dedicated to Alain Bouvier,
on the occasion of his 65th birthday,
for the long-standing collaboration and friendship

On *v*–domains: a survey

Marco Fontana

Dipartimento di Matematica
Università degli Studi "Roma Tre"

Work in progress, joint with Muhammad Zafrullah

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§1. The Genesis: Prüfer-like domains and v-domains

The v–domains generalize at the same time Prüfer domains and Krull domains and have appeared in the literature with different names.

This survey is the result of an effort to put together information on this useful class of integral domains.

In this talk, I will try to present old, recent and new characterizations of v–domains along with some historical remarks.

I will also discuss the relationship of v–domains with their various specializations and generalizations, giving suitable examples.

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Basic notation

- Let D be an integral domain with quotient field K.

- Let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D,   and let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D–submodules of K.

Then, obviously $\mathbf{f}(D) \subseteq \mathbf{F}(D)$.

- Let $A, B \in \mathbf{F}(D)$, set

$$(A : B) := \{z \in K \mid zB \subseteq A\} \text{ and } A^{-1} := (D : A).$$

- As usual, we let v (or, v_D) denote the star operation defined by $A^v := (D : (D : A)) = (A^{-1})^{-1}$ for all $A \in \mathbf{F}(D)$.

- We denote by t (or t_D), the star operation of finite type on D, associated to v, i.e., $A^t := \bigcup \{F^v \mid F \in \mathbf{f}(D) \text{ and } F \subseteq A\}$ for all $A \in \mathbf{F}(D)$.

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• The *v–domains* are precisely the integral domains D for which the v–operation is an "endlich arithmetisch brauchbar" operation (Gilmer, 1968&1972).

• Recall that a star operation $*$ on an integral domain D is *endlich arithmetisch brauchbar* (for short, *e.a.b.*) (respectively, *arithmetisch brauchbar* (for short, *a.b.*)) if for all $F, G, H \in \mathbf{f}(D)$ (respectively, $F \in \mathbf{f}(D)$ and $G, H \in \mathbf{F}(D)$)

$$(FG)^* \subseteq (FH)^* \Rightarrow G^* \subseteq H^*.$$

I asked Robert Gilmer and Joe Mott about the origins of v–domains. They had the following to say:
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• More precisely, the v–domains generalize the Prüfer v–multiplication domains, where a *Prüfer v–multiplication domains* (for short, *PvMD*; *anneau pseudo-prüférien* in Bourbaki's terminology) is an integral domain D such that every $F \in \mathbf{f}(D)$ is t–invertible, i.e., $(FF^{-1})^t = D$.

As a matter of fact, *an invertible ideal is t–invertible* and *a t–invertible ideal is v-invertible*, therefore we have the following picture:

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[General setting: Prüfer semistar domains (Houston-Malik-Mott, Fontana-Jara-Santos, Picozza, Anderson-Anderson-Fontana-Zafrullah); r–Prüfer monoids (Halter-Koch).]

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A paper by Dieudonné (1941) provides a clue to where v-domains came out as a separate class of rings, though they were not called v–domains there.

In this paper, J. Dieudonné gives an example of what we call now a v–domain that is not a Prüfer v–multiplication domain.

• Let $\mathbf{F}^v(D)$ (respectively, $\mathbf{f}^v(D)$) be the set of all nonzero fractional divisorial ideals of D (respectively, the set of all nonzero fractional divisorial ideals of finite type of D).

In general, $\mathbf{F}^v(D)$ and $\mathbf{f}^v(D)$ are not groups (with respect to the v–operation of fractional ideals).
By a classical result by Van der Waerden (1931), $\mathbf{F}^v(D)$ is a group if and only if D is completely integrally closed.

