On some classes of integral domains defined by $e.a.b.$ operations

Marco Fontana

Dipartimento di Matematica
Università degli Studi "Roma Tre"

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§0. Notation and Basic Definitions

Let D be an integral domain with quotient field K. Let

- $\overline{\mathfrak{F}}(D)$ be the set of all nonzero D-submodules of K,
- $\mathfrak{F}(D)$ be the set of all nonzero fractional ideals of D, and
- $\mathbf{f}(D)$ be the set of all nonzero finitely generated D–submodules of K.

Then, obviously,

$$\mathbf{f}(D) \subseteq \mathfrak{F}(D) \subseteq \overline{\mathfrak{F}}(D).$$

In 1994, Okabe and Matsuda introduced the notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$).

• A mapping $\star : \overline{\mathfrak{F}}(D) \to \overline{\mathfrak{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathfrak{F}}(D)$, the following properties hold:

($\star_1$) $(zE)^\star = zE^\star$;

($\star_2$) $E \subseteq F \;\Rightarrow\; E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

• When $D^\star = D$, we say that $\star$ restricted to $\boldsymbol{\mathfrak{F}}(D)$ defines *a star operation of D*

i.e. $\star : \boldsymbol{\mathfrak{F}}(D) \rightarrow \boldsymbol{\mathfrak{F}}(D)$ verifies the properties $\mathbf{(\star_2)}$, $\mathbf{(\star_3)}$ and

$\mathbf{(\star\star_1)}$ $(zD)^\star = zD\,, \quad (zE)^\star = zE^\star$.

• A *semistar operation of finite type* $\star$ is an operation such that

$$E^\star = \bigcup \{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\} \quad \text{for all } E \in \overline{\boldsymbol{\mathfrak{F}}}(D).$$

Problem

In this talk, I want to discuss the problem of characterizing integral domains for which

$$\star = \star_a \qquad or \qquad \star = b$$

for general star and semistar $\star$, or
for some distinguished star operations like, d, w, t, v,
where $\star_a$ is a canonical $a.b.$ semistar operation of finite type associated to $\star$.

§1. Background Results

• Let $\star$ be a semistar operation on D. If F is in $\mathbf{f}(D)$, we say that *F is $\star$–eab* [respectively, *$\star$–ab*] if $(FG)^\star \subseteq (FH)^\star$ implies that $G^\star \subseteq H^\star$, with $G, H \in \mathbf{f}(D)$, [respectively, with $G, H \in \overline{\mathfrak{F}}(D)$].
• *The operation $\star$* is said to be *eab* [respectively, *ab*] if each $F \in \mathbf{f}(D)$ is $\star$–eab [respectively, $\star$–ab].

An ab (star or semistar) operation is obviously an eab operation.

• W. Krull, in 1936, only considered the concept of "arithmetisch brauchbar" (for short, ab) star operation (more precisely, Krull's original notation was " ′–Operation"). He did not consider the concept of "endlich arithmetisch brauchbar" star operation.

• The *eab* star operation concept stems from the original version of Gilmer's book (1968). The results of Section 26 in that book show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings (in the general Krull's setting of integrally closed domains).

• If $\mathcal{W}$ is a given family of valuation overrings of D, then the mapping $\wedge_{\mathcal{W}}$ defined as follows: for each $E \in \overline{\mathfrak{F}}(D)$,

$$E^{\wedge_{\mathcal{W}}} := \bigcap \{EW \mid W \in \mathcal{W}\}$$

defines *an ab semistar operation of D*, since FW is principal in W, for each $F \in \mathbf{f}(D)$ and for each $W \in \mathcal{W}$.
We call a semistar operation of the previous type a *$\mathcal{W}$–operation of D*.
• If $\mathcal{W}$ coincides with the set $\mathcal{V}$ of all valuation overrings of D, then we call $\wedge_{\mathcal{V}}$ *the b–operation of D*.

If we assume that, given a family of valuation overrings overrings $\mathcal{W}$ of D, the overring $T := \bigcap\{W \mid W \in \mathcal{W}\}$ of D coincides with D, then the map $\wedge_{\mathcal{W}}$ (restricted to $\mathfrak{F}(D)$) defines a star operation of D.
In particular, if (and only if) D is integrally closed, the b–operation (restricted to $\mathfrak{F}(D)$) is a star operation of D.

