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Direct Limits and Going-down

by

David E. DOBBS,* Marco FONTANA**
and Ira J. PAPICK***

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1. Introduction

In the course of studying the flat spectral topology and a related discrete Alexandroff topology in [8], we had occasion to prove that direct limits of certain types of directed systems of GD (going-down)-homomorphisms were also GD-homomorphisms [8, Lemma 2.14]. In the present note, we establish a generalization of this result which is valid for all directed systems of GD-homomorphisms (Theorem 2.1). Analogous results are obtained for, i.e., the INC-property (Proposition 2.3). The major portion of this article is devoted to showing that various classes of going-down *rings* (in the sense of [4]) are closed under direct limit (cf. Corollary 2.7). Results of this sort are motivated in part by the well-known fact that directed unions of Prüfer domains are themselves Prüfer domains; indeed, the present work includes three proofs of a direct limit generalization of this fact.

Throughout, rings are assumed commutative, with 1; a subring must contain the 1 of the larger ring; and ring-homomorphisms are assumed unital, that is, send 1 to 1.

2. Main results

We begin with the promised sharpening of [8, Lemma 2.14].

THEOREM 2.1. *Let $(I, \leq)$ be a directed set, and let (A_i, f_{ij}) and (B_i, g_{ij}) each be direct systems of rings indexed by I. For each $i \in I$, let $h_i : A_i \to B_i$ be a ring-homomorphism satisfying GD such that, whenever $i \leq j$ in I, then $g_{ij}h_i = h_j f_{ij} : A_i \to B_j$. Set $A = \varinjlim A_i$, $B = \varinjlim B_i$ and $h = \varinjlim h_i$. Then $h : A \to B$ also satisfies GD.*

Proof. If the assertion fails, then [12], Exercise 37, p. 44 supplies $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $Q \supset h(P)B$ and

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$$h(P)B \cap h(A \backslash P)(B \backslash Q) \neq \varnothing .$$

Thus, $\sum_{i=1}^{n} h(p_i)b_i = h(a)b$ for suitable elements $p_i \in P$, $b_i \in B$, $a \in A \backslash P$ and $b \in B \backslash Q$. By the construction of direct limits (see [3], Lemma 1, p. 203), we produce an index $\alpha \in I$ and elements $x_{i\alpha} \in B_\alpha$ such that $g_\alpha(x_{i\alpha}) = b_i$ for $i = 1, \cdots, n$ and $g_\alpha(x_{n+1,\alpha}) = b$, where $g_\alpha : B_\alpha \to B$ is the canonical map. Similarly, there exist $\beta \in I$ and $y_{i\beta} \in A_\beta$ such that $f_\beta(y_{i\beta}) = p_i$ for $i = 1, \cdots, n$ and $f_\beta(y_{n+1,\beta}) = a$, where $f_\beta : A_\beta \to A$ is the canonical map. Since directedness of I yields an index γ majorizing both α and β, we may suppose that $\alpha = \beta$. (In detail, $x_{i\alpha}$ may be replaced by $g_{\alpha\gamma}(x_{i\alpha})$ since $g_\gamma g_{\alpha\gamma} = g_\alpha$, etc.) As $hf_\alpha = g_\alpha h_\alpha$ and g_α is a homomorphism, it follows that

$$g_\alpha\left(\sum_{i=1}^{n} h_\alpha(y_{i\alpha})x_{i\alpha}\right) = g_\alpha(h_\alpha(y_{n+1,\alpha})x_{n+1,\alpha}) .$$

Thus, by [3], (ii), p. 204, there is an index k in I such that $\alpha \leq k$ and

$$g_{\alpha k}(\sum h_\alpha(y_{i\alpha})x_{i\alpha}) = g_{\alpha k}(h_\alpha(y_{n+1,\alpha})x_{n+1,\alpha}) .$$

Since $g_{\alpha k}h_\alpha = h_k f_{\alpha k}$ by hypothesis, we have

$$\sum h_k(f_{\alpha k}(y_{i\alpha}))g_{\alpha k}(x_{i\alpha}) = h_k(f_{\alpha k}(y_{n+1,\alpha}))g_{\alpha k}(x_{n+1,\alpha}) .$$

To complete the proof, it suffices to show (using the preceding equation) that h_k does not satisfy GD.