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Dieudonné considers the following two properties introduced by Prüfer (1932):

$(\mathbf{V}_\beta)$ $\mathbf{f}^v(D)$ is a group (with respect to the v–operation of fractional ideals) or, equivalently, each element of $\mathbf{f}^v(D)$ has an inverse belonging to $\mathbf{f}^v(D)$ (i.e., D is a PvMD). [Note that $\mathbf{f}^v(D) = \mathbf{f}^t(D)$.]

$(\mathbf{V}_\gamma)$ the v–operation is e.a.b. (i.e., D is a v–domain).

He constructs an example of a particular domain of semigroup D that verifies $(\mathbf{V}_\gamma)$ and has a two generated ideal I such that I^{-1} is a (v–ideal) not of finite type and hence D does not satisfy $(\mathbf{V}_\beta)$.

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The picture (considered above):

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can be refined.

M. Griffin (1967), a student of Ribenboim's, showed that *D is a PvMD if and only if D_M is a valuation domain for each maximal t–ideal M of D.*

• Call a *valuation overring* V of D *essential* if $V = D_P$ for some prime ideal P of D (which is invariably the center of V over D) and call D an *essential domain* if D is expressible as an intersection of its essential valuation overrings (e.g., a Krull domain is an essential domain).

Clearly, *a Prüfer domain is essential* and, more generally, by Griffin's result, *a PvMD is essential.*

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From a local point of view, it is easy to see from the definitions that *every integral domain D that is locally essential is essential*.

The converse is not true: the first example of an essential domain having a prime ideal P such that D_P is not essential was given by Heinzer (1981).

Now add to this information the following well known result that shows that the essential domains are sitting in between PvMD's and v–domains.

Proposition 1

An essential domain is a v-domain.

This result is due to Kang (1989) and Zafrullah (1988). It can be also deduced from a general result for essential monoids due to Halter-Koch (1998).

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Remark. Since a Krull domain is locally Krull, *a Krull domain D is a locally essential domain*. Using the fact that $D = \bigcap \{D_P \mid \operatorname{ht}(P) = 1\}$, it can be shown that v is an a.b. operation (associated to the family of valuation overrings $\{D_P \mid \operatorname{ht}(P) = 1\}$) and each $F \in \mathbf{f}(D)$ is v–invertible. Therefore, *a Krull domain is a PvMD*.

A characterization of PvMD's using the essential domain property is given next.

Proposition 2

Given an integral domain D, the following are equivalent:

(i) *D is a PvMD.*

(ii) *D is an essential domain such that* $(a) \cap (b)$ *is a v–finite v–ideal, for all nonzero* $a, b \in D$.

(iii) $(a) \cap (b)$ *is t–invertible in D, for all nonzero* $a, b \in D$.

(i)⇔(ii) is due to Zafrullah (1978) (cf. also Halter-Koch (1998) for the monoid case).
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A characterization of PvMD's using the essential domain property is given next.

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Given an integral domain D, the following are equivalent:

(i) *D is a PvMD.*

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For v–domains, we have the following "v–version" of the previous characterization ((i)⇔(iii)) for PvMD's:

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Given an integral domain D,

D is a v–domain $\Leftrightarrow$ $(a) \cap (b)$ is v–invertible in D, for all nonzero $a, b \in D$.

The idea of proof is simple. Recall that every $F \in \mathbf{f}(D)$ is invertible (respectively, v–invertible; t–invertible) if and only if every nonzero two generated ideal of D is invertible (respectively, v–invertible; t–invertible) (the idea of proof dates back to Prüfer (1932)). Moreover, for all nonzero $a, b \in D$, we have:

$$(a, b)(a, b)^{-1} = (a, b)\frac{(aD \cap bD)}{ab} .$$

Therefore, in particular, the fractional ideal $(a, b)^{-1}$ (or, equivalently, (a, b)) is v–invertible if and only if the ideal $aD \cap bD$ is v–invertible.

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§2 Bézout-type domains and v–domains

Recall that an integral domain D is
- a *Bézout domain* if every finitely generated ideal of D is principal,   and
- a *GCD domain* if, for all nonzero $a, b \in D$, a greatest common divisor of a and b, $GCD(a, b)$, exists and it is in D.

