STAR OPERATIONS AND PULLBACKS

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Robert Gilmer and Joe Mott:
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In this talk I will study the star operations on a pullback of integral domains. In particular, I will characterize the star operations of a domain arising from a pullback of "a general type" by introducing new techniques for "projecting" and "lifting" star operations under surjective homomorphisms of integral domains.

I will apply part of the theory developed here to give a complete positive answer to a problem posed by D. F. Anderson in 1992 concerning the star operations on the "$D + M$" constructions.

NOTATION

Let D be an integral domain with quotient field L.

Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of L,
$\boldsymbol{F}(D)$ the set of all nonzero fractional ideals of D,
$\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of L.

Obviously, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

In this talk I will mainly consider the following situations:

(þ) *T represents an integral domain, M an ideal of T, k the factor ring T/M, D an integral domain subring of k and $\varphi : T \to T/M =: k$ the canonical projection. Set $R := \varphi^{-1}(D) =: T \times_k D$* the pullback of D inside T with respect to φ, *hence R is an integral domain (subring of T). Let K denote the field of quotients of R.*

(þ$^+$) *Let L be the field of quotients of D. In the situation* **(þ)**, *we assume, moreover, that $L \subseteq k$, and denote by $S := \varphi^{-1}(L) =: T \times_k L$* the pullback of L inside T with respect to φ. *Then S is an integral domain with field of quotients equal to K. In this situation, M, which is a prime ideal in R, is a maximal ideal in S. Moreover, if $M \neq (0)$ and $D \subsetneq k$, then M is a divisorial ideal of R, actually, $M = (R : T)$.*

(b)

$$
\begin{array}{ccccc}
R := \varphi^{-1}(D) & \xrightarrow{\varphi|_R} & D & & (\subseteq L := \mathsf{qf}(D)) \\
\downarrow & & \downarrow & & \\
T & \xrightarrow{\varphi} & k := T/M & & (\subseteq \mathsf{qf}(k)) \\
\cap & & & & \\
K := \mathsf{qf}(R) = \mathsf{qf}(T) & & & &
\end{array}
$$

(b⁺)

$$
\begin{array}{ccc}
R := \varphi^{-1}(D) & \xrightarrow{\varphi|_R} & D \\
\downarrow & & \downarrow \\
S := \varphi^{-1}(L) & \xrightarrow{\varphi|_S} & L := \mathsf{qf}(D) \\
\downarrow & & \downarrow \\
T & \xrightarrow{\varphi} & k := T/M \\
\cap & & \\
K := \mathsf{qf}(R) = \mathsf{qf}(T) & &
\end{array}
$$

Recall that a mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$, is called a *semistar operation on D* for all $0 \neq x \in L$ and $E, F \in \overline{\boldsymbol{F}}(D)$:

$(\star_1)$ $(xE)^\star = xE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^\star = (E^\star)^\star =: E^{\star\star}$.

A *star operation on D* is a map $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^\star$, that satisfies the properties $(\star_2), (\star_3)$ for all $E, F \in \boldsymbol{F}(D)$; moreover, for each $0 \neq x \in L$ and $E \in \boldsymbol{F}(D)$:

$(\star\star_1)$ $(xD)^\star = xD$; $(xE)^\star = xE^\star$.

Let $\star_D$ [respectively, $\star_T$] be a star operation on the integral domain D [respectively, T]. Our first goal is to define in a natural way a star operation on R, which we will denote by $\diamond$, associated to the given star operations on D and T. More precisely, if we denote by **Star**(A) the set of all the star operations on an integral domain A, then we want to define a map

$$\Phi : \textbf{Star}(D) \times \textbf{Star}(T) \to \textbf{Star}(R), \quad (\star_D, \star_T) \mapsto \diamond \ .$$

For each nonzero fractional ideal I of R, set

$$I^{\diamond} := \cap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right) \mid x \in I^{-1},\, x \neq 0 \right\} \cap (IT)^{\star_T},$$

where if $\frac{xI+M}{M}$ is the zero ideal of D (i.e., if $xI \subseteq M$), then we set $\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right) := M$.

Proposition 1 *Keeping the notation and hypotheses introduced in* **(þ)**, *then* $\diamond$ *defines a star operation on the integral domain* R *(*$= T \times_k D$*).*

The previous construction of the star operation $\diamond$ gives the idea for "lifting a star operation" with respect to a surjective ring homomorphim between two integral domains.