Indeed, in view of [12], Exercise 37, p. 44, it suffices to verify that $P_1 = f_k^*(P) \in \operatorname{Spec}(A_k)$ and $Q_1 = g_k^*(Q) \in \operatorname{Spec}(B_k)$ satisfy $Q_1 \supset h_k(P_1)$; $f_{\alpha k}(y_{i\alpha}) \in P_1$ for each $i = 1, \cdots, n$; $f_{\alpha k}(y_{n+1,\alpha}) \in A_k \backslash P_1$; and $g_{\alpha k}(x_{n+1,\alpha}) \in B_k \backslash Q_1$. For the first of these, one need only notice that $g_k h_k(P_1) = hf_k(P_1) \subset h(P) \subset Q$. The second and third assertions follow readily from the above information about p_i and a, since $f_k f_{\alpha k} = f_\alpha$. Similarly, the final assertion reduces to requiring $b \notin Q$, and so the proof is complete.

REMARK 2.2. (a) Some special cases of the preceding result are noted next. First, if $A_i \to B$ is a directed system of ring-homomorphisms each of which satisfies GD, then the direct limit map $\varinjlim A_i \to B$ also satisfies GD. Of course, this follows from Theorem 2.1 by setting each $B_i = B$ and $g_{ij} = 1$. Secondly, specializing to the case $A_i = A$, $f_{ij} = 1$ recovers [8, Lemma 2.14]. Thirdly, let (B_i, g_{ij}) be a directed system indexed by I and set $B = \varinjlim B_i$. If $k \in I$ is such that $g_{kj} : B_k \to B_j$ satisfies GD whenever $k \leq j$, then the canonical map $g_k : B_k \to B$ also satisfies GD. For a proof, let $B' = \varinjlim B_j$, where the indexes range over those $j \in I$ such that $k \leq j$. By the preceding observation, the canonical map $g_k' : B_k \to B'$ satisfies GD. However, since I is directed, a cofinality argument identifies B with B', whence g_k is identified with g_k', and the assertion follows.

(b) As a special case of the second observation in (a), that is of [8, Lemma 2.14], we easily recover [13], Corollary 2, whose proof was our original inspiration for Theorem 2.1. Specifically, we have that if A is a subring of B such that $A \subset A[b_1, \cdots, b_n]$ satisfies GD for each finite subset $\{b_1, \cdots, b_n\}$ of B, then $A \subset B$ also satisfies

GD. The point, of course, is that B is the direct limit of its subrings of the form $A[b_1, \cdots, b_n]$. If the A-subalgebras corresponding to $n=1$ happen to be cofinal amongst the A-subalgebras of finite type then, to use the terminology of [13], "simple going down" for $A \subset B$ implies that $A \subset B$ satisfies GD. It would be of interest to characterize those extensions $A \subset B$ for which such cofinality obtains.

(c) Note that analogues of Theorem 2.1 and the above consequences may fail when "satisfies GD" is replaced by "induces an open mapping of prime spectra with their Zariski topologies." In particular, consideration of the extension $\boldsymbol{Z} \subset \boldsymbol{Q}$ reveals the falsity of the analogue of [8, Lemma 2.14]; indeed [14], Remark 3.12 guarantees, in the terminology of [14], that $\boldsymbol{Z}$ is an FTO-domain which is not an open domain. For some positive analogues, see the next result.

First, some terminology. As in [12], p. 28, it will be convenient to let INC denote the incomparability property. Adapting from [14], p. 2, we shall say that a ring-homomorphism $f : A \to B$ is an *i*-map if $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is an injection; and an integral domain A is called an *i-domain* if the inclusion $A \subset B$ is an *i*-map for each overring B of A.

PROPOSITION 2.3. *Let (A_j, f_{jk}) and (B_j, g_{jk}) be directed systems of rings, each indexed by a directed set $(I, \leq)$. For each $j \in I$, let $h_j : A_j \to B_j$ be a ring-homomorphism satisfying INC (resp., which is an i-map) such that, whenever $j \leq k$ in I, then $g_{jk}h_j = h_k f_{jk}$. Then $h = \varinjlim h_j : A = \varinjlim A_j \to B = \varinjlim B_j$ satisfies INC (resp., is an i-map).*