Among the characterizations of the GCD domains we have that D is a GCD domain if and only if, for every $F \in \mathbf{f}(D)$, F^v is principal or, equivalently, if and only if the intersection of two (integral) principal ideals of D is still principal (see, for instance, the survey paper by D.D. Anderson (2000)).

From Proposition 3, we deduce immediately the second implication in the following picture:

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Next goal is to show that in between GCD domains and v–domains are sitting several other distinguished classes of integral domains.

First, note that, from the previous observations, it follows easily that *if D is a Prüfer domain then $(a) \cap (b)$ is invertible in D, for all nonzero $a, b \in D$. Examples show that the converse is not true. The reason for this is that $aD \cap bD$ invertible allows only that $\frac{(a,b)^v}{ab}$ (or, equivalently, $(a, b)^v$) is invertible and not necessarily the ideal (a, b).*

An important generalization of the notion of GCD domain was introduced by Anderson-Anderson (1979):

• an integral domain D is called a *Generalized GCD* (for short, *GGCD*) *domain* if the intersection of two (integral) invertible ideals of D is invertible D (or, equivalently, $(a) \cap (b)$ is invertible in D, for all nonzero $a, b \in D$).

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> Prüfer domain $\Rightarrow$ GGCD domain $\Rightarrow$ PvMD.

From the well known fact that an invertible ideal in a local domain is
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On the other hand, from the definition of PvMD, we easily deduce that *a*
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Therefore, we have the following addition to the existing picture:

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§3 Integral closures and v–domains

Recall an integral domain D with quotient field K is called a *completely integrally closed* (for short, *CIC*) *domain* if
$D = \{z \in K \mid \text{ for all } n \geq 0, \ az^n \in D \text{ for some nonzero } a \in D\}$.

It is well known that the following statements are equivalent.

  (i) D is CIC;

  (ii) for all $A \in \mathbf{F}(D)$, $(A^v : A^v) = D$;

  (ii′) for all $A \in \mathbf{F}(D)$, $(A : A) = D$;

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(see Gilmer's book for (i)⇔(ii)⇔(ii′) and Zafrullah (2008) for (ii″)⇔(iii); for a general monoid version of this characterization, see Halter-Koch (1998)).

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In Bourbaki (Exercises in Ch. 7 of Algèbre Commutative) an integral domain D is called *regularly integrally closed* if, for all $F \in \mathbf{f}(D)$, F^v is regular with respect to the v–multiplication (i.e., if $(FG)^v = (FH)^v$ for $G, H \in \mathbf{f}(D)$ then $G^v = H^v$).

Theorem 1

Let D be an integral domain, then the following are equivalent.

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(ii$_f$) For all $F \in \mathbf{f}(D)$, $(F^v : F^v) = D$.

(iii$_f$) For all $F \in \mathbf{f}(D)$ $(FF^{-1})^{-1} = D$ (or, equivalently, $(FF^{-1})^v = D$).

(iv) D is a v–domain.

A preliminary version of Theorem 1 appeared in a paper by Lorenzen (1939) (see also Dieudonné (1941)). A general monoid version of the previous characterization is given in Halter-Koch's book.

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Remark.　▷ **(a)** Note that the condition

　　$(\mathbf{ii'_f})$ *for all* $F \in \mathbf{f}(D)$, $(F : F) = D$

is equivalent to say that D is integrally closed and so it is weaker than
condition $(\mathbf{ii_f})$ of the previous Theorem 1, since
$(F^v : F^v) = (F^v : F) \supseteq (F : F)$.
On the other hand, it is easy to see that condition

　　$(\mathbf{ii''_f})$ *for all* $F \in \mathbf{f}(D)$, $(F^{-1} : F^{-1}) = D$

is equivalent to the other statements of Theorem 1.

