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Prüfer Domains and Endomorphism Rings of Their Ideals

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0. INTRODUCTION

An overring of a Prüfer domain R can be represented as the intersection of localizations of the form R_P, P a prime ideal. Recently there has been interest in representing special types of overrings. We do this for overrings of the form $(I : I)$, where I is an ideal of R. A second representation is given for $(I : I)$, when restrictions are placed on R. An example is constructed to show that the second representation cannot be extended to the class of all Prüfer domains.

This leads to the development of a general method for constructing Prüfer domains with specific characteristics. Its generality indicates that it may be useful in many other settings. We also introduce and study "absolute non-zero-divisors modulo an ideal."

Moreover, we study the fractional ideal $(R : I)$, where R is a seminormal domain an I is and ideal of R. We show that for R seminormal, the largest subring of $(R : I)$ is $(\sqrt{I} : \sqrt{I})$ (Theorem 3.3).

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Let R denote an integral domain and K its quotient field. Associated to each nonzero ideal I of R are the following algebraic objects:

$$(I:I)=\{x\in K \mid xI\subseteq I\};$$
$$I^{-1}=(R:I)=\{x\in K \mid xI\subseteq R\};$$
$$\mathcal{T}(R,I)=\mathcal{T}(I)=\bigcup\{(R:I^n)\mid n\geqslant 1\}.$$

The set $(I:I)$ is a ring, and is the largest subring of K in which I is still an ideal. In addition, $(I:I)$ is canonically isomorphic to $\mathrm{End}_R(I)$, the endomorphism ring of I. The fractional ideal I^{-1} is sometimes, but not always, a subring of K (see [8] and [2]). It is well known that I^{-1} is R-isomorphic to $\mathrm{Hom}_R(I,R)$, the *dual of* I. The ring $\mathcal{T}(I)$ is called the *Nagata transform* (or the *ideal transform*) of I.

An *overring* of R is a ring between R and its quotient field K. If R is a Prüfer domain, then each overring T of R is an intersection of the type $\bigcap\{R_P: P\in \mathbf{C}\}$, where $\mathbf{C}\subseteq \mathrm{Spec}(R)$; moreover, we can take $\mathbf{C}=\{P\in\mathrm{Spec}(R)\mid PT\neq T\}$ [6, Them. 26.1]. Apart from the general description of such an intersection in [6], it is natural to look for more "concrete" representations for specific overrings of R. This is done for $I^{-1}=(R:I)$ in [8]. There it is proved that for a Prüfer domain R,

$$(R:I)\supseteq\left(\bigcap R_P\right)\cap\left(\bigcap R_M\right), \tag{0-1}$$

where P ranges over the minimal prime ideals of I, and M over the maximal ideals not containing I; and that equality holds in (0-1) if and only if $(R:I)$ is a ring. It turns out that the intersection in (0-1) equals $(\sqrt{I}:\sqrt{I})$ (Corollary 4.15 below). See also Corollary 3.4(1).

As for $\mathcal{T}(I)$, in case R is a Prüfer domain, see [6, Sect. 26, Exercise 11].

Representations of $(I:I)$ are investigated in Section 4. We prove there that for an ideal I of a Prüfer domain R,

$$(I:I)=\left(\bigcap R_{G(Q)}\right)\cap\left(\bigcap R_M\right), \tag{0-2}$$

where M is as in (0-1), Q varies over the set of maximal zero-divisors modulo I, and $G(Q)$ is the unique prime ideal of R such that $G(Q)R_Q$ is equal to the set of zero-divisors in R_Q/IR_Q.

Can we replace $G(Q)$ by Q in (0-2) for an arbitrary Prüfer domain R? Since $\bigcap R_Q=R_{\mathcal{N}}$, where $\mathcal{N}$ is the set of non-zero-divisors modulo I, and since $R_{\mathcal{N}}\cap(\bigcap R_M)\subseteq(I:I)$ (Corollary 4.5(2)), this question is shown to be equivalent to asking whether $(I:I)$ is contained in $R_{\mathcal{N}}$. We show that the answer is positive for the class of QR-domains (which is contained in the class of Prüfer domains). It turns out that in a Prüfer domain, $\mathcal{N}$ is equal to the set of "absolute non-zero-divisors modulo Γ" introduced in Section 2 for an arbitrary domain R.

On the other hand, in order to obtain (in Section 7) a counterexample to the question raised above, we construct for a domain R, a Prüfer domain $\mathscr{P}^{\infty}(R)$ that contains R and mimics many important properties of R. The construction is carried out in Section 5 and 6. Some preliminary results for this construction are given in Section 1.

Section 3 deals with the fractional ideal $(R : I)$ for R seminormal. Thus, while Prüfer domains occupy the center stage of this paper, many results are stated in a much more general setting.

The following notation is used throughout. All rings are commutative with identity. We denote by R an integral domain, and by $\mathscr{Q}(R) = K$, its quotient field; unless otherwise stated, an ideal I of R is assumed to be a nonzero and proper ideal. Denote the set of prime ideals (resp., maximal ideals) of R by Spec(R) (resp., by Max(R)).

For general background see [6].

1. A Reformulation of the Prüfer Property

A domain R is Prüfer if and only if each ideal generated by two nonzero elements is invertible [6, Thm. 22.1]. It is convenient to reformulate this statement.

Proposition 1.1. *Let T be a domain containing R. The following conditions are equivalent*:

(1) *Each finitely generated nonzero ideal of T extended from R is invertible.*

(2) *Each ideal of T generated by two nonzero elements of R is invertible.*

(3) *For each $q \in K \setminus \{0\}$, there is an element $t \in T$ such that tq and $(1-t)/q$ are in T.*

(4) *For each $q \in K \setminus \{0\}$, there is an element $t \in T \setminus \{0\}$ such that tq and $(1-t)/q$ are in T.*

(5) *For each $q \in K \setminus \{0\}$, we have $(Tq \cap T) + (Tq^{-1} \cap T) = T$.*

Proof. (1)$\Leftrightarrow$(2): This follows from [6, Prop. 22.2(a)], since each finitely generated ideal of T which is extended from R, is generated by finitely many elements of R.

(2)$\Rightarrow$(3): Let $q = a/b$ be a nonzero element in K, a and b in R. Then the ideal $Ta + Tb$ is invertible, so there exist elements x and y in

$(Ta+Tb)^{-1}$ such that $1=xa+yb$. Set $t=yb$. Then $tq=ya\in T$, and $(1-t)/q=xb\in T$.

$(3)\Rightarrow(2)$: Let a, b be nonzero elements in R, and $q=a/b$. Choose t in T such that tq and $(1-t)/q$ are in T. Then $1=a((1-t)/a)+b(t/b)$, and the elements $(1-t)/a$ and t/b are in $(Ta+Tb)^{-1}$. Therefore the ideal $Ta+Tb$ is invertible.

For the rest of the proof assume that $q\in K\setminus\{0\}$.

$(3)\Rightarrow(4)$: If condition (3) is satisfied by $t=0$, then it is satisfied also by $t=q^{-1}\in T\setminus\{0\}$.

$(4)\Rightarrow(5)$: Choose $t\in T$ such that tq and $(1-t)/q$ are in T. Thus $t\in Tq^{-1}\cap T$ and $1-t\in Tq\cap T$. Consequently, $1\in(Tq\cap T)+(Tq^{-1}\cap T)$.

$(5)\Rightarrow(3)$: By assumption there is an element $t\in Tq^{-1}\cap T$ such that $1-t\in Tq\cap T$; so tq and $(1-t)/q$ are in T. ▮

COROLLARY 1.2. *A domain R is Prüfer if and only if for each $q\in K\setminus\{0\}$ there is an element $r_q\in R$ such that qr_q and $(1-r_q)/q$ are in R (equivalently, we may require $r_q\in R\setminus\{0\}$ for all $q\in K\setminus\{0\}$).*

Proof. Take $T=R$ in the previous proposition. ▮

Note that Corollary 1.2 immediately implies the well-known fact that an overring of a Prüfer domain is Prüfer.

DEFINITION. A domain T which contains R and satisfies the equivalent conditions of Proposition 1.1 is said to be *Prüfer over R*.

A Prüfer domain is Prüfer over each of its subrings. Moreover, a domain is Prüfer if and only if is Prüfer over itself. More generally, if $R\subseteq T$ are domains with the same quotient field, then T is Prüfer over R if and only if T is a Prüfer domain. However, if T is Prüfer over R, then $T\cap K$ is not necessarily Prüfer, even if T is a Bezout domain (indeed, if R is an arbitrary normal domain, then there is a Bezout domain T such that $R=T\cap K$ [6, Thm. (3.27)(b) and (32.15)(1)]).

If T is Prüfer over R, then a domain T_1 containing T is Prüfer over each subring of $T_1\cap K$; in particular, for each multiplicatively closed subset S of T, the domain T_S is Prüfer of $R_{R\cap S}$.

The next proposition "relatives" [6, Thm. 22.1].

PROPOSITION 1.3. *If $R\subseteq T$ are domains, the following conditions are equivalent:*

(1) *T is Prüfer over R;*

(2) *If $P\in\mathrm{Spec}(T)$, then $T_P\cap K$ is a valuation domain;*

(3) *If* $M \in \mathrm{Max}(T)$, *then* $T_M \cap K$ *is a valuation domain*;

(4) *If* $P \in \mathrm{Spec}(T)$, *then* T_P *is Prüfer over* $R_{R \cap P}$;

(5) *If* $M \in \mathrm{Max}(T)$, *then* T_M *is Prüfer over* $R_{R \cap M}$.

Proof. (1) ⇒ (2): Let $q \in K \setminus \{0\}$. There exists $t \in T$ such that tq and $(1-t)/q$ are in T. If $t \in P$, then $1/q = ((1-t)/q)/(1-t) \in T_P \cap K$; and if $t \notin P$, then $q = (tq)/t \in T_P \cap K$. Thus T_P is a valuation domain of K.

(2) ⇒ (4) ⇒ (5): Clear.

(5) ⇒ (1): Let $q \in K \setminus \{0\}$. By assumption and by Proposition 1.1(5), the ideal $(Tq \cap T) + (Tq^{-1} \cap T)$ is contained in no maximal ideal of T, and so is equal to T. By Proposition 1.1(5), T is Prüfer over R.

(2) ⇒ (3) ⇒ (5): Clear. ∎

For places and valuation domains, see [3, Chap. 6]. Following [3], if K is a field, let $\tilde{K} = K \cup \{\infty\}$. If K and L are fields and $\phi: \tilde{K} \to \tilde{L}$ is a place, we denote by V_ϕ the valuation domain associated with ϕ, that is $V_\phi = \{x \in K \mid \phi(x) \neq \infty\}$ (see [3, Ch. VI, Sect. 2, no. 3, Prop. 2]). If L_1 and L_2 are field extensions of a field K, and $\phi: \tilde{L}_1 \to \tilde{L}_2$ is a place, then ϕ is *over* K if $\phi(c) = c$ for all $c \in K$.

