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Kronecker Function Rings and Abstract Riemann Surfaces

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Let R be an integral domain, $X(R)$ the abstract Riemann surface of R, and $(R')^b$ the Kronecker function ring of the integral closure R' of R. It is proved that there exists a homeomorphism, natural in R, between $X(R)$ and $\mathrm{Spec}((R')^b)$. Ideal-theoretic and topological results are given for the extension $j\colon R \subsetneq (R')^b$, notably that R is a Prüfer domain if and only if $R = R'$ and j is universally going-down. It is also proved that each spectral space X is a closed spectral image of a treed spectral space Y; if X is irreducible, Y can be taken as an abstract Riemann surface.

1. INTRODUCTION

Let R be an integral domain with quotient field K. As in [3], the abstract Riemann surface of R is a topological space $X(R)$ whose underlying set is the collection of all valuation overrings of R. An open basis in $X(R)$ is given by the sets $E_R(x_1,..., x_n) = \{V \in X(R)\colon\ x_i \in V$ for each $i = 1,..., n\}$ as $\{x_1,..., x_n\}$ ranges over the finite subsets of K; the space $X(R)$ is called S^* in [16, p. 113]. It was shown in [3, Theorem 4.1] that $X(R)$ is a spectral space, in the sense of [11]; that is, $X(R)$ is homeomorphic to

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Spec(A), with the Zariski topology, for some (commutative) ring A. The proof of this fact in [3] did not construct an explicit A, given R. This is remedied in Theorem 2 below: $X(R)$ is homeomorphic to Spec($(R')^b$), where $(R')^b$ is the Kronecker function ring of the integral closure R' of R. (Background on Kronecker function rings appears in [13; and 10, Sects. 32–34], some of which is summarized in Sect. 2.) Moreover, in a sense made precise in Corollary 5, there is such a homeomorphism which is natural in R. Along the way, we see in Proposition 4 that "taking the Kronecker function ring" is functorial.

Theorem 2 implies that, up to order-isomorphism, abstract Riemann surfaces are just the spectral trees with unique minimal point. It therefore seems reasonable to study the treed spectral spaces. In this regard, Theorem 7 establishes that each spectral space is a closed spectral image of a treed spectral space. Proposition 9 and Examples 8 and 10 treat the converse and related questions.

The extension $j: R \subsetneq (R')^b$ merits consideration since, as Remark 3(b) shows, Spec(j): Spec($(R')^b$) $\to$ Spec(R) can be identified with the canonical surjection $f_R: X(R) \to \mathrm{Spec}(R)$. This point of view leads to a new ideal-theoretic proof, in Remark 3(c), that f_R is closed map. (This fundamental result, which implies that Spec(R) is a quotient space of $X(R)$, was first proved in [3] by appeal to topological arguments from [16].) In Theorem 11, several related results are collected, notably that R is a Prüfer domain if and only if R is integrally closed and j is a universally going-down homomorphism.

Throughout, R will denote an integral domain with integral closure R' and quotient field K. If V is a valuation overring of R, then v will denote a valuation on K having valuation ring V; v^* will denote the inf-extension of v to a valuation on the rational function field $K(X)$; and V^* will denote the trivial extension of V to $K(X)$, that is, the valuation ring of v^*. Unreferenced material is standard, typically in [10].

2. Results

Suppose, for the moment, that R is integrally closed. Let $\{V_\alpha\}$ be the set of all valuation overrings of R. The *b-operation* (also known as *completion*) converts a fractional ideal I of R to $I_b = \bigcap IV_\alpha$. By definition, the *Kronecker function ring of R* is

$$R^b = \{0\} \cup \{f/g: f, g \in R[X] \setminus \{0\} \text{ and } c(f)_b \subset c(g)_b\}.$$

(As usual, if $h \in K[X]$, then $c(h)$ denotes the fractional ideal of R generated by the coefficients of h.)