Corollary 2 *Let* R *be an integral domain with field of quotients* K, M *a prime ideal of* R, $D := R/M$ *and* $\varphi : R \to D$ *the canonical projection. Assume that* $\star$ *is a star operation on* D. *For each* $I \in \boldsymbol{F}(R)$, *set:*

$$\begin{aligned} I^{\star^{\varphi}} &:= \cap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star}\right) \mid x \in I^{-1},\ x \neq 0 \right\} \\ &= \cap \left\{ x\varphi^{-1}\left(\left(\frac{x^{-1}I+M}{M}\right)^{\star}\right) \mid x \in K,\ I \subseteq xR \right\}, \end{aligned}$$

Then $\star^{\varphi}$ *is a star operation on* R.

Let $\iota : R \hookrightarrow T$ be an embedding of integral domains with the same field of quotients K and let $*$ be a semistar operation on R. Define $*_\iota : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$ by setting:

$$E^{*_\iota} := E^*, \qquad \text{for each } \ E \in \overline{\boldsymbol{F}}(T) \ (\subseteq \overline{\boldsymbol{F}}(R)).$$

Then it is easy to see that:

(a) *If ι is not the identity map, then $*_\iota$ is a semistar, possibly non–star, operation on T, even if $*$ is a star operation on R.*

Note that, when $*$ is a star operation on R and $(R :_K T) = (0)$, a fractional ideal E of T is not necessarily a fractional ideal of R, hence $*_\iota$ is not defined as a star operation on T.

(b) *When $T := R^*$, then $*_\iota$ defines a star operation on R^*.*

Conversely, let $\star$ be a semistar operation on the overring T of R. Define $\star^\iota : \overline{\boldsymbol{F}}(R) \to \overline{\boldsymbol{F}}(R)$ by setting

$$E^{\star^\iota} := (ET)^\star \quad \text{for each} \quad E \in \overline{\boldsymbol{F}}(R)\,.$$

Then it is easy to see that

(c) $\star^\iota$ *is a semistar operation on* R.

(d) *For each semistar operation* $\star$ *on* T, *we have* $(\star^\iota)_\iota = \star$.

(e) *For each semistar operation* $*$ *on* R, *we have* $(*_\iota)^\iota \geq *$ (since $E^{(*_\iota)^\iota} = (ET)^{*_\iota} = (ET)^* \supseteq E^*$ for each $E \in \overline{\boldsymbol{F}}(R)$).

Using the notation introduced above, we immediately have the following:

Corollary 3 *With the notation and hypotheses introduced in* **(þ)** *and Proposition 1, if we use the definition given in Corollary 2, we have*

$$\diamond = (\star_D)^{\varphi} \wedge (\star_T)^{\iota}.$$

We next examine the problem of "projecting a star operation" with respect to a surjective homomorphism of integral domains.

Proposition 4 *Let R, K, M, D, φ be as in Corollary 2 and let L be the field of quotients of D. Let $*$ be a given star operation on the integral domain R. For each nonzero fractional ideal F of D, set*

$$F^{*_\varphi} := \cap \left\{ y^{-1}\varphi\left(\left(\varphi^{-1}(yF)\right)^{*}\right) \mid y \in F^{-1} = (D :_L F),\ y \neq 0 \right\}.$$

Then $*_\varphi$ is a star operation on D.

In case of a pullback of type **($\mathfrak{p}^+$)** the definition of the star operation $*_\varphi$ given above is simplified as follows:

Proposition 5 *Let T, K, M, k, D, φ, L, S and R be as in* **($\mathfrak{p}^+$)**. *Let $*$ be a given star operation on the integral domain R. For each nonzero fractional ideal F of D, we have*

$$F^{*_\varphi} = \varphi\left(\left(\varphi^{-1}(F)\right)^*\right) = \frac{\left(\varphi^{-1}(F)\right)^*}{M}.$$

Proposition 6 *Let T, K, M, k, D, φ, L, S and R be as in* **($\mathfrak{p}^+$)**. *Let $\star$ be a given star operation on the integral domain D, let $* := \star^{\varphi}$ be the star operation on R associated to $\star$ (which is defined in Corollary 2) and let $*_{\varphi}$ $(= (\star^{\varphi})_{\varphi})$ be the star operation on D associated to $*$ (which is defined in Proposition 4). Then $\star = *_{\varphi}$ $(= (\star^{\varphi})_{\varphi})$.*

Remark 7 *With the notation and hypotheses of Proposition 6, for each nonzero fractional ideal F of D, we have*

$$F^{\star} = \varphi\left(\varphi^{-1}(F)^{\star^{\varphi}}\right).$$

As a matter of fact, by the previous proof and Proposition 5, we have that $F^{\star} = F^{*_{\varphi}} = \varphi^{-1}(F)^{\star^{\varphi}}/M$.

Corollary 8 *Let T, K, M, k, D, φ, L, S and R be as in* **($\mathfrak{p}^+$)**.