▷ **(b)** By Mott-Nashier-Zafrullah (1990), condition $(\mathbf{iii_f})$ of the previous
theorem is equivalent to

　　$(\mathbf{iii_2})$ *Every nonzero fractional ideal with two generators is*
　　　　　 v–invertible.

This characterization is a variation of the Prüfer's classical result that an
integral domain is Prüfer if and only if each nonzero ideal with two
generators is invertible (and of the characterization of PvMD's also
recalled above).

▷ **(c)** Regularly integrally closed integral domains make their appearance with a different terminology in the study of a weaker form of integrality, introduced in a paper by Anderson-Houston-Zafrullah (1991).

• Recall that, given an integral domain D with quotient field K, an element $z \in K$ is called *pseudo-integral over D* if $z \in (F^v : F^v)$ for some $F \in \mathbf{f}(D)$. The terms of

• *pseudo-integral closure* (i.e., $\widetilde{D} := \bigcup\{(F^v : F^v)) \mid F \in \mathbf{f}(D)\}$)   and
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It is clear from the definition that *pseudo-integrally closed coincides with regularly integrally closed.*

For the previous observations, we deduce the following addition to the existing picture:

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▷ **(d)** By Theorem 1, we have that a v–domain is an integral domain D such that each element of $F^v \in \mathbf{f}^v(D)$ is v–invertible but, as observed by Dieudonné, F^{-1} $(= (F^v)^{-1} \in \mathbf{F}^v(D))$ does not necessarily belong to $\mathbf{f}^v(D)$.

When (and only when), in a v–domain D, $F^{-1} \in \mathbf{f}^v(D)$, D is a PvMD.

As a matter of fact, (see for instance, Zafrullah (2000)):

let $F \in \mathbf{f}(D)$, then F is t–invertible if and only if F is v–invertible and F^{-1} is v–finite, i.e., $F^v \in \mathbf{f}^v(D)$ is v–invertible and F^{-1} $(= (F^v)^{-1})$ belongs to $\mathbf{f}^v(D)$.

▷ **(e)** The "regular" teminology with respect to the v–multiplication for the elements of $\mathbf{f}^v(D)$ (used by Dieudonné and Bourbaki) is clearly different from the notion of "von Neumann regular", usually considered for elements of a ring or of a semigroup.

However, it may be instructive to record some observations showing that, in the present situation, the two notions are somehow related.

(α) Let $\mathcal{H}$ be a commutative and cancellative monoid. If any element a of $\mathcal{H}$ is *von Neumann regular* (i.e., if there is $b \in \mathcal{H}$ such that $a^2 b = a$), then a is invertible in $\mathcal{H}$ (and conversely). [A commutative semigroup in which every element is von Neumann regular is called *Clifford semigroup*.]

(β) Let D be a v–domain. If $A \in \mathbf{f}^v(D)$ is von Neumann regular in the monoid $\mathbf{f}^v(D)$ under v–multiplication, then A is t–invertible (or, equivalently, $A^{-1} \in \mathbf{f}^v(D)$). Consequently, *a v–domain D is a PvMD if and only if each element of the monoid $\mathbf{f}^v(D)$ is von Neumann regular* [i.e., $\mathbf{f}^v(D)$ is a Clifford semigroup].

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§4 v–domains and rings of fractions

Note that, *given a multiplicative set S of a PvMD, D, then D_S is still a PvMD.*

The easiest proof of this fact can be given noting that, given $F \in \mathbf{f}(D)$, if F is t–invertible in D then FD_S is t–invertible in D_S, where S is a multiplicative set of D (Bouvier-Zafrullah (1988)).

On the other hand, Mott-Zafrullah (1981) have shown that an example of a non PvMD essential domain due to Heinzer and Ohm (1973) is in fact a locally PvMD (and, hence, locally essential domain).

It is natural to ask whether D_S is a v–domain when D is a v–domain.

The answer is no.