PROPOSITION 1.4. *Assume that* T *is Prüfer over* R, L *is a field, and* $f: T \to L$ *is a homomorphism. Then there exists a unique place* $\phi: \tilde{K} \to \tilde{L}$ *extending* $f \mid T \cap K$ (*the restriction of* f *to* $T \cap K$).

Proof. To prove existence, let $P = \ker f$. Thus P is a prime ideal of T. By Proposition 1.3, the ring $T_P \cap K$ is a valuation domain of K, and thus it determines a place $\phi: \tilde{K} \to \tilde{L}$ extending $f \mid T \cap K$ (see [3, Ch. VI, Sect. 2, no. 3]). The uniqueness is now obvious. ∎

COROLLARY 1.5. *Each homomorphism of a Prüfer domain* R *into a field* L *can be uniquely extended to a place* $\tilde{K} \to \tilde{L}$.

2. ABSOLUTE NON-ZERO-DIVISORS MODULO AN IDEAL

DEFINITION. Let I be an ideal of R and $a \in R$. Then a is called *an absolute non-zero-divisor modulo* I if a is a non-zero-divisor mod IT for every domain T containing R such that $IT \neq T$.

Denote the set of absolute non-zero-divisors (resp., non-zero-divisors; zero-divisors; units) mod I by $\mathscr{A}(R, I)$ (resp.; $\mathscr{N}(R, I)$, $\mathscr{Z}(R, I)$, $\mathscr{U}(R, I)$). The set of units in a ring T is denoted by $\mathscr{I}nv(T)$. Thus $\mathscr{I}nv(R) = \mathscr{U}(R, (0))$. If T is a ring, we denote by $\mathscr{J}(T)$ the Jacobson radical of T. Let

$\mathcal{J}(R, I)$ be the intersection of all maximal ideals containing I. (Here, we allow I to possibly be (0).) Thus $\mathcal{J}(R/I) = \mathcal{J}(R, I)/I$, $\mathcal{J}(R) = \mathcal{J}(R, (0))$, and $\mathcal{J}(R, I)$ is the largest ideal of R with the property that $\mathcal{U}(R, \mathcal{J}(R, I)) = \mathcal{U}(R, I)$.

THEOREM 2.1. *Let a be a nonzero element of R and set $\mathcal{U} = \mathcal{U}(R, I)$. The following conditions are equivalent*:

(1) $a \in \mathcal{A}(R, I)$;

(2) *a is invertible in the ring* $(IR_{\mathcal{U}} : IR_{\mathcal{U}})$;

(3) $IR_{\mathcal{U}} = aIR_{\mathcal{U}}$;

(4) *For $i \in I$, there exists some $j \in I$ such that* $(1 + j)i \in aI$;

(5) $a \in \mathcal{N}(R, I)$ *and* $IR_{\mathcal{U}} \subseteq aR_{\mathcal{U}}$;

(6) *If $x \in K$ and $ax \in I$, then* $x \in IR_{\mathcal{U}}$;

(7) *If $x \in K$ and $ax \in IR_{\mathcal{U}}$, then* $x \in IR_{\mathcal{U}}$.

Proof. (1) $\Rightarrow$ (4): Let

$$S = \{s \in R \setminus \{0\} \mid \text{condition (4) is not satisfied with } a \text{ replaced by } s\}.$$

We construct a domain $T \supseteq R$ such that $IT \neq T$ and with the property that *all* elements in S are zero-divisors mod IT. Let $\{X_s \mid s \in S\}$ be a set of independent indeterminates over R. By assumption, for each $s \in S$, we may choose an element $i_s \in I$ such that $(1 + j)\, i_s \notin sI$, for all $j \in I$. Set $T = R[\{X_s, sX_s/i_s \mid s \in S\}]$.

Let $s \in S$. Then $sX_s = (sX_s/i_s)\, i_s \in IT$. Assume that $X_s \in IT$. Set $X_r = 0$ for each $s \in S$ such that $r \neq s$ and write $X = X_s$. We obtain a relation of the form

$$X = \alpha + \beta X + \gamma(sX/i_s) + g(X),$$

where α, β, γ are in I and $g(X)$ is a linear combination with coefficients in I of terms of the form $X^m(X/i_s)^n$, $m + n \geqslant 2$. Comparing the coefficients of X yields $1 = \beta + \gamma s/i_s$; or $(1 - \beta)\, i_s = \gamma s$, contradicting the choice of i_s. Therefore, $X = X_s \notin IT$, and $s \in \mathcal{Z}(T, IT)$.

(4) $\Leftrightarrow$ (3): This follows from the fact that an element u of R is in $\mathcal{U}(R, I)$ if and only if it divides in R an element of the form $1 + i$ with $i \in I$.

(3) $\Leftrightarrow$ (2) $\Rightarrow$ (5): Straightforward.

(5) $\Rightarrow$ (6): Let $x \in K$ such that $ax \in I$. Then $ax \in IR_{\mathcal{U}} \subseteq aR_{\mathcal{U}}$, and hence $x \in R_{\mathcal{U}}$. Since a is not a zero-divisor mod $IR_{\mathcal{U}}$, we obtain that $x \in IR_{\mathcal{U}}$.

(6) ⇒ (3): For $i \in I$, we have $a(i/a) \in I$ and hence $i/a \in IR_{\mathscr{U}}$. Therefore $i \in aIR_{\mathscr{U}}$, which implies that $IR_{\mathscr{U}} = aIR_{\mathscr{U}}$.

(3) ⇒ (1): Let T be a domain containing R such that $IT \neq T$. If $t \in T$ and $at \in IT$, then $at \in IT_{\mathscr{U}} = aIT_{\mathscr{U}}$; hence $t \in IT_{\mathscr{U}}$. Thus $ut \in IT$ for some $u \in \mathscr{U}$; since $u \in T$ is a unit mod IT, we obtain that $t \in IT$. Therefore $a \in \mathscr{A}(R, I)$.

(6) ⇔ (7): Clear. ∎

We restate the equivalence (1) ⇔ (2) as follows:

COROLLARY 2.2. *Set* $\mathscr{U} = \mathscr{U}(R, I)$. *Then* $\mathscr{A}(R, I) = (\mathscr{I}nv\,(IR_{\mathscr{U}} : IR_{\mathscr{U}})) \cap R$.

COROLLARY 2.3. *For a domain R we have*

(1) *If* $I \subseteq \mathscr{J}(R))$, *then* $\mathscr{A}(R, I) = \{r \in R \mid I = rI\}$;

(2) *If R is quasi-local, then* $\mathscr{A}(R, I) = \{r \in R \mid I = rI\}$.

Proof. For part (1) use Theorem 2.1(3). Part (2) follows from part (1). ∎

PROPOSITION 2.4. *Set* $\mathscr{U} = \mathscr{U}(R, I)$, *and* $\mathscr{A} = \mathscr{A}(R, I)$. *The set* $\mathscr{A}$ *is the largest multiplicative subset of R with the property* $IR_{\mathscr{A}} = IR_{\mathscr{U}}$.

Proof. Since $\mathscr{U} \subseteq \mathscr{A}$, we have $IR_{\mathscr{U}} \subseteq IR_{\mathscr{A}}$. By Theorem 2.1(3), we have $\mathscr{A} = \{a \in R \mid I \subseteq aIR_{\mathscr{U}}\}$, and so the set $\mathscr{A}$ is the largest multiplicative set with the property $IR_{\mathscr{A}} \subseteq IR_{\mathscr{U}}$. The proposition follows. ∎

Note that for the domain T constructed in the proof of Theorem 2.1 [(1) ⇒ (4)] and for each ideal L of R, we have $LT = L$ (to obtain this, set each $X_s = 0$). In particular, $IT \cap R = I$, thus $R/I \subseteq T/IT$ canonically. Hence an absolute non-zero-divisor mod I can be characterized as an element a having the property that $a + I$ is a non-zero-divisor in each ring extension of R/I of the form T/IT with $IR \cap R = I$. It turns out that the restriction to this form of ring extensions is essential. Indeed, for a ring E, an element in E is a non-zero-divisor in each ring extension of E if and only if is a unit in E. To show this, let $\underline{X} = \{X_a \mid a$ is a nonzero nonunit in $E\}$ be a set of independent indeterminates over E. Set $F = \underline{E}[\underline{X}]/L$, where L is the ideal generated by $\{aX_a \mid a$ is a nonzero nonunit in $R\}$. Clearly $E \subseteq F$ canonically, and each nonunit of E is a zero-divisor in F.

The domain T constructed above also satisfies the property $\mathscr{N}(T, IT) \cap R = \mathscr{A}(T, IT) \cap R = \mathscr{A}(R, I)$. However, for general domains $R \subseteq T$, if I is an ideal of R such that $IT \neq T$, we can state just that $\mathscr{A}(R, I) \subseteq \mathscr{A}(T, IT) \cap R$. This inclusion may be proper, even if $IT \cap R = I$, $\mathscr{Q}(R) = \mathscr{Q}(T)$ and I is a principal ideal. For example, let $R = k[X, Y]$ be the polynomial ring over a field k, $I = XR$, and $T = k[X, Y, 1/Y]$. Then $R \subseteq T$, $IT \neq T$, and Y is invertible in T (and hence is in $\mathscr{A}(T, IT) \cap R$). Since $\mathscr{U}(R, I) = k \backslash \{0\}$, we have $\mathscr{R}_{\mathscr{U}} = R$, and so $IR_{\mathscr{U}} \neq YIR_{\mathscr{U}}$. By Theorem 2.1(3), $Y \notin \mathscr{A}(R, I)$.

We have the obvious inclusions

$$\mathscr{U}(R, I) \subseteq \mathscr{A}(R, I) \subseteq \mathscr{N}(R, I). \tag{2-1}$$

Generally, both inclusions may be strict, as shown in the following example.

EXAMPLE 2.5. A domain R having a prime ideal I such that $\mathscr{U}(R, I) \subsetneq \mathscr{A}(R, I) \subsetneq \mathscr{N}(R, I)$.

Let X, Y and Z be independent indeterminates over a field k. Set $R = k[Y, Z, \{X/Y^n \mid n \geqslant 0\}]$, and let I be the ideal generated by X/Y^n for $n \geqslant 0$. We have $R/I \cong k[Z]$, so I is a prime ideal. Since $I = YI$, we obtain by Theorem 2.1(3) that $Y \in \mathscr{A}(R, I)$. Clearly $\mathscr{U}(R, I) = k \backslash \{0\}$, and $I \neq ZI$. Using Theorem 2.1(3) again, we obtain $Z \notin \mathscr{A}(R, I)$. Thus $Y \in \mathscr{A}(R, I) \backslash \mathscr{U}(R, I)$, and $Z \in \mathscr{N}(R, I) \backslash \mathscr{A}(R, I)$; hence $\mathscr{U}(R, I) \subsetneq \mathscr{A}(R, I) \subsetneq \mathscr{N}(R, I)$.