• For a domain D and a semistar operation $\star$ of D, we say that a valuation overring V of D is *a $\star$–valuation overring of D* provided $F^\star \subseteq FV$, for each $F \in \mathbf{f}(D)$.
• Set $\boldsymbol{\mathcal{V}}(\star) := \{V \mid V$ is a $\star$–valuation overring of $D\}$ and let $b(\star) := \wedge_{\boldsymbol{\mathcal{V}}(\star)}$ the ab semistar operation on D defined as follows: for each $E \in \overline{\mathbf{F}}(D)$,

$$E^{b(\star)} := \bigcap \{EV \mid V \in \boldsymbol{\mathcal{V}}(\star)\}.$$

Clearly, when $\star$ coincides with d, the identity (semi)star operation, then $b(d) = b$.
Note that, this example shows that even if $\star$ (restricted to $\mathfrak{F}(D)$) is a star operation, $b(\star)$ may be a *proper* semistar operation.

There is another construction, with a more classical origin, for associating to a semistar operation an *(e)ab* semistar operation of finite type.
In order to introduce this construction, we need first to generalize, in the semistar operation setting, one of the useful characterizations for cancellation and quasi–cancellation ideals.

Lemma 1

Let D be a domain, let $F \in \mathbf{f}(D)$ and let $\star$ be a semistar operation on D. Then, F is $\star$–eab [respectively, $\star$–ab] if and only if $((FH)^\star : F^\star) = H^\star$, for each $H \in \mathbf{f}(D)$ [respectively, for each $H \in \overline{\mathfrak{F}}(D)$].

(Note that $((FH)^\star : F^\star) = ((FH)^\star : F)$, so the previous equivalences can be stated in a formally slightly different way.)

• Using the characterization in Lemma 1, we can associate to any semistar operation $\star$ of D an *(e)ab* semistar operation of finite type $\star_a$ of D, called *the (e)ab semistar operation associated to* $\star$, defined as follows for each $F \in \mathbf{f}(D)$ and for each $E \in \overline{\mathcal{F}}(D)$:

$$F^{\star_a} := \bigcup\{((FH)^\star : H^\star) \mid H \in \mathbf{f}(D)\},$$
$$E^{\star_a} := \bigcup\{F^{\star_a} \mid F \subseteq E,\ F \in \mathbf{f}(D)\}.$$

The previous construction, in the ideal systems setting, is essentially due to P. Jaffard (1960) and F. Halter-Koch (1997-1998).

Proposition 2

Let D be a domain and let $\star$ be an eab semistar operation. Then $\star_f = b(\star) = \star_a$.

- We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* if $I^\star \cap D = I$,
- a *quasi-$\star$-maximal* if it is maximal in the set of all proper quasi-$\star$-ideals.

Note that a quasi-$\star$-maximal ideal is a prime ideal.

- It is standard to see that $\mathrm{QMax}^\star(D)$, the set of the quasi-$\star$-maximal ideals of D, is not empty, for all semistar operations $\star$ *of finite type*.
- Then, for each $E \in \overline{\mathfrak{F}}(D)$, we can consider

$$E^{\widetilde{\star}} := \bigcap \{ED_P \mid P \in \mathrm{QMax}^{\star_f}(D)\} \ .$$

It is well known that the previous definition gives rise to a semistar operation $\widetilde{\star}$ of D which is *stable* (i.e., $(E \cap F)^{\widetilde{\star}} = E^{\widetilde{\star}} \cap F^{\widetilde{\star}}$, for each $E, F \in \overline{\mathfrak{F}}(D)$) and of finite type, called the *stable semistar operation of finite type canonically associated to $\star$*.

• Recall that, if K is the quotient field of D and X is an indeterminate over K, we set

$$\mathrm{Na}(D,\star) := \{f/g \in K(X) \mid f,g \in D[X],\ g \neq 0,\ \mathbf{c}(g)^\star = D^\star\}.$$

• It is known that $E^{\widetilde{\star}} = E\mathrm{Na}(D,\star) \cap K$ for all $E \in \overline{\mathfrak{F}}(D)$ (see for instance Fontana-Loper (2003)).

Note that a similar property holds for $\star_a$:

• $E^{\star_a} = E\mathrm{Kr}(D,\star) \cap K$ for all $E \in \overline{\mathfrak{F}}(D)$ (see for instance Fontana-Loper (2003)), where:

$$\mathrm{Kr}(D,\star) := \{f/g \in K(X) \mid f,g \in D[X],\ g \neq 0,\ \text{and there exists } h \in D[X]\setminus\{0\} \text{ with } (\mathbf{c}(f)\mathbf{c}(h))^\star \subseteq (\mathbf{c}(g)\mathbf{c}(h))^\star\}.$$

§2. Some Results

As a first application of the previous techniques, we recall some of the characterizations given by Fontana-Jara-Santos (2003) of a *Prüfer $\star$–multiplication domain* (i.e., an integral domain in which every nonzero finitely generated ideal is $\star_f$–invertible).