Rather than the above definition, we shall need the following facts, collected from [10, Sect. 32]. R^b is a Bézout (and, hence, a Prüfer) domain with quotient field $K(X)$; $R^b \cap K = R$; and the function $g_R: X(R^b) \to X(R)$, $W \mapsto W \cap K$, is a bijection, with inverse given by $V \mapsto V^*$.

We begin by showing that g_R remains an isomorphism in a richer categorical setting.

LEMMA 1. *If R is an integrally closed integral domain, then the canonical bijection $g_R: X(R^b) \to X(R)$ is a homeomorphism.*

Proof. The typical quasi-compact open subset of $X(R)$ is the union of finitely many sets of the form $E_R(x_1,..., x_n)$. Since

$$g_R^{-1}(E_R(x_1,..., x_n)) = E_{R^b}(x_1,..., x_n),$$

it follows that g_R is continuous (and spectral). As g_R is a bijection, it now suffices to show that g_R sends the typical subbasic open set $E_{R^b}(\alpha)$ to an open set Y. The case $\alpha = 0$ is evident. If $\alpha \neq 0$, write

$$\alpha = (a_0 + a_1 X + \cdots + a_n X^n)/(b_0 + b_1 X + \cdots + b_m X^m)$$

in lowest terms, with all $a_i, b_j \in K$. Ignoring vanishing coefficients, we find that

$$Y = \{V \in X(R): \alpha \in V^*\} = \{V \in X(R): \inf v(a_i) \geqslant \inf v(b_j)\}$$

$$= \bigcup_{i,j} \{V \in X(R): v(a_i) \leqslant v(a_\lambda) \text{ for all } \lambda,\ v(b_j) \leqslant v(b_\mu) \text{ for all } \mu,\ v(a_i) \geqslant v(b_j)\}$$

$$= \bigcup_{i,j} E_R(\{a_\lambda/a_i: 1 \leqslant \lambda \leqslant n\} \cup \{b_\mu/b_j: 1 \leqslant j \leqslant m\} \cup \{a_i/b_j\})$$

is indeed open, completing the proof.

We next recover the fact [3, Theorem 4.1] that each $X(R)$ is a spectral space. More explicitly, we have

THEOREM 2. *Let R be an integral domain. Then $X(R)$ is homeomorphic to* $\mathrm{Spec}((R')^b)$.

Proof. The identity map $X(R') \to X(R)$ is a homeomorphism. Thus, without loss of generality, R is integrally closed. By Lemma 1, it is enough to prove, for $S = (R')^b$, that $X(S)$ is homeomorphic to $\mathrm{Spec}(S)$. However this is evident since S is Prüfer domain (cf. [3, Proposition 2.2]).

Remark 3. (a) Each $X(R)$ is a T_0-space, and in the usual way [11, p. 53] thus acquires the structure of a partially ordered set: $V_1 \leqslant V_2$ if and only if V_2 is in the closure of V_1, that is, if and only if $V_2 \subset V_1$. It follows, in case $R = R'$, that the bijection g_R studied in Lemma 1 is an order-isomorphism; in other words, valuation overrings V_i of R $(=R')$ satisfy $V_2 \subset V_1$ if and only if $V_2^* \subset V_1^*$. This order-isomorphism may also be established without appeal to Lemma 1: the reader may fashion a direct proof using [10, Theorem 32.10]. In this way, one finds a new proof of Theorem 2 which uses [3, Theorem 4.1] but not the full force of Lemma 1. The point is that, for $R = R'$, g_R is then a homeomorphism by virtue of [11, Proposition 15], since g_R is an order-isomorphism and a spectral map of spectral spaces.

(b) For each integral domain R, it was shown in [3, Corollary 2.6] that $\mathrm{Spec}(R)$ is a quotient space of $X(R)$. This followed since the canonical surjection $f_R\colon X(R) \to \mathrm{Spec}(R)$, $V \mapsto$ center of V on R, is a closed map [3, Theorem 2.5]. We next give a ring-theoretic translation of this topological fact, namely that the inclusion map $j\colon R \subsetneq (R')^b$ satisfies the going-up property (GU, in the notation of [12, p. 28]).