Note that if I is a maximal ideal of R, then $\mathscr{U}(R, I) = \mathscr{N}(R, I)$, and so both inclusions in (2-1) become equalities.

For certain classes of domains, one of the inclusions in (2-1) becomes an equality (see Theorem 2.6 and Proposition 2.8 below).

THEOREM 2.6. *For each ideal I of a Prüfer domain R, we have $\mathscr{A}(R, I) = \mathscr{N}(R, I)$.*

Proof. Let $a \in \mathscr{N}(R, I)$, and $i \in I$. There is an element $j \in R$ such that $j(a/i)$ and $(1 + j)(i/a)$ are in R (Corollary 1.2). Since $ja = j(a/i)i \in I$, we have $j \in I$. Also, $(1 + j)i = ar \in I$ for some $r \in R$, hence $r \in I$, and $(1 + j)i \in aI$. By Theorem 2.1(4), $a \in \mathscr{A}(R, I)$. ∎

Using Proposition 2.4 and Theorem 2.6 we obtain

COROLLARY 2.7. *Let R be a Prüfer domain. Set $\mathscr{U} = \mathscr{U}(R, I)$, $\mathscr{A} = \mathscr{A}(R, I)$, and $\mathscr{N} = \mathscr{N}(R, I)$. Then $IR_{\mathscr{U}} = IR_{\mathscr{A}} = IR_{\mathscr{N}}$.*

A domain R is *archimedean* if $\bigcap_{n=0}^{\infty} a^n R = (0)$ for every nonunit $a \in R$; see [15].

PROPOSITION 2.8. *Let R be a domain such that every quotient ring of R is archimedean. Then $\mathscr{A}(R, I) = \mathscr{U}(R, I)$ for each ideal I of R.*

Proof. Let I be an ideal of R, and let $\mathscr{U} = \mathscr{U}(R, I)$. If $a \in \mathscr{A}(R, I)$, then $aIR_{\mathscr{U}} = IR_{\mathscr{U}}$. Thus $(0) \neq IR_{\mathscr{U}} \subseteq \bigcap_{n=0}^{\infty} a^n R_{\mathscr{U}}$. Since $R_{\mathscr{U}}$ is archimedean, it follows that a is invertible in $R_{\mathscr{U}}$; that is, $a \in \mathscr{U}$. ∎

It is not clear if the converse of Proposition 2.8 is true. However, we have

PROPOSITION 2.9. *If $\mathcal{A}(R, I) = \mathcal{U}(R, I)$ for each ideal I of R, then R is archimedean.*

Proof. (cf. Example 2.5 above). Let a and b be nonzero elements of R such that $a \in \bigcap_{n=1}^{\infty} b^n R$. Then $a/b^n \in R$ for all $n \geqslant 1$. Let I be the ideal generated by $\{a/b^n \mid n \geqslant 0\}$. Since $I = bI$, we obtain that $b \in \mathcal{A}(R, I) = \mathcal{U}(R, I)$. Thus $Rb = I + Rb = R$. It follows that b is a unit in R, and so R is archimedean. ∎

A domain is *Mori* in case it satisfies the ascending chain condition on (integral) divisorial ideals; in particular, a noetherian domain is Mori. Examples of domains which satisfy the assumption of Proposition 2.9 are one-dimensional domains [12, Coro. 1.4], and Mori domains (see [2, Sect. 3] and [13, Sect. 3, Coro. 1]).

For one-dimensional Prüfer domains or one-dimensional quasi-local domains we have

$$\mathcal{U}(R, I) = \mathcal{N}(R, I)$$

and so, in (2-1) we have equalities throughout. Thus Theorem 2.6 does not characterize Prüfer domains.

PROPOSITION 2.10. *Let I be a finitely generated ideal of R. Then $\mathcal{A}(R, I) = \mathcal{U}(R, I)$.*

Proof. Set $\mathcal{U} = \mathcal{U}(R, I)$. Let $a \in \mathcal{A}(R, I)$. By Theorem 2.1(3), $aIR_{\mathcal{U}} = IR_{\mathcal{U}}$. Since I is finitely generated, by [11, Ch. I, (3.10)] a is a unit in $R_{\mathcal{U}}$, that is $a \in \mathcal{U}$. ∎

COROLLARY 2.11. *Let a and b be nonzero nonunits in R. The following conditions are equivalent:*

(1) $a \in \mathcal{A}(R, bR)$;

(2) $b \in \mathcal{A}(R, aR)$;

(3) $aR + bR = R$.

3. ON $(R : I)$

In this section we deal with the R-module $(R : I)$ in the general context of seminormal domains. We use the fact that a domain R is *seminormal* if and only if for each $x \in K$, if $x^n \in R$ for $n \gg 0$, then $x \in R$. See [14] and [16] for seminormality in a general setting.

THEOREM 3.1. *If R is a seminormal domain, then the ring $(\sqrt{I}:\sqrt{I})$ is seminormal and*

$$(\sqrt{I}:\sqrt{I}) = \{x \in K \mid x^n \in (R:I) \text{ for all } n \geqslant 1\}$$
$$= \{x \in K \mid x^n \in (R:I) \text{ for } n \gg 0\}.$$

Proof. Let $x \in K$ such that $x^n \in (R:I)$ for $n \gg 0$. If $t \in \sqrt{I}$, then $t^m \in I$ for $m \gg 0$. Hence, for $m \gg 0$, we obtain that $(xt)^m \in R$, and by seminormality that $xt \in R$. Also $(xt)^{m+1} = (x^{m+1}t^m)t \in \sqrt{I}$. Thus $xt \in \sqrt{I}$, and consequently $x \in (\sqrt{I}:\sqrt{I})$. This implies both above equalities.

To show that $(\sqrt{I}:\sqrt{I})$ is seminormal, let $x^n \in (\sqrt{I}:\sqrt{I})$ for $n \gg 0$. Thus $x^n \in (R:I)$ for $n \gg 0$, and so $x \in (\sqrt{I}:\sqrt{I})$. ∎

COROLLARY 3.2. *If R is a seminormal domain, and I is a radical ideal, then the ring $(I:I)$ is seminormal.*

We wonder to what extent the hypothesis on I being radical in Corollary 3.2 can be relaxed.

Another corollary of Theorem 3.1 is the following theorem.

THEOREM 3.3. *If R is a seminormal domain, then $(\sqrt{I}:\sqrt{I})$ is the largest subring of $(R:I)$.*

In contrast to the seminormal case, in general the union of all subrings of $(R:I)$ may not be a ring, even if I is a principal ideal. For example, consider the ring $k[\{X^nZ \mid n \geqslant 0\}, \{Y^nZ \mid n \geqslant 0\}]$, where k is a field , and X, Y, Z are independent indeterminates over k. Set $I = ZR$. For every $n \geqslant 0$, X^n and Y^n are in $(R:I)$; hence the rings $k[X]$ and $k[Y]$ are contained in $(R:I)$. Nevertheless, $XY \notin (R:I)$ since $XYZ \notin R$. By Theorem 3.1, R is not seminormal. Indeed, $XYZ \notin R$, although $(XYZ)^n \in R$ for all $n \geqslant 2$.

Let $I_v = (I^{-1})^{-1}$. If R is seminormal, then

$$(I:I) \subseteq (I_v:I_v) \subseteq (\sqrt{I}:\sqrt{I}) \subseteq (R:I).$$

If in addition $(R:I)$ is a ring, using [8, Prop. 1.2], we obtain

$$(I:I) \subseteq (I_v:I_v) = (\sqrt{I}:\sqrt{I}) = (R:\sqrt{I}) = (R:I).$$

It is not clear if $(I_v:I_v) = (\sqrt{I}:\sqrt{I})$, assuming just that R is seminormal.

Concerning [8, Prop. 1.2], note that for arbitrary R and I, we have $(II^{-1}:II^{-1}) = (I_v:I_v)$.

From Theorem 3.3 we obtain

COROLLARY 3.4. *Let R be a seminormal domain. Then:*

(1) *$(R:I)$ is a ring if and only if* $(R:I)=(\sqrt{I}:\sqrt{I})$;

(2) *If $(R:I)$ is a ring, then $(R:I)$ is seminormal.*

Corollary 3.4(1) is false for a general domain R. For example, let $R=k[[X^2, X^5]]$, where X is an indeterminate over the field k, and $I=(X^4, X^5)$. Thus $X^n \in I$ for all $n \geqslant 4$. It is easy to see that $(R:I)=k[[X]]$, $\sqrt{I}=(X^2, X^5)$ and $(\sqrt{I}:\sqrt{I})=k[[X^2, X^3]]$. Therefore $(\sqrt{I}:\sqrt{I}) \neq (R:I)$, although $(R:I)$ is a ring. Note that R is noetherian, quasi-local, and one-dimensional, $\sqrt{I}$ is a maximal ideal, and $(R:I)$ is a discrete valuation domain.

Corollary 3.4(1) holds for an arbitrary domain R if the ideal I is radical [1, Prop. 3.3(1)].

The next two lemmas lead to a characterization as to when, for seminormal domains, $(R:I)=\mathscr{T}(I)$ (Theorem 3.7).

LEMMA 3.5. *Let R be a seminormal domain and $I_1 \subseteq I_2$ ideals of R with the same radical. If $(R:I_1)$ is a ring, then* $(R:I_1)=(R:I_2)=(\sqrt{I_1}:\sqrt{I_1})$.

Proof. By Corollary 3.4(1),

$$(R:I_1)=(\sqrt{I_1}:\sqrt{I_1})=(\sqrt{I_2}:\sqrt{I_2}) \subseteq (R:I_2) \subseteq (R:I_1).$$

Thus we have equality everywhere. ∎

For the next lemma, recall that $I^0=R$.

LEMMA 3.6. *For a domain R, if $(R:I^m)=(R:I^{m+1})$ for some $m \geqslant 0$, then $(R:I^n)=(R:I^m)$ for all $n \geqslant m$.*

Proof. We proceed by induction on $n \geqslant m$ using the equality $(R:I^{n+1})=((R:I^n):I)$. ∎

THEOREM 3.7. *For a seminormal domain R, the following conditions are equivalent*:

(1) $(R:I)=\mathscr{T}(I)$;

(2) *$(R:I^n)$ is a ring for some $n \geqslant 2$*;

(3) *$(R:I^n)$ is a ring for all $n \geqslant 1$.*

Proof. Obviously, (1) ⇒ (3) ⇒ (2).