Example: D v–domain $\not\Rightarrow$ D_S v–domain

An example of an essential domain D with a prime ideal P such that D_P is not essential was given by Heinzer (1981).

What is interesting is that an essential domain is a v–domain by Proposition 1 and that, in this example, D_P is a (non essential) overring of the type $k + XL[X]_{(X)} = (k + XL[X])_{XL[X]}$, where L is a field and k a proper subfield that is algebraically closed in L.

Now, a domain of type $k + XL[X]_{(X)}$ is an integrally closed (not CIC) local Mori domain (see Barucci (1983) or Gabelli-Houston (1997)).

It is well known that if a Mori domain is a v–domain then it must be CIC, i.e., a Krull domain (Nishimura (1967)), and hence, in particular, an essential domain.

Therefore, Heinzer's construction provides an example of an essential (v–)domain D with a prime ideal P such that D_P is not a v–domain.

Remark. Note that a similar situation holds for CIC domains; i.e., if D is CIC then it may be that for some multiplicative set S of D the ring of fractions D_S is not a completely integrally closed domain.

A well known example in this connection is the ring **E** of entire functions. For **E** is a completely integrally closed Bézout domain that is infinite dimensional (Henriksen (1952, 1953)).

Localizing **E** at one of its prime ideals of height greater than one would give a valuation domain of dimension greater than one, which is obviously not completely integrally closed.

For another example of a CIC domain that has non–CIC rings of fractions, look at the integral domain of integer-valued polynomials $\text{Int}(\mathbb{Z})$.

This is a two-dimensional Prüfer non-Bézout domain.

It is well known that *if $\{D_\lambda \mid \lambda \in \Lambda\}$ is a family of overrings of D with $D = \bigcap_{\lambda \in \Lambda} D_\lambda$ and if each D_λ is a completely integrally closed (respectively, integrally closed) domain then so is D.*

It is natural to ask if in the above statement "completely integrally closed/integrally closed domain" is replaced by "v–domain" the statement is still true.

The answer in general is no, because by Krull's theorem every integrally closed integral domain is expressible as an intersection of a family of its valuation overrings and of course a valuation domain is a v–domain.

But, an integrally closed domain is not necessarily a v–domain. (A very explicit example is given by $\overline{\mathbb{Q}} + X\mathbb{R}[X]$, where $\mathbb{R}$ is the field of real numbers and $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$.)

If however each of D_λ is a ring of fractions of D, then the answer is yes.
A slightly more general statement is given next.

Proposition 4

Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a family of flat overrings of D such that $D = \bigcap_{\lambda \in \Lambda} D_\lambda$. If each of D_λ is a v–domain then so is D.

From the previous considerations, we have the following addition to the existing picture:

$$P v MD \;\Rightarrow\; \text{locally } P v MD \;\Rightarrow\; \text{locally } v\text{–domain} \;\Rightarrow\; v\text{–domain}.$$

§5 v–domains, polynomials and rational functions

As for the case of integrally closed domains and of completely integrally closed domains, it is well known that, *given an integral domain D and an indeterminate X over D,*

$$D[X] \text{ is a } P\nu MD \Leftrightarrow D \text{ is a } P\nu MD.$$

A similar statement holds for v–domains.
It follows from the fact that, *the following statements are equivalent* (D.D. Anderson-Kwak-Zafrullah (1995)).

(i) *For every $F \in \mathbf{f}(D)$, F^v is v–invertible in D.*

(ii) *For every $G \in \mathbf{f}(D[X])$, G^v is v–invertible in D[X].*

This equivalence is essentially based on a polynomial characterization of integrally closed domains given by Querré (1980).

From the previous equivalence, we deduce immediately that every $F \in \mathbf{f}(D)$ is v–invertible if and only if every $G \in \mathbf{f}(D[X])$ is v–invertible and this proves the following:

Theorem 5

Given an integral domain D and an indeterminate X over D, D is a v-domain if and only if D[X] is a v–domain.