An equivalent assertion is that $\mathrm{Spec}(j)$ is a closed map (cf. [8, Proposition 2.7(b)]). However, this assertion holds since [3, Corollary 4.5(b)] provides a commutative diagram

$$\begin{tikzcd}
X((R')^b) \arrow[r, "f_{(R')^b}"] \arrow[d, "g_{R'}"'] & \mathrm{Spec}((R')^b) \arrow[d] \arrow[dd, bend left=60, "\mathrm{Spec}(j)"] \\
X(R') \arrow[r, "f_{R'}"] \arrow[d, equal] & \mathrm{Spec}(R') \arrow[d] \\
X(R) \arrow[r, "f_R"] & \mathrm{Spec}(R)
\end{tikzcd}$$

which now, by Theorem 2, permits us to identify $\mathrm{Spec}(j)$ with f_R.

By [12, Theorem 42], it therefore follows that j also satisfies the lying-over property (LO). Theorem 11 will present a thorough study of j with respect to related properties such as incomparability (INC), going-down (GD), etc.

(c) We next give a direct proof that $j\colon R \subsetneq (R')^b$ satisfies GU. By the remarks in (b), this yields a new proof of [3, Theorem 2.5], namely that $f_R\colon X(R) \to \mathrm{Spec}(R)$ is a closed map.

$R \subsetneq R'$, being an integral extension, satisfies GU. By factoring j through R', we may therefore take R integrally closed. Consider primes $P_1 \subset P_2$ of R and a prime Q_1 of R^b such that $Q_1 \cap R = P_1$. By [10, Corollary 19.7(2)], $W_1 = (R^b)_{Q_1}$ contains a valuation ring (W, M) of $K(X)$ such that $R \subset W$ and $M \cap R = P_2$. (Note that the maximal ideal of W_1 lies over P_1.) Set

$W_2 = (W \cap K)^*$. By the order-isomorphism $X(R^b) \to X(R)$ noted in (a), we have

$$R^b \subset W_2 \subset (W_1 \cap K)^* = W_1.$$

Since R^b is a Prüfer domain, $W_2 = (R^b)_{Q_2}$ for some prime Q_2 of R^b. Evidently, $Q_1 \subset Q_2$. Moreover, $Q_2 \cap R = P_2$ because

$$Q_2 \cap R = (\text{center of } W_2 \text{ on } R) = \text{center of } W \text{ on } R.$$

This completes the proof.

We next fix some useful notation. Let $\mathbf{D}$ be the category whose objects form the class of all integral domains and whose morphisms are the inclusion maps. Let $\mathbf{Z}$ be the category of all abstract Riemann surfaces of integral domains, viewed as a full subcategory of the category $\mathbf{S}$ of spectral spaces and spectral maps.

It was noted in [3, Corollary 4.5(a)] that the object assignment $R \mapsto X(R)$ extends to a contravariant functor $X: \mathbf{D} \to \mathbf{Z}$. Moreover, if $I: \mathbf{Z} \to \mathbf{S}$ denotes the inclusion functor, then $\{f_R: R \in \mathrm{Ob}(\mathbf{D})\}$ gives a natural transformation $IX \dot{\to}$ Spec (of contravariant functors $\mathbf{D} \to \mathbf{S}$). These facts explain the genesis of the diagram in Remark 3(b).

Our next categorical goal is more substantial, namely to describe X up to natural equivalence. Let $\mathbf{C}$ be the full subcategory of $\mathbf{D}$ whose objects are the integrally closed integral domains. An easy first step asserts that integral closure ("normalization") gives a functor $N: \mathbf{D} \to \mathbf{C}$; that is, given integral domains $R \subset S$, one then has $R' \subset S'$. The key step is given next. It shows that "taking the Kronecker function ring" is functorial and thus introduces the functor $(-)^b$: $\mathbf{C} \to \mathbf{D}$.