(2) ⇒ (1): Lemma 3.5 implies that $(R:I^m)=(\sqrt{I}:\sqrt{I})$ for $1 \leqslant m \leqslant n$. In particular, $(R:I)=(R:I^2)$. By Lemma 3.6, $(R:I^n)=(R:I)$ for all $n \geqslant 1$, that is $(R:I)=\mathscr{T}(I)$. ∎

Theorem 3.7 generalizes [5, Thm. 3.1(e)].

As shown by the next example, the implication (2)$\Rightarrow$(3) in the Theorem 3.8 fails if we allow $n = 1$ in (2), even when R is a Prüfer domain and I is a prime ideal of R (see also [5]).

EXAMPLE 3.8. A Prüfer domain R having a prime ideal P such that $(R : P)$ is a ring, but $(R : P) \neq \mathcal{T}(P)$.

Let $R = \mathbb{Z} + X\mathbb{Q}[X]$, and $P = X\mathbb{Q}[X]$. By [4, Coro. 4.15], R is Prüfer. We have $(R : P) = \mathbb{Q}[X]$ and $(R : P^2) = (1/X)\,\mathbb{Q}[X]$. Hence, $1/X \in (R : P^2) \backslash (R : P)$, and so $(R : P) \neq \mathcal{T}(P)$.

COROLLARY 3.9. *Let R be a seminormal domain. Then $(R : I) \neq \mathcal{T}(I)$ if and only if*

$$R \subsetneq (R : I) \subsetneq (R : I^2) \subsetneq \cdots$$

Proof. Assume that $(R : I) \neq \mathcal{T}(I)$. By Theorem 3.7, $(R : I^n)$ is not a ring for $n \geq 2$. By Lemma 3.6, we obtain $R \subsetneq (R : I) \subsetneq (R : I^2) \subsetneq \cdots$

The converse is obvious. ∎

4. ON $(I : I)$

Let I be an ideal of a domain R. This section is concerned with representing $(I : I)$ as an intersection of localizations of R at prime ideals. General results are given. However, the Prüfer hypothesis on a domain yields the best results.

Denote $\mathcal{U}(R, I)$ by $\mathcal{U}$, $\mathcal{N}(R, I)$ by $\mathcal{N}$, and $\mathcal{A}(R, I)$ by $\mathcal{A}$. The sets $\mathcal{U}$, $\mathcal{A}$, and $\mathcal{N}$ are saturated multiplicative sets. The set of prime ideals, all of whose elements are zero-divisors modulo I, is denoted by $\mathbf{Z}(R, I)$. The elements of $\mathbf{Z}(R, I)$ are called *prime divisors* of I. The set of maximal (resp., minimal) elements of $\mathbf{Z}(R, I)$ is denoted by $\mathbf{MZ}(R, I)$ (resp., by $\mathbf{mZ}(R, I)$). Similar notation is used for each set $\mathbf{C} \subseteq \operatorname{Spec}(R)$; e.g., $\mathbf{MC}$ is the set of ideals that are maximal in $\mathbf{C}$. The set of maximal ideals containing I is denoted by $\mathbf{M}(R, I)$, and the set of maximal ideals not containing I, by $\mathbf{M}'(R, I)$.

The prime ideals which are disjoint from $\mathcal{A}$ are called *weak prime divisors of I*. The set of these ideals is denoted by $\mathbf{W}(R, I)$. If R is Prüfer, then $\mathbf{W}(R, I) = \mathbf{Z}(R, I)$ by Theorem 2.6. If R is Mori, then $\mathbf{W}(R, I)$ is the set of prime ideals containing I by Proposition 2.8.

For a multiplicative subset S of R, we denote by $\mathbf{M}(S) = \mathbf{M}(R, S)$ the set of maximal elements in the family of ideals not intersecting S. Thus $\mathbf{M}(\mathcal{U}) = \mathbf{M}(R, I)$, $\mathbf{M}(\mathcal{A}) = \mathbf{MW}(R, I)$, and $\mathbf{M}(\mathcal{N}) = \mathbf{MZ}(R, I)$.

We denote by $\mathscr{C} = \mathscr{C}(R, I)$ the ring

$$\bigcap \{IR_M \mid M \in \mathbf{M}'(R, I)\}$$

(if $\mathbf{M}'(R, I) = \varnothing$, that is, if $I \subseteq \mathscr{J}(R)$, then $\mathscr{C} = K$.)

A basic result for this section is the following well-known theorem.

THEOREM 4.0 ([cf., e.g., [6, Coro. 4.6 and Thm. 4.10]). *For a multiplicative subset S of R, the ideals in* $\mathbf{M}(S)$ *are prime, and for each fractional ideal L we have*

$$LR_S = \bigcap \{LR_Q \mid Q \in \mathbf{M}(S)\}.$$

COROLLARY 4.1. $I = R_{\mathscr{U}} \cap \mathscr{C}$.

Proof. The corollary follows from the previous theorem, since $IR_M = R_M$ for $M \in \mathbf{M}'(R, I)$. ∎

LEMMA 4.2. *Let R be a domain, F and L fractional ideals of R, and* $\mathscr{S}$ *a family of multiplicative subsets of R such that* $F = \bigcap_{S \in \mathscr{S}} FR_S$. *Then*

$$(F : L) = \bigcap_{S \in \mathscr{S}} (F : L) R_S = \bigcap_{S \in \mathscr{S}} (FR_S : LR_S).$$

Proof. We have $(F : L) = (\bigcap FR_S) \colon L = \bigcap (FR_S : L) = \bigcap (FR_S : LR_S)$. The lemma follows. ∎

PROPOSITION 4.3. *Let S be a multiplicative subset of R such that* $\mathscr{U} \subseteq S \subseteq \mathscr{A}$. *Then*:

(1) $IR_{\mathscr{U}} = IR_S$;

(2) $(I : I) = (IR_{\mathscr{U}} : IR_{\mathscr{U}}) \cap \mathscr{C} = (\bigcap \{(IR_M : IR_M) \mid M \in \mathbf{M}(S)\}) \cap \mathscr{C}$;

(3) $R_S \cap \mathscr{C} \subseteq (I : I)$;

(4) $(I : I) = R_S \cap \mathscr{C}$ *if and only if* $(I : I) \subseteq R_S$;

(5) $R_S \cap \mathscr{C} = R_S \cap (R : I) = R_S \cap (I : I) = R_S \cap \mathscr{T}(I)$
$= R_S \cap (\sqrt{I} : \sqrt{I})$.

Proof. (1) follows from Proposition 2.4.

(2) For $M \in \mathbf{M}'(R, I)$, we have $(IR_M : IR_M) = R_M$. Hence, by Corollary 4.1, Theorem 4.0 and Lemma 4.2, we obtain

$$\begin{aligned}(I : I) &= (IR_{\mathscr{U}} : IR_{\mathscr{U}}) \cap \mathscr{C} = (IR_S : IR_S) \cap \mathscr{C} \\ &= \left(\bigcap \{(IR_M : IR_M) \mid M \in \mathbf{M}(S)\}\right) \cap \mathscr{C}.\end{aligned}$$

(3) follows from (2) since $R_S \subseteq (IR_S : IR_S) = (IR_{\mathscr{U}} : IR_{\mathscr{U}})$.

(4) follows from (3) since $(I : I) \subseteq \mathscr{C}$.

(5) follows from (3) since $\mathscr{T}(I) \subseteq \mathscr{C}$, $\mathscr{U}(R, \sqrt{I}) = \mathscr{U}(R, I)$ and $\mathscr{C}(R, \sqrt{I}) = \mathscr{C}(R, I)$. ∎

COROLLARY 4.4. $R_{\mathscr{A}} \cap \mathscr{C} \subseteq (I : I)$.

Note that $R_{\mathscr{N}} \cap (R : I) \subseteq (I : I)$ for an arbitrary domain R.
Since $\mathscr{A} = \mathscr{N}$ for R Prüfer (Theorem 2.6), we obtain

COROLLARY 4.5. *Let I be an ideal of a Prüfer domain R. Then*:

(1) $I = IR_{\mathscr{N}} \cap \mathscr{C}$;

(2) $R_{\mathscr{N}} \cap \mathscr{C} \subseteq (I : I)$;

(3) $(I : I) = R_{\mathscr{N}} \cap \mathscr{C}$ *if and only if* $(I : I) \subseteq R_{\mathscr{N}}$.

LEMMA 4.6. *If S is a multiplicative subset of R containing $\mathscr{U}$, then each ideal in $\mathbf{M}(S)$ contains I.*

Proof. We have $\mathscr{U}(R_S, IR_S) \subseteq \mathscr{U}(R_S)$, that is $IR_S \subseteq \mathscr{J}(R_S)$. The lemma follows. ∎

For a Prüfer domain R and a prime ideal Q containing I, we denote by $G(Q) = G(R, Q, I)$ the unique prime ideal of R such that $G(Q)\, R_Q = \mathscr{Z}(R_Q, IR_Q)$. Thus $G(Q) = \mathscr{Z}(R_Q, IR_Q) \cap R$.

THEOREM 4.7. *Let R be a Prüfer domain and S a multiplicative subset of R, such that $\mathscr{U} \subseteq S \subseteq \mathscr{N}$. Then*:

(1) $(I : I) = \bigcap \{R_{G(M)} \mid M \in \mathbf{M}(S)\} \cap \mathscr{C}$;

(2)
$$\begin{aligned}(I : I) &= \bigcap \{R_{G(Q)} \mid Q \in \mathbf{MZ}(R, I)\} \cap \mathscr{C}\\ &= \bigcap \{R_{G(Q)} \mid Q \in \mathbf{Z}(R, I)\} \cap \mathscr{C}\\ &= \bigcap \{R_{G(M)} \mid M \in \mathbf{M}(R, I)\} \cap \mathscr{C}.\end{aligned}$$

Proof. (1) By Lemma 4.6, $G(M)$ is defined for $M \in \mathbf{M}(S)$. Since $\mathscr{A} = \mathscr{N}$, we obtain by Proposition 4.3(2)

$$(I : I) = \bigcap \{(IR_M : IR_M) \mid M \in \mathbf{M}(S)\} \cap \mathscr{C}.$$

By [7, Lemma 1.3], $(IR_M : IR_M) = R_{G(M)}$ for all M.

(2) follows from (1) applied first to $S = \mathscr{N}$, and then, to $S = \mathscr{U}$. ∎

The intersection $\bigcap \{R_{G(M)} \mid M \in \mathbf{M}(S)\}$ in Theorem 4.7(1) has a natural meaning: it equals $(IR_{\mathscr{U}} : IR_{\mathscr{U}})$. Indeed, $IR_{\mathscr{U}} = IR_S = \bigcap \{IR_M \mid M \in \mathbf{M}(S)\}$, so the assertion follows from the proof of Theorem 4.7(1). In particular,

$$\bigcap \{R_{G(Q)} \mid Q \in \mathbf{MZ}(R, I)\} = (IR_{\mathscr{U}} : IR_{\mathscr{U}}).$$

We now turn to the question under which assumptions on R or on I, we may replace $G(Q)$ by Q in Theorem 4.7(2) (see Theorem 4.11 below). This is equivalent to asking, when the inclusion $R_{\mathscr{N}} \cap \mathscr{C} \subseteq (I : I)$ becomes an equality, or, when $(I : I)$ is contained in $R_{\mathscr{N}}$. We need some preliminary lemmas.