Note that a much more interesting and general result was proved in terms of pseudo-integral closures by Anderson-Houston-Zafrullah (1991), i.e., *let $\mathcal{H}$ be a commutative cancellative monoid and set*
$\widetilde{\mathcal{H}} := \{x \in \mathbf{qg}(\mathcal{H}) \mid \exists L \in \mathbf{f}(\mathcal{H}) \text{ with } xL^v \subseteq L^v\}$, *then* $\widetilde{D[\mathcal{H}]} = \widetilde{D}[\widetilde{\mathcal{H}}]$.
Therefore, $D[\mathcal{H}]$ is a v–domain if and only if D is a v–domain and $\mathcal{H}$ is pseudo-integral closed.

Theorem 6

Let D be an integral domain with quotient field K and let X be an indeterminate over D. Set

$\qquad V_D := \{g \in D[X] \mid \mathbf{c}_D(g) \text{ is } v\text{–invertible}\}$

$\qquad T_D := \{g \in D[X] \mid \mathbf{c}_D(g) \text{ is } t\text{–invertible}\}.$

(a) T_D and V_D are multiplicative sets of $D[X]$ with $T_D \subseteq V_D$. Furthermore, V_D (or, equivalently, T_D) is saturated if and only if D is integrally closed.

(b) Suppose that D is an integrally closed domain, then the following are equivalent:
  (i) D is a v-domain (respectively, a PvMD).
  (ii) $V_D = D[X]\setminus\{0\}$ (respectively, $T_D = D[X]\setminus\{0\}$).
  (iii) $D[X]_{V_D}$ (respectively, $D[X]_{T_D}$) is a field (or, equivalently, $D[X]_{V_D} = K(X)$ (respectively, $D[X]_{T_D} = K(X)$)).
  (iv) Each nonzero element $z \in K$ satisfies a polynomial $f \in D[X]$ such that $\mathbf{c}_D(f)$ is v–invertible (respectively, t–invertible).

§6 v–domains and domains with a divisor theory: a brief account

Borevich and Shafarevich (1966) introduced "domains with a divisor theory" in order to generalize Dedekind domains and unique factorization domains, along the lines of Kronecker's classical theory of "algebraic divisors" (see Kronecker (1882) and also H. Weyl (1940) and Edwards (1990)).

Let D be an integral domain and set $D^\bullet := D \setminus \{0\}$. An integral domain D is said to have a *divisor theory* if there is a *factorial monoid* $\mathcal{H}$ and a semigroup homomorphism, denoted by $(-)$: $D^\bullet \to \mathcal{H}$, given by $a \mapsto (a)$, satisfying some properties related to the divisibility properties in D and $\mathcal{H}$.

[Recall that a *factorial monoid* $\mathcal{H}$ is a commutative cancellative monoid such that every element $\mathfrak{a} \in \mathcal{H}$ can be uniquely represented as a finite product of atomic (= irreducible) elements of $\mathcal{H}$, i.e., $\mathfrak{a} = \mathfrak{q}_1\mathfrak{q}_2...\mathfrak{q}_r$, with $r \geq 0$ and this factorization is unique up to the order of factors (for $r = 0$ this product is set equal to identity of $\mathcal{H}$.]

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After showing that Dedekind domains and UFD are particular domains with a divisor theory, Borevich and Shafarevich do not enter into the details of the determination of those integral domains for which a theory of divisors can be constructed, but it is known that they coincide with the Krull domains (see Aubert (1983) and Lucius (1998)).