PROPOSITION 4. *If $R \subset S$ are integrally closed integral domains, then $R^b \subset S^b$.*

Proof. Let K (resp. L) be the quotient field of R (resp. S); view $K \subset L$. Let $\{V_\alpha\}$ be the set of valuation overrings of R; for each index α, let v_α be a valuation on K with valuation ring V_α. As above, let V_α^* be the trivial extension of V_α to $K(X)$, obtained from v_α^*, the inf-extension of v_α to a valuation on $K(X)$. Since R is integrally closed, [10, Theorem 32.11] yields that $R^b = \bigcap V_\alpha^*$.

For each index α, let $\{W_{\alpha,\beta}\}$ be the set of valuation rings of L such that $W_{\alpha,\beta} \cap K = V_\alpha$; β runs over an index set depending on α. Let $W_{\alpha,\beta}^*$, $w_{\alpha,\beta}$ and $w_{\alpha,\beta}^*$ have the obvious meanings. Then $S^b = \bigcap_{\alpha,\beta} W_{\alpha,\beta}^*$ since $\{W_{\alpha,\beta}\}$ is the set of valuation overrings of S. Hence, it will suffice to prove that $V_\alpha^* \subset W_{\alpha,\beta}^*$ whenever β is in the index set depending on α.

Let v be the restriction of $w_{\alpha,\beta}^*$ to K; then v is equivalent to v_α since each has valuation ring V_α. (The point is that $W^*_{\alpha,\beta} \cap K = V_\alpha$.) Let v^* be the inf-extension of v to a valuation on $K(X)$. Then, by construction of the inf-extension, v^* is just the restriction of $w^*_{\alpha,\beta}$ to $K(X)$. Thus $W^*_{\alpha,\beta} \cap K(X)$ is the valuation ring of v^*. However, it is easy to see that inf-extensions of equivalent valuations remain equivalent. Therefore, v^* and v^*_α have the same valuation ring, and so $V^*_\alpha = W^*_{\alpha,\beta} \cap K$, completing the proof.

We next give a functorial version of Theorem 2.

COROLLARY 5. *$X \cong (\mathrm{Spec}) \circ (-)^b \circ N$. In other words, the abstract Riemann surface functor X is naturally equivalent to the composite of the functors N: $\mathbf{D} \to \mathbf{C}$, $(-)^b$: $\mathbf{C} \to \mathbf{D}$, and Spec: $\mathbf{D} \to \mathbf{S}$.*

Proof. We shall show that a natural equivalence is given by the homeomorphisms $X(R) \to \mathrm{Spec}((R')^b)$ established in Theorem 2. Recall their construction: the identity $X(R) = X(R')$ is composed with the homeomorphisms $g_{R'}^{-1}$: $X(R') \to X((R')^b)$ and $f_{(R')^b}$: $X((R')^b) \to \mathrm{Spec}((R')^b)$. The effect is to send the typical $V \in X(R)$ to the center of V^* on $(R')^b$.

The asserted naturality means that, for integral domains $R \subset S$, the canonical diagram

$$\begin{array}{ccc} X(S) & \longrightarrow & \mathrm{Spec}((S')^b) \\ \downarrow & & \downarrow \\ X(R) & \longrightarrow & \mathrm{Spec}((R')^b) \end{array}$$

is commutative. There is no harm in assuming that $R = R'$ and $S = S'$.

Let K (resp. L) be the quotient field of R (resp. S). Chase $W \in X(S)$ through the above diagram. If V denotes $W \cap K$, the "vertical, followed by horizontal" path sends W to P, the center of V^* on R^b. The other path sends W to Q, the intersection of R^b with the center of W^* on S^b. To verify commutativity, that is to prove $P = Q$, it is enough to show $(R^b)_P = (R^b)_Q$. Note that $(R^b)_P = V^*$ and $(R^b)_Q = W^* \cap K(X)$. Applying [10, Theorem 32.10], we have $W^* \cap K(X) = (W^* \cap K(X) \cap K)^*$ which is, of course, just $(W^* \cap K)^*$. Since $W^* \cap K = V$, the proof is complete.