LEMMA 4.8. *Let a and b be nonzero elements in R and $q = a/b$. Then*

$$Rq^{-1} \cap R \subseteq R(1-a) + Rb; \text{ and if } (1-a)q \in R,$$

then

$$Rq^{-1} \cap R = R(1-a) + Rb.$$

Proof. Let $x \in Rq^{-1} \cap R$. Then $x = x(1-a) + (xq)b \in R(1-a) + Rb$. For the second part, assume that $(1-a)q \in R$; so $1 - a \in Rq^{-1} \cap R$. Since $bq \in R$, we obtain $Rq^{-1} \cap R = R(1-a) + Rb$. ▮

LEMMA 4.9. *If R is a Prüfer domain and $q \in K \setminus \{0\}$, then there are elements a and b in $R \setminus \{0\}$ such that $q = a/b$ and $Rq^{-1} \cap R = R(1-a) + Rb$.*

Proof. By Corollary 1.2, there is a nonzero element $a \in R$ such that a/q and $(1-a)q$ are in R. Let $b = a/q$. Then $b \in R$, and $q = a/b$. By Lemma 4.8, $Rq^{-1} \cap R = R(1-a) + Rb$. ▮

LEMMA 4.10. *If R is a Prüfer domain, x is a nonzero element in $(I : I)$, and $L = Rx^{-1} \cap R$, then $I = LI$.*

Proof. By Lemma 4.9, there are nonzero elements a and b in R such that $x = a/b$ and $L = (1-a)R + bR$. If $i \in I$, then $i = (1-a)i + ia = i(1-a) + b(xi) \in LI$. ▮

A domain R is a QR-*domain* if each overring of R is a quotient ring of R. Each QR-domain is Prüfer, and each Bezout domain is a QR-domain (see [6, Sect. 27]).

We say that an ideal I *has no embedded primes* if each prime ideal $P \in \mathbf{Z}(R, I)$ is minimal over I. Clearly, this property is equivalent to each of the following conditions:

(1) $\mathscr{X}(R, I) = \mathscr{X}(R, \sqrt{I})$;

(2) $\mathbf{Z}(R, I) = \mathbf{Z}(R, \sqrt{I})$;

(3) $\mathscr{N}(R, I) = \mathscr{N}(R, \sqrt{I})$.

THEOREM 4.11. *We have*

$$(I : I) = R_{\mathscr{N}} \cap \mathscr{C}$$

under each of the following assumptions:

(1) *R is a* QR-*domain*;

(2) *R is Prüfer and I has no embedded primes.*

Moreover, under assumption (2), *we have* $(I : I) = (\sqrt{I} : \sqrt{I})$.

Proof. (1) We show that $(I : I) \subseteq R_{\mathscr{N}}$. Let x be a nonzero element of R, and set $L = Rx^{-1} \cap R$. By Lemma 4.9, L is generated by two elements. By [6, Thm. 27.5], there exists an element $t \in R$ such that $\sqrt{L} = \sqrt{tR}$. Thus $L^m \subseteq tR$ for some $m \geq 1$. By Lemma 4.10, $I = L^m I \subseteq tI$. This implies that $t \in \mathscr{A}(R, I) = \mathscr{N}(R, I)$. For some $n \geq 1$, we have $t^n \in L$. Therefore $t^n x \in R$, so $x \in R_{\mathscr{N}}$.

(2) follows from Theorem 4.7(2), for then $G(Q) = Q$ for all $Q \in \mathbf{MZ}(R, I)$. ∎

The next two propositions provide conditions which ensure that I has no embedded primes.

PROPOSITION 4.12. *For a domain R, an ideal I has no embedded primes under each of the following assumptions*:

(1) *R is one-dimensional*;

(2) *I is radical*;

(3) *I has just one prime divisor* (*for example if I is primary*).

For the next proposition, note that the following conditions are equivalent:

(1) $I \subseteq \mathscr{J}(R)$;

(2) $\mathscr{U}(R, I) = \mathscr{U}(R)$;

(3) $\mathscr{U}(R, I) \subseteq \mathscr{U}(R)$;

(4) $\mathbf{M}(R, I) = \mathrm{Max}(R)$.

Also the following conditions are equivalent:

(1) Each nonunit of R is a zero-divisor mod I;

(2) $\mathcal{N}(R, I) = \mathcal{U}(R)$;

(3) $\mathcal{N}(R, I) \subseteq \mathcal{U}(R)$;

(4) $\mathbf{Z}(R, I) = \operatorname{Spec}(R)$;

(5) $\mathbf{MZ}(R, I) = \operatorname{Max}(R)$.

PROPOSITION 4.13. *If R is a Prüfer domain, then I has no embedded primes under each of the following assumptions*:

(1) $(R_{\mathcal{U}} : IR_{\mathcal{U}}) = (IR_{\mathcal{U}} : IR_{\mathcal{U}})$;

(2) $I \subseteq \mathcal{J}(R)$ *and* $(R : I) = (I : I)$;

(3) *Each nonunit of R is in* $\mathcal{Z}(R, I)$ *and* $(R : I) = (I : I)$.

Proof. (1) By Corollary 3.4(1),

$$(IR_{\mathcal{U}} : IR_{\mathcal{U}}) = (\sqrt{IR_{\mathcal{U}}} : \sqrt{IR_{\mathcal{U}}}) = (\sqrt{I}R_{\mathcal{U}} : \sqrt{IR_{\mathcal{U}}}).$$

Clearly $\mathcal{U}(R, \sqrt{I}) = \mathcal{U}(R, I) = \mathcal{U}$. By Corollary 2.2, $\mathcal{A}(R, I) = \mathcal{A}(R, \sqrt{I})$, and by Theorem 2.6, $\mathcal{N}(R, I) = \mathcal{N}(R, \sqrt{I})$. Therefore, I has no embedded primes.

(2) follows from (1) since $\mathcal{U}(R, I) = \mathcal{U}(R)$.

(3) follows from (2) since $I \subseteq \mathcal{J}(R)$. ∎

COROLLARY 4.14. *If R is a Prüfer domain and I is a radical ideal, then* $(I : I) = R_{\mathcal{N}} \cap \mathcal{C}$.

COROLLARY 4.15. *If R is a Prüfer domain, then* $(\sqrt{I} : \sqrt{I}) = \bigcap \{R_P \mid P \in \mathbf{mZ}(R, I)\} \cap \mathcal{C}$.

PROPOSITION 4.16. *Let I be an ideal of a Prüfer domain R such that each nonunit of R is a zero-divisor* mod I. *The following conditions are equivalent*:

(1) $(R : I)$ *is a ring*;

(2) $(R : I) = R$;

(3) $(R : I) = (I : I)$;

(4) $(R : I) = \mathcal{T}(I)$;

(5) $(R : I^n)$ *is a ring for some* $n \geqslant 1$.

Moreover, if the above equivalent conditions hold, then $(I : I) = R$.

Proof. In view of Theorem 3.7, it is enough to prove the implication $(1) \Rightarrow (2)$:

By Proposition 4.13(2), I has no embedded primes, so $\mathbf{Z}(R, \sqrt{I}) = \mathbf{Z}(R, I)$, and by assumption, $\mathbf{Z}(R, I) = \text{Spec}(R)$. Hence, $\mathbf{Z}(R, \sqrt{I}) = \text{Spec}(R)$. By Corollary 4.14, $(\sqrt{I} : \sqrt{I}) = R$, and by Corollary 3.4(1), $(R : I) = (\sqrt{I} : \sqrt{I}) = R$, and by Corollary 3.4(1), $(R : I) = (\sqrt{I} : \sqrt{I}) = R$. ▮

COROLLARY 4.17. *Let I be an ideal of a Prüfer domain R such that each nonunit of R is a zero-divisor* mod I. *If $(R : I) \neq R$, then*

$$(I : I) \subsetneq (R : I) \subsetneq (R : I^2) \subsetneq \cdots$$

Proof. By Proposition 4.16, we have $(I : I) \subsetneq (R : I)$ and by Corollary 3.9, $(R : I) \subsetneq (R : I^2) \subsetneq \cdots$ ▮

Our next goal is to characterize the family $\{G(Q) \mid Q \in Z(R, I)\}$ occurring in Theorem 4.7(2) (see Proposition 4.19 and Theorem 4.20 below).

LEMMA 4.18. *Let S be a multiplicative set of R such that $I \cap S = \emptyset$. Then*:

(1) $\mathscr{Z}(R_S, IR_S) \cap R \subseteq \mathscr{Z}(R, I)$;

(2) *If $S_1 \subseteq S_2$ are multiplicative subsets of R not intersecting I, then* $\mathscr{Z}(R_{S_2}, IR_{S_2}) \cap R_{S_1} \subseteq \mathscr{Z}(R_{S_1}, IR_{S_1})$;

(3) *If $P \subseteq Q$ are prime ideals containing I, then $\mathscr{Z}(R_P, IR_P) \cap R_Q \subseteq \mathscr{Z}(R_Q, IR_Q)$.*

Proof. (1) is immediate and the implications (1) ⇒ (2) ⇒ (3) are obvious. ▮

PROPOSITION 4.19. *Let P be a prime ideal containing I. The following conditions are equivalent*:

(1) $PR_P = \mathscr{Z}(R_P, IR_P)$;

(2) *For some prime ideal Q containing P, we have $P \subseteq \mathscr{Z}(R_Q, IR_Q)$*;

(3) *For each prime ideal Q containing P, we have $P \subseteq \mathscr{Z}(R_Q, IR_Q)$.*

Proof. (1) ⇒ (3) Let Q be a prime ideal containing I. By Lemma 4.18, we have $P \subseteq \mathscr{Z}(R_P, IR_P) \cap R_Q \subseteq \mathscr{Z}(R_Q, IR_Q)$.
The implications (3) ⇒ (2) ⇒ (1) are trivial. ▮

We denote by $\mathbf{G}(R, I)$ the set of prime ideals containing I which satisfy the equivalent conditions of Proposition 4.19.

It follows from Proposition 4.19 that if $P \subseteq Q$ are in $\text{Spec}(R)$, and $Q \in \mathbf{G}(R, I)$, then $P \in \mathbf{G}(R, I)$. Note that each ideal in $\mathbf{G}(R, I)$ is a Nagata prime of I [11, Sect. 7], although the converse fails even if R is Prüfer. Indeed, by the remarks below, the converse would imply that $G(Q) = Q$ for each $Q \in \mathbf{MZ}(R, I)$, and so that $(I : I) = R_{\mathcal{N}} \cap \mathscr{C}$ for each ideal I in a Prüfer domain R, contradicting Example 7.2.