Proof. If the assertion concerning INC is assumed to fail, then there exist distinct comparable prime ideals $Q \subset W$ of B such that $h^*(Q) = h^*(W) = P \in \operatorname{Spec}(A)$. Select $b \in W \backslash Q$. By the construction of direct limits, there exist an index $k \in I$ and an element $x \in B_k$ such that $g_k(x) = b$, where $g_k : B_k \to B$ is the canonical map. Let $W_1 = g_k^*(W)$, $Q_1 = g_k^*(Q)$ and $P_1 = f_k^*(P)$, where f_k: $A_k \to A$ is the canonical map. Evidently, $Q_1 \subset W_1$ are comparable prime ideals of B_k, distinct since $x \in W_1 \backslash Q_1$. But the condition $g_k h_k = h f_k$ readily yields that $h_k^*(Q_1) = f_k^*(P) = h_k^*(W_1)$, contradicting the assumption that h_k satisfies INC. The preceding argument also applies, *mutatis mutandis*, to give the assertion about *i*-maps.

To avoid unnecessary repetition, let us *fix notation for* (2.4)–(2.10). Data will consist of a directed system (A_j, f_{jk}) of rings indexed by a directed set $(I, \leq)$; and its direct limit, $A = \varinjlim A_j$, together with the canonical maps $f_j : A_j \to A$.

COROLLARY 2.4. *If A_j is an i-domain for each $j \in I$, then A is also an i-domain.*

Proof. If not, then as in the preceding proof (also cf. [14], Proposition 2.10), there exists an element u in the quotient field of A such that $B = A[u]$ has distinct prime ideals Q, W such that $u \in W \backslash Q$ and $Q \cap A = W \cap A = P \in \operatorname{Spec}(A)$. Write $u = ab^{-1}$ for appropriate nonzero a, $b \in A$. By the construction of direct limits, there exists $k \in I$ and c, $d \in A_k$ such that $f_k(c) = a$ and $f_k(d) = b$.

We claim that there exists a ring-homomorphism $H : D = A_k[cd^{-1}] \to B$ which restricts to f_k on A_k and sends cd^{-1} to u. For this, it is enough to show that if h is the homomorphism $A_k[X]\to B$ which restricts to f_k on A_k and sends X to u, then h vanishes on the kernel of the evaluation map $A_k[X]\to D$. Now, if

$$g=a_0X^n+a_1X^{n-1}+\cdots+a_n\in A_k[X]$$

is an n-th degree polynomial in that kernel, i.e. satisfies $g(cd^{-1})=0$, then,

$$0=d^ng(cd^{-1})=a_0c^n+a_1c^{n-1}d+\cdots+a_nd^n\,.$$

Applying f_k and dividing by b^n results in

$$0=f_k(a_0)u^n+f_k(a_1)u^{n-1}+\cdots+f_k(a_n)=h(g)\,,$$

as claimed.

Observe that $Q_1=H^*(Q)$ and $W_1=H^*(W)$ are prime ideals of D, distinct since $cd^{-1}\in W_1\backslash Q_1$. But the conditions satisfied by H imply that $Q_1\cap A_k=f_k^*(P)=W_1\cap A_k$, whence the inclusion $A_k\subset A_k[cd^{-1}]$ is not an i-map, contradicting the assumption that A_k is an i-domain. This completes the proof.

Corollary 2.4 is reminiscent of the well-known fact (cf. [9], Proposition 22.6) that a directed union of Prüfer domains is itself a Prüfer domain. Indeed, Prüfer domains may be characterized as the integrally closed i-domains [9], Theorem 26.2. Recall that, in general, each overring of an i-domain is a going-down ring [14], Corollary 2.13. Accordingly, one might conjecture that the classes of Prüfer domains and of going-down rings are closed under taking direct limits. We shall soon establish these conjectures, together with their analogue for locally divided domains, a type of going-down ring figuring intimately in the analysis of arbitrary going-down rings (cf. [5], Theorem 2.5 and Corollary 2.8). For completeness, we recall that an integral domain D is called *divided* in case $P=PD_P$ for each $P\in\operatorname{Spec}(D)$; and D is said to be *locally divided* if D_P is divided for each $P\in\operatorname{Spec}(D)$.

PROPOSITION 2.5. (a) *If A_j is a Prüfer domain for each $j\in I$, then A is also a Prüfer domain.*

(b) *If A_j is divided for each $j\in I$, then A is divided.*

(c) *If A_j is locally divided for each $j\in I$, then A is locally divided.*

Proof. In any event, A is an integral domain (cf. [2], Proposition 3, p. 122).