(a) *The map* $(-)_\varphi : \mathbf{Star}(R) \to \mathbf{Star}(D)$, $* \mapsto *_\varphi$, *is order–preserving and surjective.*

(b) *The map* $(-)^\varphi : \mathbf{Star}(D) \to \mathbf{Star}(R)$, $\star \mapsto \star^\varphi$, *is order–preserving and injective.*

(c) *Let $\star$ be a star operation on D. Then for each nonzero ideal I of R with $M \subset I \subseteq R$,*

$$I^{\star^\varphi} = \varphi^{-1}\left((\varphi(I))^\star\right) .$$

The next result shows how the composition map

$$(-)^{\varphi} \circ (-)_{\varphi} : \textbf{Star}(R) \to \textbf{Star}(R)$$

compares with the identity map.

Theorem 9 *Let T, K, M, k, D, φ, L, S and R be as in* **(b$^+$)**. *Assume that $D \subsetneq k$. Then for each star operation $*$ on R,*

$$* \leq ((*)_{\varphi})^{\varphi} .$$

We will show that in general $* \lneq ((*)_{\varphi})^{\varphi}$. However, in some relevant cases, the inequality is, in fact, an equality:

Corollary 10 *Let T, K, M, k, D, φ, L, S and R be as in Theorem 9. Then*

$$v_R = ((v_R)_{\varphi})^{\varphi} ; \qquad (v_D)^{\varphi} = v_R ; \qquad (v_R)_{\varphi} = v_D .$$

Our next goal is to apply the previous results for giving a componentwise description of the "pullback" star operation $\diamond$ considered in Proposition 1.

Proposition 11 *Let T, K, M, k, D, φ, L, S and R be as in* **($\mathfrak{p}^+$)**. *Assume that $M \neq (0)$ and $D \subsetneq k$. Let*

$$\Phi : \mathbf{Star}(D) \times \mathbf{Star}(T) \to \mathbf{Star}(R)\,, \quad (\star_D, \star_T) \mapsto \diamond := (\star_D)^{\varphi} \wedge (\star_T)^{\iota},$$

be the map considered in Proposition 1 and Corollary 3. The following properties hold:

(a) $\diamond_{\varphi} = \star_D$.

(b) $\diamond_{\iota} = (v_R)_{\iota} \wedge \star_T$ $(\in \mathbf{Star}(T))$.

(c) $\diamond = (\diamond_{\varphi})^{\varphi} \wedge (\diamond_{\iota})^{\iota}$.

Example 12 *With the same notation and hypotheses of Proposition 11, we show that, in general,* $\diamond_\iota \neq \star_T$ *(even if* $L = k$*).*

Let D be any integral domain (not a field) with quotient field L. Let $T := L[X,Y]_{(X,Y)}$ and let $M := (X,Y)T$. Note that T is a 2-dimensional local UFD, thus $M^{v_T} = T$. Set $\diamond := (v_D)^{\varphi} \wedge (v_T)^{\iota}$ (thus $\star_T = v_T$). Then $M^{\diamond_\iota} = M^{\diamond} = M^{(v_D)^{\varphi}} \cap M^{(v_T)^{\iota}} = M^{v_R} \cap M^{(v_T)^{\iota}} = M$, because $M^{v_R} = M$ and $M^{(v_T)^{\iota}} = (MT)^{v_T} = M^{v_T} = T$.

Remark 13 (a) Note that, with the same notation and hypotheses of Proposition 11, *the map* Φ *is not one-to-one in general.*

This fact immediately follows from Example 12 and Proposition 11 (b) and (c), since

$$(\star_D)^{\varphi} \wedge (\star_T)^{\iota} = \diamond = (\diamond_{\varphi})^{\varphi} \wedge (\diamond_{\iota})^{\iota} \,.$$

(b) In the same setting as above, *the map* Φ *is not onto in general.*

An example, even in case $L = k$, is given next.

Example 14 *Let D be a 1-dimensional discrete valuation domain with quotient field L. Set $T := L[X^2, X^3]$, $M := (X^2, X^3)T = XL[X] \cap T$ and $K := L(X)$. Let φ and R be as in* **(p$^+$)**. *Then $v_R \notin$ Im(Φ).*

Note that, for each $\diamond \in$ Im(Φ), $\diamond \leq (v_D)^{\varphi} \wedge (v_T)^{\iota} \leq v_R$. In order to show that $v_R \notin$ Im(Φ), it suffices to prove that $(v_D)^{\varphi} \wedge (v_T)^{\iota} \neq v_R$. The fractional overring T of R is not a divisorial ideal of R, since $T^{v_R} = (R :_K (R :_K T)) = (R :_K M) \supseteq L[X] \supsetneq T$. Therefore, $T^{(v_D)^{\varphi} \wedge (v_T)^{\iota}} = T^{v_R \wedge (v_T)^{\iota}} = T^{v_R} \cap T^{(v_T)^{\iota}} = T^{v_R} \cap T^{v_T} = T^{v_R} \cap T = T \subsetneq T^{v_R}$.