Assume that R is a Prüfer domain. If Q is a prime ideal containing I, then $Q \in \mathbf{G}(R, I)$ if and only if $Q = G(Q)$. Hence, for every prime ideal Q containing I, we have $GG(Q) = G(Q)$; also, by Lemma 4.18(3), if $Q_1 \subseteq Q_2$ are prime ideals containing I, then $G(Q_1) \subseteq G(Q_2)$. The function $G(Q) = G(R, Q, I)$ is not necessarily one-to-one even if restricted to $\mathbf{M}(R, I)$. Indeed, let R be a Prüfer domain having a nonzero prime ideal I contained in at least two distinct maximal ideals (explicitly, in Example 3.8, the prime ideal $P = X\mathbb{Q}[X]$ of the Prüfer domain $R = \mathbb{Z} + X\mathbb{Q}[X]$ is contained in infinitely many maximal ideals; namely, $\mathbf{M}(R, P) = \{nR \mid n \geqslant 2\}$). For each prime ideal Q containing I, we have $G(Q) = I$, so the restriction $G \mid \mathbf{M}(R, I)$ is not one-to-one. We leave open the question whether $G \mid \mathbf{MZ}(R, I)$ is one-to-one.

THEOREM 4.20. *Let R be a Prüfer domain, and S a multiplicative subset of R, such that $\mathscr{U} \subseteq S \subseteq \mathcal{N}$. Then:*

(1) $\mathbf{G}(R, I) = \{P \in \operatorname{Spec}(R) \mid P \subseteq G(M)$ *for some* $M \in \mathbf{M}(S)\}$;

(2) $(I : I) = (\bigcap \{R_P \mid P \in \mathbf{G}(R, I)\}) \cap \mathscr{C}$.

Proof. (1) If $P \in \mathbf{G}(R, I)$, then $P \subseteq \mathscr{Z}(R, I)$. Since $S \subseteq \mathcal{N}$, we obtain that $P \subseteq M$ for some $M \in \mathbf{M}(S)$. Hence $P \subseteq G(M)$, and (1) follows.

(2) follows from (1) and from Theorem 4.7(1). ∎

PROPOSITION 4.21. *Let R be a Prüfer domain. Then:*

(1) *For each multiplicative set S containing $\mathscr{U}$,*

$$\mathbf{MG}(R, I) \subseteq \{G(M) \mid M \in \mathbf{M}(S)\};$$

(2) We have

$$\mathbf{MG}(R, I) \subseteq \{G(Q) \mid Q \in \mathbf{MZ}(R, I)\} \subseteq \{G(M) \mid M \in \mathbf{M}(R, I)\}.$$

Proof. (1) follows from Theorem 4.20(1).

(2) The first inclusion follows from part (1) applied to $S = \mathcal{N}$. As for the second inclusion, let $Q \in \mathbf{MZ}(R, I)$, thus $Q \subseteq M$ for some $M \in \mathbf{M}(R, I)$. Hence $G(Q) \subseteq G(M)$. Since R is Prüfer, and Q and $G(M)$ are contained in M, we obtain that either $Q \subseteq G(M)$, or $G(M) \subseteq Q$. In the first case, since $Q \in \mathbf{MZ}(R, I)$ and $G(M) \in \mathbf{Z}(R, I)$, we obtain $Q = G(M)$, and so $G(Q) = Q = G(M)$. In the second case, we obtain $G(M) \subseteq G(Q)$, and again we have equality. This proves the second inclusion. ∎

It is not clear if either of the inclusions in Proposition 4.21(2) is actually an equality. We do not even know whether each ideal in $G(R, I)$ is contained in an ideal belonging to $\mathbf{MG}(R, I)$.

5. The Domain $\mathscr{P}(R)$

This section and the next contains the details of the construction of the ring $\mathscr{P}^{\infty}(R)$. Let R be a fixed integral domain with quotient field K. Let $\underline{X} = \{X_q \mid q \in K \setminus \{0\}\}$ be a set of independent indeterminates over K. Denote by $\mathscr{S}(K)$ the set of all elements X_q, qX_q and $(1 - X_q)/q$, for $q \in K \setminus \{0\}$. Then $\mathscr{S}(K)$ is a subset of the polynomial ring $K[\underline{X}]$. Note that $\mathscr{S}(K)$, as well as the set $\underline{X}$, depends just on K rather than on R.

Define $\mathscr{P}(R)$ to be the domain $R[\mathscr{S}(K)]$. If there is no danger of confusion write $\mathscr{P}$ for $\mathscr{P}(R)$. Clearly $\mathscr{P}(R) \subseteq K[\underline{X}]$ and the quotient field of $\mathscr{P}(R)$ is $K(\underline{X})$.

The first two propositions follow from the definitions.

Proposition 5.1. (1) *Let $R_1 \subseteq R_2$ be domains with the same quotient field. Then $\mathscr{P}(R_2) = \mathscr{P}(R_1)[R_2]$.*

(2) *If R is a domain with quotient field K and S is a subset of K, then $\mathscr{P}(R[S]) = \mathscr{P}(R)[S]$.*

Proposition 5.2. *The domain $\mathscr{P}(R)$ is Prüfer over R.*

For an arbitrary domain D, let D' be the integral closure of D in its quotient field.

Proposition 5.3. $\mathscr{P}' \cap K = R'$.

Proof. If $x \in \mathscr{P}' \cap K$, then x satisfies an equation

$$x^r + f_{r-1}x^{r-1} + \cdots + f_0 = 0$$

with all f_i in $\mathscr{P}$. For some $q_1, ..., q_n$ in $K \setminus \{0\}$, all f_i are in $R[X_{q_1}, q_1 X_{q_1}, (1 - X_{q_1})/q_1, ..., X_{q_n}, q_n X_{q_n}, (1 - X_{q_n})/q_n]$. We prove by induction on n that $x \in R'$. It suffices to consider the case $n = 1$. Set $q_1 = q$ and $X_1 = X$. Thus we assume that all f_i are in $R[X, qX, (1 - X)/q]$. Setting $X = 0$ yields $x \in R[1/q]'$, and setting $X = 1$ yields $x \in R[q]'$; thus $x \in R[q]' \cap R[1/q]'$. To complete the proof, we show that $R[q]' \cap R[1/q]' = R'$. If V is a valuation ring of K containing R, then V contains q or $1/q$. Thus $V \supseteq R[q]' \cap R[1/q]'$. It follows that $R' \supseteq R[q]' \cap R[1/q]'$; so we have equality. ∎

PROPOSITION 5.4. $\mathscr{U}(\mathscr{P}') = \mathscr{U}(R')$.

Proof. Since $\mathscr{P}' \subseteq K[\underline{X}]$, we have $\mathscr{U}(\mathscr{P}') \subseteq K$. By Proposition 5.3, $\mathscr{U}(\mathscr{P}') \subseteq R'$. Hence $\mathscr{U}(\mathscr{P}') \subseteq \mathscr{U}(R')$. ∎

COROLLARY 5.5. $\mathscr{U}(\mathscr{P}) \cap R = \mathscr{U}(R)$.

Note that Corollary 5.5 also follows from Corollary 5.7(1) below.

PROPOSITION 5.6. *For an ideal I of R, we have* $\sqrt{I\mathscr{P}} \cap R = \sqrt{I}$.

Proof. It is enough to show that $\sqrt{IR[X, qX, (1-X)/q]} \cap R \subseteq \sqrt{I}$ for $q \in K \backslash \{0\}$ (compare the proof of Proposition 5.3). Let $s \in \sqrt{IR[X, qX, (1-X)/q]} \cap R$. Hence $IR_s[X, qX, (1-X)/q]$ is the unit ideal. Set $X=0$ to obtain $1 \in IR_s[1/q]$, and set $X=1$ to get $1 \in IR_s[q]$. Then $IR_s \neq R_s$ would contradict the fact that the ideal IR_s survives in one of the rings $R_s[q]$, $R_s[1/q]$ [10, Thm. 55]. This $IR_s = R_s$, that is $s \in \sqrt{I}$. ∎

COROLLARY 5.7. *Let I be an ideal of R.*

(1) *If* $I \neq R$, *then* $I\mathscr{P} \neq \mathscr{P}$;

(2) *If I is a radical ideal, then* $I\mathscr{P} \cap R = I$.

It is shown in Remark 7.3 that if I is a radical ideal of R, then $I\mathscr{P}$ may not be a radical ideal of $\mathscr{P}$.

We will use repeatedly the following well-known theorem.

THEOREM 5.8. *Let K be a field,* $\underline{X}$ *a set of independent indeterminates over K,* $\phi: \tilde{K} \to \tilde{L}$ *a place, and* $\lambda: \underline{X} \to L$ *a map. Then there exists a place* $\Phi: (K(\underline{X}))^{\sim} \to \tilde{L}$ *extending* ϕ *and* λ.

Proof. Consider $\underline{X}$ also as a set of indeterminates over L. Let $\underline{Y} = \{X - \lambda(X) \mid X \in \underline{X}\}$, thus $L(\underline{X}) = L(\underline{Y})$. There exists a place $\Psi: (L(\underline{Y}))^{\sim} \to \tilde{L}$ over L which satisfies $\Psi(Y) = 0$ for all $Y \in \underline{Y}$. Indeed, by Zorn's Lemma, we reduce this assertion to the case that the set $\underline{Y}$ contains a single element Y. In this case, the valuation domain $L[Y]_{YL[Y]}$ determines a place Ψ as required.

There exists a place $\Theta: (K(\underline{X}))^{\sim} \to (L(\underline{X}))^{\sim}$ which extends ϕ and satisfies $\Theta(X) = X$ for all $X \in \underline{X}$. Indeed, if (V, P) is the valuation domain determined by the place ϕ, then $V[\underline{X}]_{PV[\underline{X}]}$ is a valuation domain which determines a place Θ as required [6, Prop. (18.7)].

Set $\Phi = \Psi \circ \Theta$. ∎

LEMMA 5.9. *Let K and L be fields,* $\phi: \tilde{K} \to \tilde{L}$ *a place,* $\underline{X}$ *a set of indeterminates over K and* $\lambda: \underline{X} \to L$ *a map. Then there exists a place* $\Phi: (K(\underline{X}))^{\sim} \to \tilde{L}$ *extending* ϕ *and* λ, *which satisfies for* $X \in \underline{X}$ *and* $c \in K$:

(1) $\Phi(cX)=0$, *if* $\lambda(X)=0$;

(2) $\Phi(c(X-1))=0$, *if* $\lambda(X)=1$.