It is natural to say that a T_0-space X is a *treed space* in case the induced partially ordered set structure on the points of X is a tree. Treed spectral spaces abound: consider, for instance, Spec(R), R any Prüfer domain. It is also easy to show directly that each abstract Riemann surface is treed, using the fact that the overrings of any given valuation domain are linearly ordered by inclusion. Another proof of this is indicated next.

LEMMA 6. (a) *A nonempty topological space X is an irreducible spec-*

tral space if and only if X is homeomorphic to Spec(R), *R an integral domain.*

(b) *Each abstract Riemann surface is an irreducible treed spectral space.*

(c) *A partially ordered set X is a spectral set and a tree with a unique minimal point if and only if X is order-isomorphic to an abstract Riemann surface.*

Proof. (a) This assertion is well known: cf. [1, Corollary 1, p. 102].

(b) Combine Theorem 2 with (a) and the preceding remarks, bearing in mind that each Kronecker function ring is a Prüfer domain.

(c) If X is a spectral tree with a unique minimal element, a result of Lewis [14, Theorem 3.1] supplies a Prüfer domain R with Spec(R) order-isomorphic to X. As $\mathrm{Spec}(R) \cong X(R)$, the "only if" assertion follows. The converse follows directly from (b) and (a).

THEOREM 7. *Let X be a spectral space. Then there exists a treed spectral space Y and a closed spectral* (*continuous*) *surjection $Y \to X$. If, in addition, X is irreducible, then one can also arrange that Y is an abstract Riemann surface.*

Proof. If X is irreducible, Lemma 6(a) provides an integral domain R such that X is homeomorphic to Spec(R). By [3, Lemma 2.1, Theorem 2.5 and Theorem 4.1], the canonical map $f_R: X(R) \to \mathrm{Spec}(R)$ is continuous, surjective, closed and spectral. By Lemma 6(b), choosing $Y = X(R)$ dispatches this case.

For the general case, we proceed via a construction suggested by [15, Example 5.6]. Let ∞ denote a point outside X, and topologize $X^* = X \cup \{\infty\}$ as follows. The closed subsets of X^* are just X^* and the closed subsets of X. Now the space X^* is spectral, since it is straightforward to verify that X^* inherits the criteria of Hochster [11, Proposition 4] from X. Moreover, X^* is irreducible, since each $x \in X$ satisfies $\infty \leqslant x$ in the induced partial order. By the first case, there exist a treed spectral space Z and a closed spectral (continuous) surjection $h: Z \to X^*$. Note that $Y = h^{-1}(X)$ is spectral, being a closed subspace of a spectral space. Thus $h|_Y: Y \to X$ meets the asserted conditions.

We next show that the converse of Theorem 7 is false.

EXAMPLE 8. There exists a closed continuous surjective map $h: Y \to X$ such that Y is a treed spectral space, although the space X is not spectral. One can arrange, in addition, that Y is (homeomorphic to) an abstract Riemann surface and X is irreducible.

To begin the construction, we use [14, Corollary 3.6] to find a valuation domain V such that $Y = \mathrm{Spec}(V)$ is order-isomorphic to the denumerable totally ordered set $S = \{y_n\} \cup \{\infty\}$ satisfying

$$y_1 < y_2 < \cdots < y_n < \cdots < \infty.$$

Of course, $Y \cong X(V)$ is an irreducible treed spectral space.

Next, let X be the three-element space $\{x_1, x_2, x\}$ whose open subsets are ϕ, X, and $\{x_1, x_2\}$. Note that X is not spectral since X is not a T_0-space. Moreover X is irreducible.