Theorem 3

Let $\star$ be a semistar operation of an integral domain D. The following properties are equivalent:

(i) *D is a Prüfer $\star$–multiplication domain.*

(ii) *$\widetilde{\star}$ is (e)ab (i.e., $\widetilde{\star} = (\widetilde{\star})_a$).*

(iii) *$\mathrm{Na}(D, \star)$ is a Prüfer domain (i.e., $\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \star)$).*

We collect in the following lemma some elementary facts about the b-operation.

Lemma 4

Let D be an integral domain.

(1) $b = d_a$.

(2) *b is a semistar operation of finite type.*

(3) $\widetilde{b} = d$. *In particular, an ideal of D is b-invertible if and only if it is invertible.*

(4) $P^b \cap D = P$ *for each nonzero prime ideal P of D.*

(5) $(\sqrt{I})^b \cap D = \sqrt{I}$ *for each nonzero ideal I of D.*

Remark 5

Note that if I is a nonzero ideal of a domain D, then I^b coincides with *the integral closure $\overline{I}$ of the ideal I* (in the sense of Zariski-Samuel), where

$$\overline{I} := \{x \in K \mid x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \text{ and } a_k \in I^k, \text{ for some } n \geq 1\}.$$

Note that there is another relative ("weaker") notion of integral closure of an ideal I in an overring T of D considered in the books by Atiyah-Macdonald (1969) and Kunz (1985) and precisely given by

$$\overline{I}^T := \{x \in T \mid x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n = 0 \text{ and } a_k \in I, \text{ for some } n \geq 1\},$$

but, here, I will always refer to the "classical" Zariski-Samuel context.

I was informed very recently by Irena Swanson that Darij Grinberg has studied a general version of integral closure of ideals with respect to ideal filtrations.

I start now to consider the problem of comparing a semistar operation $\star$ with $\star_a$, the *eab*-semistar operation canonically associated.

I start by the following facts.

Proposition 6

(1) *$d = d_a$ is equivalent to Prüfer domain;*

(2) *$w = w_a$ is equivalent to PvMD;*

(3) *$t = t_a$ is equivalent to v-domain.*

I do not give a sketch of the proof, but I wish to emphasize that there is a "fil rouge" (common thread) in the previous statements: for the three operations d, w and t there is an equivalence between the (apparently weaker) property of "*eab* cancellation" and the (apparently stronger) property of "(semi)star invertibility".

The next step is the study of domains for which $v = v_a$ (note that, by definition, $v_a = t_a$).
We start by considering the case when $v = v_a$, as star operations.

Proposition 7

Given an integral domain D, $v = v_a$ on $\mathfrak{F}(D)$ if and only if PvMD with t-finite character such that each (nonzero) t-prime is contained in only one t-maximal ideal and t-maximal ideals are t-finite (and, therefore, t-invertible).

Sketch of the proof.
Since v_a is an operation of finite type, then clearly $v = v_a$ on $\mathfrak{F}(D)$ is equivalent to $v = t$ on $\mathfrak{F}(D)$ and $t = t_a$ (on $\mathfrak{F}(D)$). Since $t = t_a$ (on $\mathfrak{F}(D)$) is equivalent to v-domain (Lemma 6(3)), $v = v_a$ on $\mathfrak{F}(D)$ is equivalent to v-domain which is also a TV-domain (i.e., a domain for which $t = v$ on $\mathfrak{F}(D)$), see papers by Houston-Zafrullah (1988), Hwang-Chang (1998) and El Baghdadi (2009)). The conclusion follows from the characterizations given by Houston-Zafrullah (1988) of TV-domains. □

Remark 8

An integral domain in which $v = v_a$ (on $\mathfrak{F}(D)$) is a PvMD with the property that PD_P is principal ideal (or, equivalently, a divisorial ideal) in the essential valuation overring D_P, for every (nonzero) t-prime ideal P of D. However, a PvMD (or, even, a Prüfer domain) with this property not necessarily has $v = v_a$ even on $\mathfrak{F}(D)$. For instance, take an almost Dedekind domain which is non Dedekind.

As a consequence we have the following.

Corollary 9

Let D be an integral domain. Then, $v = v_a$ on $\mathfrak{F}(D)$ and $\dim_t(D) = 1$ if and only if D is a Krull domain. In particular, in case of $\dim(D) = 1$, D is a Dedekind domain.

From Proposition 7 and from results due to Houston-Zafrullah (1988) and Kang (1989) (see also Hwang-Chang (1998), Gabelli-El Baghdadi (2005) and El Baghdadi (2009)), we obtain

Corollary 10

Let D be an integral domain. The following are equivalent.

(i) $v = v_a$ *on* $\mathfrak{F}(D)$.

(ii) D *is a PvMD and* $v = t$ *on* $\mathfrak{F}(D)$.

(iii) D *is an essential domain and* $v = t$ *on* $\mathfrak{F}(D)$.

(iv) D *is a v-domain and* $v = t$ *on* $\mathfrak{F}(D)$.