Proof. By Theorem 5.8, there is a place Θ: $(K(\underline{X}))^{\sim} \to (K(\underline{X}))^{\sim}$ over K, which satisfies for $X \in \underline{X}$,

$$\Theta(X)=\begin{cases} 0 & \text{if } \lambda(X)=0; \\ 1 & \text{if } \lambda(X)=1; \\ X & \text{otherwise.} \end{cases}$$

By Theorem 5.8 again, there is a place Ψ: $(K(\underline{X}))^{\sim} \to \tilde{L}$ which extends ϕ and λ. Set $\Phi=\Psi \circ \Theta$. ∎

Return to the notation $\underline{X}=\{X_q \mid q \in K \backslash \{0\}$.

LEMMA 5.10. *Let K and L be fields, and let ϕ: $\tilde{K} \to \tilde{L}$ be a place. Assume that R is a subring of V_ϕ, K is the quotient field of R, and T is a subring of L which is Prüfer over $\phi(R)$. Then there exists a place Φ: $(\mathscr{Q}(\mathscr{P}(R))^{\sim} \to (\mathscr{Q}(T))^{\sim}$ extending ϕ and such that $\Phi(\mathscr{P}(R)) \subseteq T$.*

Proof. Let q be a nonzero element of K. Since T is Prüfer over $\phi(R)$, if $\phi(q) \neq 0$ or ∞, we can choose $x_q \in T$ such that $\phi(q)\, x_q$ and $(1-x_q)/\phi(q)$ are in T. If $\phi(q)=0$, let $x_q=1$; and if $\phi(q)=\infty$, let $x_q=0$. By Lemma 5.9, there is a place ϕ: $(K(\underline{X}))^{\sim} \to \tilde{L}$ extending ϕ and such that $\Phi(X_q)=x_q$ for all $q \in K \backslash \{0\}$. Consequently, the elements $\Phi(X_q)$, $\Phi(qX_q)$ and $\Phi((1-X_q)/q)$ are in T. Therefore $\Phi(\mathscr{P}(R)) \subseteq T$. ∎

COROLLARY 5.11. *Let K and L be fields. Every place ϕ: $\tilde{K} \to \tilde{L}$ is extendable to a place $(\mathscr{Q}(\mathscr{P}(V_\phi)))^{\sim} \to \tilde{L}$ which is finite on $\mathscr{P}(V_\phi)$.*

Proof. In the previous lemma take $R=V_\phi$ and $T=\phi(V_\phi)$. ∎

PROPOSITION 5.12. *Let R and T be domains, and let f: $R \to T$ be a homomorphism such that T is Prüfer over $f(R)$. Then f is extendable to a homomorphism $\mathscr{P}(R) \to T \Leftrightarrow f$ is extendable to a place $(\mathscr{Q}(R))^{\sim} \to (\mathscr{Q}(T))^{\sim}$.*

Proof. ($\Rightarrow$) Since $\mathscr{P}(R)$ is Prüfer over R (Proposition 5.2), this direction follows from Proposition 1.4.

($\Leftarrow$) Let ϕ: $\tilde{K} \to (\mathscr{Q}(T))^{\sim}$ be a place extending f. By Lemma 5.10, there is a place Φ: $(K(\underline{X}))^{\sim} \to (\mathscr{Q}(T))^{\sim}$ extending ϕ and such that $\Phi(\mathscr{P}(R)) \subseteq T$. Hence $\Phi \mid \mathscr{P}(R)$ is a homomorphism from $\mathscr{P}(R)$ into T extending f. ∎

PROPOSITION 5.13. *A homomorphism from R into a domain T is extendable to a homomorphism $\mathscr{P}(R) \to \mathscr{P}(T)$ if and only if it is extendable to a place $(\mathscr{Q}(R))^{\sim} \to (\mathscr{Q}(T))^{\sim}$.*

Proof. Let $f: R \to T$ be a homomorphism. Since $f(R) \subseteq T$, and $\mathscr{P}(T)$ is Prüfer over T, it is Prüfer also over $f(R)$. By Proposition 5.12, f is extendable to a homomorphism $\mathscr{P}(R) \to \mathscr{P}(T)$ if and only if it is extendable to a place $\tilde{K} \to (\mathscr{Q}(\mathscr{P}(T)))^{\sim}$.

To complete the proof, consider a place $\Phi: \tilde{K} \to (\mathscr{Q}(\mathscr{P}(T)))^{\sim}$ extending f. Since $\mathscr{Q}(\mathscr{P}(T))$ is a transcendental extension of $\mathscr{Q}(T)$, by Theorem 5.8 there exists a place $\Psi: (\mathscr{Q}(\mathscr{P}(T)))^{\sim} \to (\mathscr{Q}(T))^{\sim}$ over $\mathscr{Q}(T)$. Clearly $\Psi \circ \Phi: \tilde{K} \to (\mathscr{Q}(T))^{\sim}$ is a place extending f. ∎

6. The Domain $\mathscr{P}^{\infty}(R)$

Let R be a domain with quotient field K. Define $\mathscr{P}^n = \mathscr{P}^n(R)$ inductively as follows: $\mathscr{P}^0 = \mathscr{P}^0(R) = R$ and for $n > 0$, let $\mathscr{P}^n = \mathscr{P}^n(R) = \mathscr{P}(\mathscr{P}^{n-1})$. Set

$$\mathscr{P}^{\infty} = \mathscr{P}^{\infty}(R) = \bigcup_{n=0}^{\infty} \mathscr{P}^n(R).$$

Here is an alternative definition for $\mathscr{P}^{\infty}(R)$. Define inductively $\mathscr{S}^0(K) = \mathscr{S}(K)$, and $\mathscr{S}^n(K) = \mathscr{S}(\mathscr{S}^{n-1}(K))$ for $n > 0$. Set $\mathscr{S}^{\infty}(K) = \bigcap_{n=0}^{\infty} \mathscr{S}^n(K)$. Note that $\mathscr{S}^{\infty}(K)$ depends just on K. It is easy to show that $\mathscr{P}^{\infty}(R) = R[\mathscr{S}^{\infty}(K)]$.

Theorem 6.1. *The following properties holds for* $\mathscr{P}^{\infty}$:

(1) $\mathscr{P}^{\infty}$ *is a Prüfer domain containing* R;

(2) $\mathscr{P}^{\infty} \cap K = R'$;

(3) $\sqrt{I\mathscr{P}^{\infty}} \cap R = \sqrt{I}$;

(4) *If* I *is a radical ideal of* R, *then* $I\mathscr{P}^{\infty} \cap R = I$;

(5) *If* $I \neq R$, *then* $I\mathscr{P}^{\infty} \neq \mathscr{P}^{\infty}$;

(6) $\mathscr{U}(\mathscr{P}^{\infty}) = \mathscr{U}(R')$;

(7) $\mathscr{U}(\mathscr{P}^{\infty}) \cap R = \mathscr{U}(R)$.

Proof. (1) Clearly $R = \mathscr{P}^0 \subseteq \mathscr{P}^1 \subseteq \cdots$, so $\mathscr{P}^{\infty}$ is a domain containing R. For each $n > 0$, let $\underline{X}^{(n)}$ denote the set of indeterminates adjoined to $\mathscr{P}^{n-1}$, that is $\underline{X}^{(n)} = \{X_q^{(n)} \mid q \in \mathscr{Q}(\mathscr{P}^{(n-1)}) \setminus \leqslant 0\}\}$.

Let q be a nonzero element in $\mathscr{Q}(\mathscr{P}^{\infty}) = K(\bigcup_{n=1}^{\infty} \underline{X}^{(n)})$. Thus $q \in \mathscr{Q}(\mathscr{P}^n)$, for some $n \geqslant 0$. Hence, $X_q^{(n+1)}$, $qX_q^{(n+1)}$ and $(1 - X_q^{(n+1)})/q$ are in $\mathscr{P}^{n+1} \subseteq \mathscr{P}^{\infty}$. By Corollary 1.2, $\mathscr{P}^{\infty}$ is a Prüfer domain.

(2) Proposition 5.3 yields inductively that $(\mathscr{P}^n)' \cap K = R'$ for all integers $n \geqslant 0$. Since $\mathscr{P}^{\infty}$ is normal, we have $\mathscr{P}^{\infty} = \bigcup (\mathscr{P}^n)'$. Hence $\mathscr{P}^{\infty} \cap K = R'$.

(3) Note that $\sqrt{I\mathscr{P}^{\infty}} = \bigcup \sqrt{I\mathscr{P}^{n}}$. Thus (3) follows from Proposition 5.6 using induction.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). Clear.

(6) follows from Proposition 5.4 using induction, since $\mathscr{P}^{\infty}$ is normal.

(7) follows from (5) or from (6). ∎

PROPOSITION 6.2. *Let K and L be fields, $\phi: \tilde{K} \sim \tilde{L}$ a place, R a subring of V_{ϕ} having K as its quotient field, and T a Prüfer subring of L containing $\phi(R)$. Then there exists a place $\Phi: (\mathscr{Q}(\mathscr{P}^{\infty}(R)))^{\sim} \to \tilde{L}$ extending ϕ and such that $\Phi(\mathscr{P}^{\infty}(R)) \subseteq T$.*

Proof. Using Lemma 5.10, we inductively define places $\Phi_n: (\mathscr{Q}(\mathscr{P}^{n}))^{\sim} \to \tilde{L}$ such that $\Phi_0 = f$ and for $n > 0$, Φ_n is an extension of Φ_{n-1}, which $\Phi_n(\mathscr{P}^{n}) \subseteq T$. Let Φ be the least common extension of the places Φ_n. ∎

The next three results are the analogs for $\mathscr{P}^{\infty}$ of Corollary 5.11, Proposition 5.12 and Proposition 5.13, and can be obtained in a similar way or using induction.

COROLLARY 6.3. *Let K and L be fields. Every place $\phi: \tilde{K} \to \tilde{L}$ is extendable to a place $(\mathscr{Q}(\mathscr{P}^{\infty}(R)))^{\sim} \to \tilde{L}$ which is finite on $\mathscr{P}^{\infty}(V_{\phi})$.*

PROPOSITION 6.4. *A homomorphism of R into a Prüfer domain T is extendable to a homomorphism $\mathscr{P}^{\infty} \to T$ if and only if it is extendable to a place $(\mathscr{Q}(R))^{\sim} \to (\mathscr{Q}(T))^{\sim}$.*

PROPOSITION 6.5. *A homomorphism from R into a domain T is extendable to a homomorphism $\mathscr{P}^{\infty}(R) \to \mathscr{P}^{\infty}(T)$ if and only it is extendable to a place $(\mathscr{Q}(R))^{\sim} \to (\mathscr{Q}(T))^{\sim}$.*

The analog of Proposition 5.1 for $\mathscr{P}^{\infty}$ is:

PROPOSITION 6.6. (1) *Let $R_1 \subseteq R_2$ be domains with the same quotient field. Then*

$$\mathscr{P}^{\infty}(R_2) = \mathscr{P}^{\infty}(R_1)[R_2].$$

(2) *If R is a domain with quotient field K and S is a subset of K, then $\mathscr{P}^{\infty}(R[S]) = \mathscr{P}^{\infty}(R)[S]$.*

The previous proposition and Theorem 6.1(1) imply

COROLLARY 6.7. $\mathscr{P}^{\infty}(R) = \mathscr{P}^{\infty}(R')$.