Finally, identify Y with S and define h: $Y \to X$ as follows: $h(y_{2n+1}) = x_1$, $h(y_{2n}) = x_2$, and $h(\infty) = x$. It is straightforward to verify the assertions concerning h.

Example 8 should be contrasted with the result of Hochster [11, Theorem 7] that S has images. Since the space X in Example 8 satisfies all of Hochster's criteria for spectral spaces except the T_0-condition, it seems natural to ask the following question. Under what conditions on a closed continuous surjection h: $Y \to X$ from a treed spectral space Y to a T_0-space X may one conclude that X is spectral? The next result is a small contribution in this direction. For additional motivation, note in Example 8 that $h^{-1}(U) = \{y_n\}$ is not quasi-compact for the quasi-compact open $U = \{x_1, x_2\}$.

PROPOSITION 9. *Let h: $Y \to X$ be a continuous surjective map such that $h^{-1}(U)$ is quasi-compact for each quasi-compact open subset U of X, Y is a spectral space, and X is a T_0-space having a basis of quasi-compact open sets. Then X is a spectral space.*

Proof. Let Y^H (resp. X^H) denote Y (resp. X) with the patch topology, in the sense of [11, p. 52]. By [11, Corollary, p. 54], it is enough to show that X^H is compact. Since the quasi-compact open sets in X (resp. Y) and their complements form an open sub-basis for X^H (resp. Y^H), it is straightforward to verify that X^H is Hausdorff and that h: $Y^H \to X^H$ is continuous. However Y^H is (quasi-) compact [11, Theorem 1], so the assertion follows.

Remark 10. One cannot replace the "continuous" hypothesis in Proposition 9 with "closed." To illustrate this, begin with the set $\mathbb{N}$ of positive integers, equipped with the discrete topology. Select $b \notin \mathbb{N}$ and let $W = \mathbb{N} \cup \{b\}$ be the one-point compactification of $\mathbb{N}$. It is well known that W is a Boolean space; in particular, W is a spectral space. Let ∞ denote a point outside W, and let $Y = W \cup \{\infty\}$ be the spectral space topologized as in the proof of Theorem 7. Select $c \notin \mathbb{N}$ and set $X = \mathbb{N} \cup \{c\}$, equipped with the discrete topology. Then the asserted behavior is exhibited by $h: Y \to X$ defined by $h(n) = n$ for all $n \in \mathbb{N}$ and $h(b) = c = h(\infty)$.

Indeed, h is evidently surjective and closed, and X is a T_0-space whose singleton subspaces form a quasi-compact open basis. Moreover, $h^{-1}(U)$ is quasi-compact for each quasi-compact (open) subset U of X since U (and $h^{-1}(U)$) must be finite.

However, X is not a spectral space. Unlike the situation in Example 8, the culprit here is not the T_0-property: the present X is not quasi-compact, the reason being that X is infinite and discrete.

The infinitude of $\mathbb{N}$ (not to mention Proposition 9) also explains why h is not continuous: $\mathbb{N}$ is closed in X and $\mathbb{N} = h^{-1}(\mathbb{N})$ is not closed in Y. To see this, namely that $\mathbb{N}$ is not closed in W, or equivalently that $\{b\}$ is not open in W, one need only recall the construction of the one-point compactification. An open neighborhood V of b in W must be such that $W \setminus V$ is finite, but $W \setminus \{b\} = \mathbb{N}$.

In closing, we shall pursue a theme mentioned in Remark 3(b). As usual, if P is a property of some ring-homomorphisms, then a ring-homomorphism $f: R \to T$ is said to be (have, satisfy) *universally* P if, for each change of base $R \to S$, the induced homomorphism $S \to S \otimes_R T$ satisfies P. The various kinds of P and other background needed below are discussed conveniently in the following references, with which we assume familiarity. For Theorem 11(a), see [12]; for (b), see [4, 7]; for (d), see [12, 9]; and for (e), see [12, 4, 5]. We need not treat "universally LO" and "universally flat" explicitly since LO and flat are universal properties (cf. [6, Corollary 2.2] and [1, Corollary 2, p. 19]). Finally, for partial motivation of Theorem 11(e), recall the following result of J. T. Arnold (cf. [10, Theorem 33.4]). If R is an integrally closed integral domain, then $R^b = R(X)$ if and only if R is a Prüfer domain.