Theorem 15 *With the notation and hypotheses of Proposition 11, set*

$$\mathbf{Star}(T; \upsilon_R) := \{\star_T \in \mathbf{Star}(T) \mid \star_T \leq (\upsilon_R)_\iota\}.$$

Then

(a) $\mathbf{Star}(T; \upsilon_R) = \{\star_T \in \mathbf{Star}(T) \mid (\upsilon_R \wedge (\star_T)^\iota)_\iota = \star_T \}$
$= \{*_\iota \mid * \in \mathbf{Star}(R)\} \cap \mathbf{Star}(T)$
$= \{*_\iota \mid * \in \mathbf{Star}(R)$ *and* $T^* = T\}$.

(b) *The restriction* $\Phi' := \Phi|_{\mathbf{Star}(D) \times \mathbf{Star}(T;\upsilon_R)}$ *is one-to-one.*

(c) $\mathrm{Im}(\Phi') = \mathbf{Star}(R; (\mathfrak{p}^+)) := \{* \in \mathbf{Star}(R) \mid T^* = T$ *and* $* = (*_\varphi)^\varphi \wedge (*_\iota)^\iota\}$.

We next apply some of the theory developed above for answering a problem posed by D. F. Anderson in 1992 [A-1992].

Example 16 (“$D + M$”–constructions).

Let T be an integral domain of the type $k+M$, where M is a maximal ideal of T and k is a subring of T canonically isomorphic to the field T/M, and let D be a subring of k with field of quotients L ($\subseteq k$). Set $R := D + M$. Note that R is a faithfully flat D–module.

Given a star operation $*$ on R, D.F. Anderson [A-1988, page 835] defined a star operation on D in the following way: for each nonzero fractional ideal F of D, set

$$F^{*_D} := (FR)^* \cap L\,.$$

From [A-1988, Proposition 5.4 (b)] it is known that:

For each nonzero fractional ideal F of D,

(a) $F^{*_D} + M = (F + M)^*$;

(b) $F^{*_D} = (F + M)^* \cap L = (F + M)^* \cap k$.

David F. Anderson in [A-1992] observed that the previous construction gives rise to a map $\boldsymbol{\alpha}$: **Star**$(D + M) \to$ **Star**(D), $* \mapsto *_D$, which is order-preserving but not injective. He poses the question whether $\boldsymbol{\alpha}$ may be surjective or, more precisely, whether $\boldsymbol{\alpha}$ may have a right inverse $\boldsymbol{\beta}$: **Star**$(D) \to$ **Star**$(D + M)$, which is an (injective) order-preserving map. He gave an answer in a particular situation, considering just the star operations defined by families of overrings.

The theory developed above gives a complete answer to these questions.

We start by comparing the operation $*_D$ defined in [A-1988] with the "projection", $*_\varphi$, considered above in a general pullback setting.

Claim. *If $\varphi : R \to D$ is the canonical projection and if $*_\varphi$ is the star operation defined in Proposition 4, then $*_D = *_\varphi$ (i.e. the map $\boldsymbol{\alpha}$ coincides with the map $(-)_\varphi :$ **Star**$(R) \to$ **Star**(D)).*

In particular, by [A-1992, Proposition 2 (a), (c)], *we deduce that*

(1) $(d_R)_\varphi = d_D\,, \quad (t_R)_\varphi = t_D\,, \quad (v_R)_\varphi = v_D\,,$ *and*

(2) $(*_f)_\varphi = (*_\varphi)_f\,.$

By applying Proposition 6 and Corollary 8 (a) to the particular case of $R = D + M$ (special case of **($\mathfrak{p}^+$)**), we know that the map

$$(-)_\varphi : \textbf{Star}(D + M) \to \textbf{Star}(D), \quad * \mapsto *_\varphi = *_D \ ,$$

is surjective and order-preserving and it has the injective order-preserving map

$$(-)^\varphi : \textbf{Star}(D) \to \textbf{Star}(D + M), \quad \star \mapsto \star^\varphi \ ,$$

as a right inverse.

This fact gives a complete positive answer to the problem posed by D.F. Anderson.

[A-1988] David F. Anderson, *A general theory of class groups.* Comm. Algebra **16** (1988), 805–847.

[A-1992] David F. Anderson, *Star operations and the $D+M$ constructions.* Rend. Circ. Mat. Palermo **41** II Serie (1992), 221–230.