Taking the above definition as a starting point, Lucius (1998) introduces a more general class of domains, called the domains with GCD–theory. An integral domain D is said to have a *GCD–theory* if there is a GCD–monoid $\mathcal{G}$ and a semigroup homomorphism, denoted by $(-): D^\bullet \to \mathcal{G}$, given by $a \mapsto (a)$, verifying essentially the same axioms of a divisor theory (i.e., **(1)** $a|b$ (in D) $\Leftrightarrow$ $(a)|(b)$ (in $\mathcal{G}$); **(2)** $\mathfrak{a} = \mathfrak{b}$ (in $\mathcal{G}$) $\Leftrightarrow$ $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$ (in the set of ideals of D), where $\overline{\mathfrak{a}} := \{0 \neq x \in D \mid \mathfrak{a}|(x)$ (in $\mathcal{G})\} \cup \{0\}$).

One of the main results obtained by Lucius is the following.

Theorem 7

Given an integral domain D, D is a ring with GCD–theory if and only if D is a v–domain.

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One of the main results obtained by Lucius is the following.

Theorem 7

Given an integral domain D, D is a ring with GCD–theory if and only if D is a v–domain.

The proof of the "if part" of the previous theorem is constructive and provides explicitly the GCD–theory. The GCD–monoid is constructed, *via Kronecker function rings*.

Recall that, when v is an e.a.b. operation (i.e., when D is a v–domain), the Kronecker function ring with respect to v, $\mathrm{Kr}(D, v)$, is well defined and it is a Bézout domain.

Let $\mathcal{K}$ be the monoid $\mathrm{Kr}(D, v)^\bullet$, let $\mathcal{U} := \mathcal{U}(\mathrm{Kr}(D, v))$ be the group of invertible elements in $\mathrm{Kr}(D, v)$ and set $\mathcal{G} := \mathcal{K}/\mathcal{U}$.

The canonical map:

$$[-] : D^\bullet \to \mathcal{G} = \frac{\mathrm{Kr}(D, v)^\bullet}{\mathcal{U}}, \ \ a \mapsto [a] \ (= \text{the equivalence class of } a \text{ in } \mathcal{G})$$

defines a GCD–theory for D, called the *Kroneckerian GCD–theory* for the v–domain D.

In particular, the GCD of elements in D is realized by the equivalence class of a polynomial; more precisely, under this GCD–theory, let $a_0, a_1, ..., a_n \in D^\bullet$,
$$\mathrm{GCD}(a_0, a_1, ..., a_n) := \mathrm{GCD}([a_0], [a_1], ..., [a_n]) = [a_0 + a_1 X + ... + a_n X^n].$$

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§7 Ideal-theoretic characterizations of v–domains

Nowadays, we know a very long list of equivalent statements concerning ideal–theoretic properties, providing further characterizations of (several classes of) v–domains. A first important step in this direction was made with the paper the "A to Z" paper (Anderson-Anderson-Costa-Dobbs-Mott-Zafrullah, 1989).

The next goal is to explore briefly this aspect of the v–domains theory.

Proposition 8

Let D be an integral domain. Then, D is a v–domain if and only if D is integrally closed and $(F_1 \cap F_2 \cap ... \cap F_n)^v = F_1^v \cap F_2^v \cap ... \cap F_n^v$ for all $F_1, F_2, ..., F_n \in \mathbf{f}(D)$ (i.e., the v–operation distributes over finite intersections of finitely generated fractional ideals).

The "if part" is contained in the "A to Z" paper where the converse was left open). The converse was proved few years later by Matsuda-Okabe (1993).

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Note that, even for a Noetherian 1-dimensional domain, the v–operation may not distribute over finite intersections of (finitely generated) fractional ideals. For instance, here is an example due to W. Heinzer cited in (D.D. Anderson-Cook, 2006).

Example 2

Let k be a field, X an indeterminate over k and set $D := k[\![X^3, X^4, X^5]\!]$, $F := (X^3, X^4)$ and $G := (X^3, X^5)$.

Clearly, D is a non-integrally closed 1-dimensional local Noetherian domain with maximal ideal $M := (X^3, X^4, X^5) = F + G$.

It is easy to see that $F^v = G^v = M$, and so
$F \cap G = (X^3) = (F \cap G)^v \subsetneq F^v \cap G^v = M$.

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