THEOREM 11. *Let R be an integrally closed integral domain with quotient field K and Kronecker function ring R^b. Let $j: R \to R^b$ be the inclusion map. Then*:

(a) *j satisfies* GU *and* LO.

(b) *j satisfies none of the following five properties: universally* GU, *universally* INC, *universally i-, universally unibranched, universally mated.*

(c) $\mathrm{Spec}(j): \mathrm{Spec}(R^b) \to \mathrm{Spec}(R)$ *is an open map if and only if* $\mathrm{Spec}(R[x_1, \ldots, x_n]) \to \mathrm{Spec}(R)$ *is an open map for each finite subset* $\{x_1, \ldots, x_n\}$ *of K.*

(d) *j satisfies* GD *if and only if R is a* GD-*domain.*

(e) *The following seven conditions are equivalent*:

(1) *R is a Prüfer domain;*

(2) *j is unibranched;*

(3) *j is mated;*

(4) *j is an i-extension;*

(5) *j satisfies* INC;

(6) *R^b is R-flat;*

(7) *j is universally* GD.

Proof. (a) This was established in Remark 3(b), as a consequence of the fact that Spec(j) is a closed map. (A direct proof was given in Remark 3(c).)

(b) It is well known that universally GU is equivalent to integral (cf. [2, Lemma, p. 160]). Hence, j is not universally GU since the induced extension of quotient fields, $K \subset K(X)$, is not algebraic.

Each of the other four universal properties in the assertion is known to entail algebraic (in fact, for the last three properties, purely inseparable) residue field extensions: see [7, Theorem 2.2; 4, Theorem 2.1]. Thus it suffices to show that if $Q \in \mathrm{Spec}(R^b)$ and $P = Q \cap R$, then $\mathbf{k}_R(P) \subset \mathbf{k}_{R^b}(Q)$ is not algebraic. Let (W, M) denote the valuation domain $(R^b)_Q$ and set $(V, N) = W \cap K$. Since [10, Theorem 32.10] assures that $W = V^*$, it follows from [1, Proposition 2, p. 436] that W/M (which is just $\mathbf{k}_{R^b}(Q)$) is the rational function field $F(Y)$, where $F = V/N$. But $\mathbf{k}_R(P) = R_P/PR_P$ canonically embeds in F, giving the assertion since Y is not even algebraic over F.

(c) As noted in Remark 3(b), Spec(j) may be identified with the canonical surjection $f_R\colon X(R) \to \mathrm{Spec}(R)$. Thus, (c) may be viewed as a translation of [3, Proposition 3.1].

(d) The characterizations of GD-domains [9, Theorem 1] give the "if" half, and reduce its converse to showing, in case j has GD, that $R \subset V$ has GD for each valuation overring V of R. Accordingly, consider primes $P_2 \subset P_1$ of R and a prime Q_1 of V such that $Q_1 \cap R = P_1$. We must find $Q_2 \in \mathrm{Spec}(V)$ such that $Q_2 \subset Q_1$ and $Q_2 \cap R = P_2$.

Let (W, N) denote $V^*_{Q_1}$. Since R^b is a Prüfer domain, W is just $(R^b)_P$, where $P = N \cap R^b$. Note that $P \cap R = N \cap R = N \cap V_{Q_1} \cap R = Q_1 \cap R = P_1$. Thus, since j has GD, there exists a prime Q of R^b contained in P and satisfying $Q \cap R = P_2$. Set $T = (R^b)_Q$ and $S = T \cap K$, valuation domains whose maximal ideals we denote by M and m, respectively. Since S contains $W \cap K = V_{Q_1}$, it follows that S is the localization of V_{Q_1} at some prime $Q_2 V_{Q_1}$. Then $Q_2 \subset Q_1$, $S = V_{Q_2}$, and $Q_2 \cap R = m \cap R = M \cap R = Q \cap R = P_2$, as desired.