(v) D *is integrally closed and* $\mathrm{Na}(D, v)$ *is a divisorial domain.*

(vi) D *is integrally closed and* $v = w$ *on* $\mathfrak{F}(D)$.

(vii) $w = v_a$ *(on* $\mathfrak{F}(D)$*) and* $v = t$ *on* $\mathfrak{F}(D)$.

(viii) $v = w_a$ *on* $\mathfrak{F}(D)$.

(ix) $w = t = v = w_a = t_a = v_a$ *on* $\mathfrak{F}(D)$.

Recall that a *domain* is called *divisorial* if every nonzero ideal is divisorial (i.e., if $d = v$ as star operations).

Heinzer (1968) characterized the integrally closed divisorial domains as

- h-local
(i.e., $D = \bigcap\{D_M \mid M \in \mathrm{Max}(D)\}$ has finite character and every nonzero prime is contained in a unique maximal)
- Prüfer domains such that
- the maximal ideals are finitely generated.

Note that a domain in which $\star = \star_a$ is not necessarily a P$\star$MD.

For example, in any integral domain $b = b_a$ and, on the other hand, a PbMD is a Prüfer domain.
More generally, for a semistar operation $\star$ of finite type which is $(e.)a.b.$, we have $\star = \star_a$, but the domain is not necessarily a P$\star$MD, since as we mentioned above Fontana-Jara-Santos (2003)):

$$\text{P}\star\text{MD} \quad \Leftrightarrow \quad \widetilde{\star} = (\widetilde{\star})_a\,.$$

Therefore, $\star = \star_a$ does not imply $\widetilde{\star} = (\widetilde{\star})_a$ and, conversely, $\widetilde{\star} = (\widetilde{\star})_a$ does not imply $\star = \star_a$ even on $\mathfrak{F}(D)$ (for instance, take $\star = v$ in a PvMD which does not verify the other conditions listed in Proposition 7).

We have already observed that the domains for which $d = b$ are exactly the Prüfer domains (Proposition 6).
The next goal is to understand the domains for which $v = b$. This is a stronger condition than $v = v_a$, since we also require that $v_a = b$.
First, we consider the case when $v = b$ as star operations.

Proposition 11

The semistar operations b and v coincide on $\mathfrak{F}(D)$ if and only if D is an integrally closed divisorial domain.

Sketch of the proof.
Note that if $b = v$ on $\mathfrak{F}(D)$, then in particular $v = v_a$ on $\mathfrak{F}(D)$. In this situation, by Proposition 7, D is a PvMD (with further properties). On the other hand, $b = v$ on $\mathfrak{F}(D)$ also implies that D is a PbMD, i.e., D is Prüfer domain. Furthermore, D is Prüfer domains (i.e., $b = d$), and so $d = b = v$ on $\mathfrak{F}(D)$, thus D is a divisorial integrally closed (Prüfer) domain.

If we require $b = v$ as semistar operations, i.e., if we require that they coincide on $\overline{\mathfrak{F}}(D)$, we can say something more.

In fact, if V is a valuation overring such that $(D : V) = (0)$, we have $V^v = K$ and $V^b = V$. So, if $b = v$ as semistar operations, such a valuation overring of D cannot exist, hence D must be a conducive domain and also, by Proposition 11, a Prüfer divisorial domain. Therefore, D is a valuation domain (cf. Dobbs-Fedder (1984) or Picozza's thesis (2004)). In this case, we also have $d = v$ as semistar operations. Moreover, if $d = v$ then $b = v_a$ and $v = t$ (since d is obviously of finite type). In particular $b = v$ $(= t)$ since $b \leq t$ $(= v) \leq t_a$ $(= v_a)$. Therefore, we can conclude that

Proposition 12

For an integral domain D, the following are equivalent.

- **(i)** $b = v$ *(as semistar operations).*
- **(ii)** D *is a valuation domain with principal maximal ideal.*
- **(iii)** $d = v$ *(as semistar operations) and D is integrally closed.*

A similar argument shows that

Proposition 13

For an integral domain D, the following are equivalent.

(i) $v = v_a$ *(as semistar operations).*

(ii) D *is a valuation domain with principal maximal ideal.*

Therefore,

$$v = v_a \text{ (as semistar operations) } \Leftrightarrow v = b \text{ (as semistar operations).}$$

Sketch of the proof. Under (i), D is PvMD (Corollary 10) and conducive, since for each $Q \in \mathrm{Max}^t(D)$, D_Q is a v-valuation overring of D, because in this case $FD_Q = F^w D_Q = F^t D_Q$ $(= F^v D_Q)$. Therefore $(D : D_Q) \neq (0)$ (otherwise, $(D_Q)^v = (D : (D : D_Q)) = K$, but $(D_Q)^{v_a} = D_Q$).