Remarks 6.8. (1) Generally, a homomorphism of a domain R into a field L is not extendable to a place from K into L, but just into the algebraic closure of L. Thus by Proposition 6.2, generally a homomorphism $R \to T$ cannot be extended to a homomorphism $\mathscr{P}^{\infty}(R) \to \mathscr{P}^{\infty}(T)$. A similar remark holds for $\mathscr{P}(R)$. Hence, like $\mathscr{Q}(R)$, the domains $\mathscr{P}^{\infty}(R)$ and $\mathscr{P}(R)$ are not defined as functors acting in the category of domains having as morphisms all the ring homomorphisms.

Even if we restrict the morphisms in the category of domains of homomorphisms $R \to T$ extendable to places $\mathscr{Q}(R) \to \mathscr{Q}(T)$, it is not clear how the make $\mathscr{P}^{\infty}(R)$ (or $\mathscr{P}(R)$) into a functor. Indeed, a place extending a given homomorphism is not uniquely determined and it is not clear how the choose such a place functorially.

On the other hand, if we restrict morphisms to monomorphisms, then $\mathscr{P}^{\infty}(R)$ and $\mathscr{P}(R)$ are functors, like $\mathscr{Q}(R)$.

(2) The domain $\mathscr{P}(R)$ is Prüfer if and only if R is the field of 2 elements. (If R contains more than 2 elements, then the set $\underline{X}$ contains at least 2 indeterminates; and so the ring $K[\underline{X}]$ is not Prüfer. Since $K[\underline{X}]$ is an overring of $\mathscr{P}(R)$, we obtain that $\mathscr{P}(R)$ itself is not Prüfer.) For each domain R and for $2 \leqslant n \leqslant \infty$, the domain $\mathscr{P}^n(R)$ is not Prüfer, since $\mathscr{P}^{n-1}(R)$ is infinite.

(3) One can be more "economical" in the definition of $\mathscr{P}(R)$ (and so of $\mathscr{P}^{\infty}(R)$). Define

$$\mathscr{F}(R) = R[\{X_q, qX_q, (1-X_q)/q \mid q \in K\setminus\{0\} \text{ and there is no element } x \text{ in } R \text{ such that } qx \text{ and } (1-x)/q \text{ are in } R\}].$$

Define $\mathscr{F}^n(R)$ for $0 < n \leqslant \infty$ analogously to the definition of $\mathscr{P}^n(R)$. Thus $\mathscr{F}(R)$ is Prüfer over R and $\mathscr{F}^{\infty}(R)$ is a Prüfer domain containing R. Also $\mathscr{F}^n(R) \subseteq \mathscr{P}^n(R)$ for all $0 \leqslant n \leqslant \infty$. Moreover, R is Prüfer if and only if $R = \mathscr{F}^{\infty}(R)$. In many respects the domains $\mathscr{F}^n(R)$ behave like the domains $\mathscr{P}^n(R)$, but apparently, this is not so with respect to homomorphisms.

(4) We can obtain a Prüfer overring of R using a similar definition to that of $\mathscr{P}(R)$ (or of $\mathscr{F}(R)$). For each $q \in K\setminus\{0\}$ pick an element $x_q \in K$. Let T be the domain obtained by adjoining to R the elements x_q, qx_q, $(1-x_q)/q$ for $q \in K\setminus\{0\}$. By Corollary 1.2, T is a Prüfer overring of R. Moeover, an overring of R is Prüfer if and only if it contains a domain T constructed in this way.

(5) The reformulation of the Prüfer property in Section 1 is not essential for constructions of the type $\mathscr{P}(R)$ (and so also of $\mathscr{P}^{\infty}(R)$), although it seems to be very convenient. For example, let us start with the definition of a Prüfer domain as a domain in which every nonzero finitely generated ideal is invertible. We construct a domain $\mathscr{E}(R)$ by "closing" the

domain R with respect to the above mentioned property. More precisely, let $X_i^{(s)}$ be independent indeterminates over R for each finite sequence $s=(r_1, ..., r_{m(s)})$ in $R\setminus\{0\}$ and $1 \leqslant i \leqslant m(s)$. For each such sequence, adjoin to R the rational functions $r_j X_i^{(s)}$ $(1 \leqslant j,\ i \leqslant m(s))$ and $1/(r_1 X_1^{(s)} + \cdots + r_{m(s)} X_{m(s)}^{(s)})$ in order to obtain a domain $\mathscr{E}(R)$. Clearly $\mathscr{E}(R)$ is Prüfer over R. Define $\mathscr{E}^n(R)$ for $0 \leqslant n \leqslant \infty$ analogously to the definition of $\mathscr{P}^n(R)$. In this construction we may use just sequences of length 2. This construction seems to be more cumbersome than $\mathscr{P}(R)$ because it uses rational functions over K and not just polynomials.

7. Counterexample

This section contains an example of a Prüfer domain A for which equality in Corollary 4.5(2) fails. The domain A is a quotient ring of $\mathscr{P}^\infty(R)$ for an appropriate domain R.

Lemma 7.1. *Let R be a normal domain containing a field k, and let K be the quotient field of R. Let $c \in R\setminus\{0\}$, and $q \in K\setminus R$. Assume*:

(1) *$cq^n \in R$ for all $n \geqslant 1$;*

(2) *$\mathscr{U}(R[q]') = k\setminus\{0\}$;*

(3) *There is a place $\phi: \tilde{K} \to \tilde{K}$ such that $\phi(R) \subseteq R$, $\phi(q) = q$, and $\phi(c) = 0$.*

Set $B = \mathscr{P}^\infty(R)$. Let L be the ideal of B generated by the elements cq_n for $n \geqslant 0$. Let $\mathscr{N} = \mathscr{N}(B, L)$. Set $A = B_{\mathscr{N}}$, and $I = LA$.

Then the domain A is Prüfer, each nonunit of A is a zero-divisor mod I, *but $(I:I) \neq A$. Moreover, $A \subsetneqq (I:I) \subsetneqq (A:I) \subsetneqq (A:I^2) \subsetneqq \cdots$*

Proof. Note that $L \neq B$ by condition (3) and by Proposition 6.1(5), and that $L = cB[q]$.

Since B is a Prüfer domain, so is A. Clearly $q \in (I:I)$, and by construction each nonunit of A is a zero-divisor mod I. We now show that $q \notin A$. Assume that $q \in A$. There is an element $s \in \mathscr{N}$ such that $sq \in B$. But $\mathscr{N} = \mathscr{A}$ (Theorem 2.6); thus $s \in \mathscr{A}(B[q], cB[q]) = \mathscr{U}(B[q], cB[q])$ (Corollary 2.11).

View ϕ as a place $\tilde{K} \to (\mathscr{Q}(B))^\sim$. By Proposition 6.2, there exists a place Φ: $(\mathscr{Q}(B))^\sim \to (\mathscr{Q}(B))^\sim$ extending ϕ and such that $\Phi(B) \subseteq B$. Since $\phi(q) = q$, we have $\Phi B[q]) \subseteq B[q]$. Also $\Phi(c) = 0$, and $s \in \mathscr{U}(B[q], cB[q])$; so $\Phi(s) \in \mathscr{U}(B[q])$. By Proposition 6.4(2), $B[q] = \mathscr{P}^\infty(R[q])$. Theorem 6.1(6) implies that $\mathscr{U}(B[q]) = \mathscr{U}(R[q]') = k\setminus\{0\}$. Thus $\Phi(s) \in k\setminus\{0\}$.

Since $sq \in B$ and $\Phi(B) \subseteq B$, we obtain $\Phi(s)q = \Phi(sq) \in B$. Therefore $q \in B \cap K = R$, a contradiction. This proves that $q \notin A$, so $(I:I) \neq A$.

By Corollary 4.17, we obtain $A \subsetneqq (I:I) \subsetneqq (A:I) \subsetneqq (A:I^2) \subsetneqq \cdots$ ∎

In the setting of Lemma 7.1, $(A : I)$ is not a ring by Proposition 4.16. It is easy to show that $1/c^{n-1} \in (A : I^n) \setminus (A : I^{n-1})$ for all $n \geq 1$.

EXAMPLE 7.2., A Prüfer domain A containing a proper ideal I such that each nonunit in A is a zero-divisor mod I, but $A \subsetneq (I : I) \subsetneq (A : I) \subsetneq (A : I^2) \subsetneq \cdots$

Let k be a field, X and Y independent indeterminates over k. Set $R = k[\{XY^n \mid n \geq 0\}]$. The domain R is normal (see [10, Sect. 3-3, Exercise 8, p. 114] and [9, Ex. 3, p. 51]). Let $K = \mathscr{Q}(R) = k(X, Y)$. Set $c = X$ and $q = Y$, thus $c \in R \setminus \{0\}$ and $q \notin K \setminus R$. Condition (1) of Lemma 7.1 is clearly satisfied. Condition (2) is satisfied, since $\mathscr{U}(R[Y]') = \mathscr{U}(k[X, Y] = k \setminus \{0\}$. Condition (3) is satisfied, since there is a (unique) place ϕ: $(K(X, Y))^\sim \to (K(X, Y))^\sim$ over $k(Y)$ such that $\phi(X) = 0$ (Theorem 5.8). By Lemma 7.1, for $B = \mathscr{P}^\infty(R)$, the domain $A = B_{\mathscr{N}}$ yields the desired example.

Remark 7.3. If M is a maximal ideal of a domain R, then $M\mathscr{P}^\infty(R)$ is not necessarily radical, let alone prime (cf. Theorem 6.1(4)). Indeed, in the setting of Lemma 7.1, let M be the ideal of R generated by $\{cq^n \mid n \geq 0\}$, thus M is maximal. Nevertheless, the ideal MB of $B = \mathscr{P}^\infty(R)$ is not radical; otherwise, the ideal $I = MA$ of $A = B_{\mathscr{N}}$ is radical, so $(I : I) = A$ by Corollary 4.14, a contradiction. It follows that if P is a radical ideal of a domain R, then $P\mathscr{P}(R)$ is not necessarily a radical ideal of $\mathscr{P}(R)$ (cf. Corollary 5.7(2)); otherwise, using induction, that would imply a similar property for $\mathscr{P}^\infty$, which is not the case. Similarly, if I is a prime ideal of a domain R, then the ideal $I\mathscr{P}(R)$ is not necessarily prime. It is not clear whether for each prime ideal P of R, the ideal $P\mathscr{P}(R)$ is radical.

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