(a) One proof proceeds by applying Corollary 2.4, since any direct limit of integrally closed integral domains is itself integrally closed. For a more direct proof, we shall use the criterion that an integral domain is a Prüfer domain if and only if each of its ideals is flat. Let J be any ideal of A and, for each $j\in I$, set $J_j=f_j^{-1}(J)$. Since A_j is a Prüfer domain, J_j is A_j-flat. Thus, by [1], Proposition 9, p. 35, $\varinjlim J_j$ is A-flat. However, according to [10], Proposition 6.1.2 (ii), p. 128, $\varinjlim J_j\cong J$, and so the assertion follows.

(b) Let $P\in\operatorname{Spec}(A)$ and, for each $j\in I$, set $P_j=f_j^*(P)\in\operatorname{Spec}(A_j)$. As above, $P\cong\varinjlim P_j$. However, each A_j is supposed divided, and so $P_j=P_j(A_j)_{P_j}\cong P_j\otimes_{A_j}(A_j)_{P_j}$. Since tensor product commutes with direct limit, the proof that $P = PA_P$ will be complete if $A_P \cong \varinjlim (A_j)_{P_j}$. However, this needed isomorphism *does* hold, by virtue of [10], Proposition 6.1.6 (ii), p. 130, whose applicability is a consequence of noticing that $P_j = f_{jk}^*(P_k)$ whenever $j \leq k$ in I.

(c) We shall offer two proofs. First, let $P \in \operatorname{Spec}(A)$ and, for each $j \in I$, set $P_j = f_j^*(P)$. As in the proof of (b), an appeal to [10] reveals $A_P = \varinjlim (A_j)_{P_j}$. Since A_j is assumed to be locally divided, $(A_j)_{P_j}$ is divided, and so an appeal to (b) shows that A_P is divided. Since P was an arbitrary prime of A, the assertion follows.

To sketch another proof of (c), we recall from [6], Theorem 2.4 that an integral domain D is locally divided if and only if $D + QD_Q$ is D-flat for each $Q \in \operatorname{Spec}(D)$. Now, let P, P_j be as above. Since A_j is locally divided for each j and direct limit preserves flatness, it follows that $B = \varinjlim (A_j + P_j(A_j)_{P_j})$ is A-flat. However, one may verify routinely that the canonical D-module epimorphism $B \to A + PA_P$ is an isomorphism, from which the assertion (again) follows.

LEMMA 2.6. *Assume that each A_i is a domain. Then each overring B of A may be expressed as $B = \varinjlim B_j$, where B_j is an overring of A_j for each $j \in I$, such that the canonical diagram*

$$\begin{array}{ccc} A_j & \longrightarrow & B_j \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

commutes whenever $j \in I$.

Proof. Let K be the quotient field of A, and consider an overring B of A (so that $A \subset B \subset K$). For each $j \in I$, let $P_j = f_j^*(0) \in \operatorname{Spec}(A_j)$. As in the proofs of parts (b) and (c) of Proposition 2.5, we have $\varinjlim (A_j)_{P_j} \cong K$. In particular, $((A_j)_{P_j}, g_{jk})$ is a directed system indexed by I; note that g_{jk} restricts to f_{jk} on A_j. Let $g_j : (A_j)_{P_j} \to K$ be the canonical structure map, and set $B_j = g_j^{-1}(B)$. Then $A_j \subset B_j$ since g_j restricts to f_j on A_j. The idea of the proof is now to verify the following assertions:

(1) $(B_j, g_{jk}|_{B_j})$ is a directed set of rings indexed by I;

(2) Whenever $j \leq k$ in I, one has $(g_{jk}|_{B_j})h_j = h_k f_{jk}$; and

(3) The direct limit of the system in (1) may be identified with B in such a way that $\varinjlim h_j$ becomes identified with the inclusion map $A \to B$.

Now, (1) follows readily from the condition $g_k g_{jk} = g_j$. The compatibility condition (2) is a consequence of the above remarks. Finally, to establish (3), observe that the direct limit of the system in (1) may be viewed as $B' = \bigcup(\operatorname{im}(g_j|_{B_j}))$, the union indexed by I. Evidently, $B' \subset B$, by the definition of B_j. For the reverse inclusion, view any given $b \in B$ inside $\varinjlim (A_j)_{P_j}$ and use the construction of direct limits (cf. [3], Lemma 1(i), p. 204) to find $k \in I$ and $x \in (A_k)_{P_k}$ such that $g_k(x) = b$; then $x \in g_k^{-1}(B) = B_k$, whence $b \in B'$. Thus $B = B'$. For the final assertion in (3), one has to verify that the inclusion map $A \to B$ is compatible with the composite maps $A_j \to B_j \to B$, and this holds since g_j restricts to f_j. The proof is complete.

