, to conclude that f is universally UGD.

As for the asserted equivalences, Corollary 3.12(c) yields (iv) $\Rightarrow$ (iii); Corollary 3.12(b) and Proposition 3.14 combine to give (ii) $\Rightarrow$ (*i*); and (*iii*) $\Rightarrow$ (ii). The last of these follows from the general fact that a ring-homomorphism $g: A \to B$ inherits UGD from its induced map g_1: $A[X] \to B[X]$, which is itself a consequence of the above calculation that UGD satisfies condition (b) of Proposition 2.2. It therefore remains only to show that (i) $\Rightarrow$ (iv).

By induction on n, we need only prove (i) $\Rightarrow$ (ii). Let f be as in (i). By Corollary 3.12(b) and Theorem 2.5, $f_1: R[X] \to T[X]$ is strongly mated. Now consider $P \in \operatorname{Spec}(R[X])$ such that $f_1(P)\, T[X] \neq T[X]$. Since f_1 is mated, $(f_1)_P: R[X]_P \to T[X]_P$ is evidently unibranched. Moreover, f_1 satisfies GD, by virtue of being both mated and QGU. One may now repeat the argument in the proof of Proposition 3.14, *mutatis mutandis*, to show first that $T[X]_P$ has unique maximal ideal F_P and ultimately that $R[X]_P^* = T[X]_P$. This establishes (ii) and completes the proof.

COROLLARY 3.16. *If a ring-homomorphism $f: R \to T$ is UGD and f_n: $R[X_1,..., X_n] \to T[X_1,..., X_n]$ satisfies QGU for each $n \geqslant 1$, then f is universally strongly GD.*

Proof. Combining Theorem 3.15 with Corollary 2.3, Corollary 3.12(b) and Theorem 2.5, we see that f is universally going-down and universally strongly mated. This completes the proof.

We can now give our principal application. We reiterate that it accomodates possibly nonintegral extensions. (Cf. Remark 3.5(b); contrast [13, Theorem C] and [14, Theorem 4].) It is in the spirit of Theorem 2.5 (and [13]).

THEOREM 3.17. *Let R be an integral domain, T an overring of R and $f: R \to T$ the inclusion map. Then the following are equivalent*:

(i) *f is universally going-down.*

(ii) *f is universally strongly going-down.*

(iii) *For each $n \geqslant 0$, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ is UGD.*

(iv) *For each $n \geqslant 0$, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ satisfies going-down.*

(v) *f is UGD and, for each $n \geqslant 1$, $f_n: R[X_1,..., X_n] \to T[X_1,..., X_n]$ is QGU.*

(vi) *f is UGD and universally QGU.*

(vii) *f is universally UGD.*

(viii) *f is universally strongly UGD.*

Proof. In view of Corollary 2.3, Theorem 3.15 and Corollary 3.16, it is enough to prove that (iv) $\Rightarrow$ (iii). Accordingly, assume (iv). One sees easily that it suffices to show f is UGD. Therefore, by Proposition 3.14, it is enough to prove that $f_1: R[X] \to T[X]$ is mated. However, this is a direct consequence of [4, Theorem 2.1] since f_1 satisfies GD and T is an overring of R. The proof is complete.

Remark 3.18. (a) To underscore the importance of the "overring" hypothesis in Theorem 3.17 (and shed further light on its role in [13], [14] and [4]), consider the following example. If k is a field and $f: k \to k[X]$ is the inclusion map, then f is universally going-down (because flat) but is not UGD (by Remark 3.5(c)).

(b) The example in Remark 3.5(a) shows, by virtue of Remark 3.10(c) and Theorem 3.17, that one cannot remove entirely the "integrality" hypothesis in [13, Theorem C, (vi) $\Rightarrow$ (ii)] relating universally going-down and universally unibranched.

It seems interesting to combine our results with the relevant literature for the integral case. (An instructive example in this regard is the inclusion map $f: k[[X^2, X^3]] \to k[[X]]$, where k is a field. Since f is integral and UGD, Theorem 3.17 implies that f is universally going-down. To motivate (ii) below, notice that the domain of f is not seminormal.) This results in

COROLLARY 3.19. *Let R be an integral domain, T an integral overring of R, and $f: R \to T$ the inclusion map. Then the following are equivalent:*

(i) *f is universally going-down.*

(ii) *T is the weak normalization of R inside T (with respect to f).*

(iii) *$f_1: R[X] \to T[X]$ satisfies going-down.*

(iv) *$f_1: R[X] \to T[X]$ is unibranched.*

(v) *f is universally unibranched.*

Proof. Since integrality is a universal property, Remark 3.10(d) and Theorem 3.17 imply that (i) is equivalent to f being UGD. Accordingly, in order to derive (i) $\Leftrightarrow$ (ii) from Remark 3.6, we can assume that T is the weak normalization of R with respect to f and need only show that f satisfies GD. However [1, Theorem 1] implies that f induces a homeomorphism $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$ (and hence an isomorphism of the corresponding partially ordered sets: cf. [11, p. 53]), whence f satisfies GD.

By Corollary 2.3, Theorem 2.1, [13, Theorem C] and [14, Theorem 4], the remaining equivalences follow, to complete the proof.

In closing, we summarize the impact of Theorem 3.17 for integral extensions, which form the most important family of universally QGU homomorphisms. In view of Example 3.1 and Remark 3.18(a), Corollary 3.20 seems a pleasant GD-analogue of Theorem 2.5.

COROLLARY 3.20. *Let R be an integral domain, T an integral overring of R, and $f: R \to T$ the inclusion map. Then the following conditions are equivalent*:

(i) *f is universally going-down.*

(ii) *f is universally strongly going-down.*

(iii) *f is UGD.*

(iv) *f is universally UGD.*

(v) *f is universally strongly UGD.*

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