(e) In general, unibranched $\Rightarrow$ mated $\Rightarrow$ i-extension. However, we recalled in (a) that j satisfies LO, and so these implications reverse when applied to j. In other words, (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

(2) is just the requirement that Spec(j) be a bijection. As Spec(j) may be identified with $f_R: X(R) \to \mathrm{Spec}(R)$, a translation of [3, Proposition 2.2] yields that (1) $\Leftrightarrow$ (2).

(4) $\Rightarrow$ (5) Each i-extension satisfies INC.

(5) $\Rightarrow$ (1) Deny. By [10, Theorem 26.2 and Corollary 19.7(1)], there exists a valuation overring V of R such that $R \subset V$ does not satisfy INC; that is, $Q_1 \cap R = P = Q_2 \cap R$ for some distinct primes $Q_1 \subset Q_2$ of V. Let $W_i = V^*_{Q_i}$ for $i = 1, 2$. By the order-isomorphism $X(R^b) \to X(R)$ noted in Remark 3(a), it follows that $W_2 \subsetneqq W_1$. Then $P_1 \subsetneqq P_2$, where $P_i \in \mathrm{Spec}(R^b)$ satisfies $W_i = (R^b)_{P_i}$. But each P_i lies over P since

$$P_i \cap R = (\text{maximal ideal of } W_i) \cap R = Q_i V_{Q_i} \cap R = Q_i \cap R.$$

This contradicts (5), as desired.

(1) $\Rightarrow$ (6) Each torsion-free module over a Prüfer domain is flat. For an alternative proof, appeal to the above-cited result of Arnold and factor j as the composite of the flat maps $R \hookrightarrow R[X]$ and $R[X] \hookrightarrow R(X)$.

(6) $\Rightarrow$ (7) Each flat map is universally GD.

(7) $\Rightarrow$ (1) Prüfer domains are just the integrally closed universally GD-domains [5, Corollary 2.3]. Hence by [5, Theorem 2.6], it suffices to prove that $R \hookrightarrow V$ is universally GD for each valuation overring V of R. Thus by [4, Corollary 2.3], it suffices to prove that $R[X_1,..., X_n] \subset V[X_1,..., X_n]$ satisfies GD for each positive integer n.

Consider primes $P_2 \subset P_1$ of $R[X_1,..., X_n]$ and a prime Q_1 of $V[X_1,..., X_n]$ such that $Q_1 \cap R[X_1,..., X_n] = P_1$. Let $P = Q_1 \cap V$ and $W = V_P$. Since W^* dominates W and $W \subset W^*$ satisfies GD, it follows that $W \subset W^*$ satisfies LO. Since LO is a universal property, $W[X_1,..., X_n] \subset W^*[X_1,..., X_n]$ also satisfies LO. Therefore, there exists a prime N of $W^*[X_1,..., X_n]$ which lies over $Q_1 W[X_1,..., X_n] = Q_1 V[X_1,..., X_n]_{V \setminus P}$. Observe that $N \cap V[X_1,..., X_n] = Q_1$. Hence, it suffices to show that $R[X_1,..., X_n] \subset W^*[X_1,..., X_n]$ satisfies GD; for if $M \in \mathrm{Spec}(W^*[X_1,..., X_n])$ is contained in N and lies over P_2, then $M \cap V[X_1,..., X_n]$ is the desired prime inside Q_1. Thus, it is enough to show that $R \hookrightarrow W^*$ is universally GD. But this factors as the composite of j: $R \hookrightarrow R^b$ and $R^b \hookrightarrow W^*$, each of which is a universally GD-map. The proof is complete.

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