Lemma 2.6 is perfectly suited for our present purposes. For example, it immediately recovers Corollary 2.4 since a direct limit of (abelian group) monomorphisms is itself a monomorphism. Similarly, Lemma 2.6 also leads to a (third) proof of Proposition 2.5(a), since Prüfer domains may be characterized as the integral domains each of whose overrings is flat **[15]**, Theorem 4. More to the point, we now give the promised result.

COROLLARY 2.7. *If A_j is a going-down ring for each $j \in I$, then A is also a going-down ring.*

Proof. Use the criterion that an integral domain D is a going-down ring if and only if the inclusion $D \subset E$ satisfies GD for each overring E of D. Apply Lemma 2.6 and Theorem 2.1, to complete the proof.

Our next result concerns QR-domains. Recall that an integral domain D is said to be a QR-domain in case each overring of D is a quotient ring (ring of fractions) of D. As quotient rings are flat, any QR-domain must be a Prüfer domain and, in particular, a going-down ring.

COROLLARY 2.8. *If A_j is a QR-domain for each $j \in I$, then A is also a QR-domain.*

Proof. Let B be an overring of A. By Lemma 2.6, $B = \varinjlim B_j$, where B_j is a suitable overring of A_j for each $j \in I$. By hypothesis, $B_j = (A_j)_{T_j}$, where we may suppose that T_j is a saturated multiplicative subset of A_j. Observe, using (2) in the proof of Lemma 2.6, that $f_{jk}(T_j) \subset T_k$ whenever $j \leq k$ in I. Letting T be the multiplicative set $\bigcup \{f_j(T_j) \mid j \in I\} \subset A$, one readily verifies (cf. [10], Proposition 6.1.5, p. 129) that the canonical ring-homomorphism $\varinjlim B_j \to A_T$ is an isomorphism, completing the proof.

Our final results concern strong extensions [7] and pseudo-valuation domains [11]. Recall that an extension $D \subset E$ of rings is said to be *strong* if, whenever $xy \in P$ for some $x \in E$, $y \in E$ and $P \in \operatorname{Spec}(D)$, then either x or y is in P; and that a domain D is a *pseudo-valuation domain* (PVD) in case $D \subset K$ is strong, where K is the quotient field of D. Any PVD is a divided ring and, hence, a going-down ring.

PROPOSITION 2.9. *Let (A_j, f_{jk}) and (B_j, g_{jk}) be directed systems of rings, each indexed by a directed set $(I, \leq)$. For each $j \in I$, let $h_j : A_j \to B_j$ be a strong extension such that, whenever $j \leq k$ in I, then $g_{jk}h_j = h_k f_{jk}$. Then $h = \varinjlim h_j : A = \varinjlim A_j \to B = \varinjlim B_j$ is a strong extension.*

Proof. Suppose that $xy \in P$ for some $x \in B$, $y \in B$, $P \in \operatorname{Spec}(B)$. By the nature of direct limits, there exists an index j and elements x_j, y_j of B_j such that $g_j(x_j) = x$, $g_j(y_j) = y$ and $x_j y_j \in P_j = g_j^{-1}(P)$ (cf. [10], Proposition 6.1.2). Since h_j is assumed strong, either x_j or y_j is in P_j, and so either x or y is in P, as desired.

COROLLARY 2.10. *If A_j is a PVD for each $j \in I$ and if f_{jk} is a monomorphism whenever $j \leq k$ in I, then A is a PVD and the quotient field of A is $\varinjlim K_j$, where K_j*

denotes the quotient field of A_j for each $j \in I$.

Proof. By the definition of PVD's, $A_j \to K_j$ is a strong extension for each $j \in I$. Thus, by Proposition 2.9, $A \to \varinjlim K_j$ is also strong. Since $\varinjlim K_j$ is readily shown to be the quotient field of A, the assertions follow.

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Department of Mathematics
University of Tennessee
Knoxville, TN 37916
U.S.A.

Istituto Matematico
Università di Roma
00185 Rome
Italy

Department of Mathematics
University of Missouri
Columbia, MO 65211
U.S.A.