An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations

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1 Introduction: The Genesis

Toward the middle of the XIXth century, E.E. Kummer discovered that the ring of integers of a cyclotomic field does not have the unique factorization property and he introduced the concept of "ideal numbers" to re-establish some of the factorization theory for cyclotomic integers [45, Vol. 1, 203-210, 583-629].

As R. Dedekind wrote in 1877 to his former student E. Selling, *the goal* of *a* general theory was immediately clear after Kummer's solution in the special case of cyclotomic integers: to extend Kummer's theory to the case of general algebraic integers.

Dedekind admitted to having struggled unsuccessfully for many years before he published the first version of his theory in 1871 [45] (XI supplement to Dirichlet's "Vorlesungen über Zahlentheorie" [12]).

The theory of Dedekind domains, as it is known today, is based on Dedekind's original ideas and results. Dedekind's point of view is based on ideals ("ideal numbers") for generalizing the algebraic numbers; he proved that, *in the ring of the integers of an algebraic number field, each proper ideal factors uniquely into a product of prime ideals.*

L. Kronecker essentially achieved this goal in 1859, but he published nothing until 1882 [41].

Kronecker's theory holds in a larger context than that of rings of integers of algebraic numbers and solves a more general problem. The primary objective of his theory was to extend the set of elements and the concept of divisibility in such a way that any finite set of elements has a GCD (greatest common divisor) in an extension of the original ring which still mirrors as closely as possible the ideal structure of the original ring. It is probably for this reason that the basic objects of Kronecker's theory –corresponding to Dedekind's "ideals"– are called "divisors".

Let D_0 be a PID with quotient field K_0 and let K be a finite field extension of K_0. Kronecker's *divisors* are essentially all the possible GCD's of finite sets of elements of K that are algebraic over K_0; a divisor is *integral* if it is the GCD of a finite set of elements of the integral closure D of D_0 in K.

One of the key points of Kronecker's theory is that it is possible to give an explicit description of the "divisors". The divisors can be represented as equivalence classes of polynomials and a given polynomial in $D[X]$ represents the class of the integral divisor associated with the set of its coefficients.

More precisely, we can give the following definition.

The classical Kronecker function ring. Let D be as above. *The Kronecker function ring of D* is given by:

$$\begin{aligned}\mathrm{Kr}(D) :=& \left\{ \tfrac{f}{g} \mid f, g \in D[X],\ g \neq 0 \text{ and } \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g) \right\} \\ =& \left\{ \tfrac{f'}{g'} \mid f', g' \in D[X] \text{ and } \boldsymbol{c}(g') = D \right\},\end{aligned}$$

(where $\boldsymbol{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e. the ideal of D generated by the coefficients of h).

Note that we are assuming that D is a Dedekind domain (being the integral closure of D_0, which is a PID, in a finite field extension K of the quotient field K_0 of D_0 [26, Theorem 41.1 and Theorem 37.8]).

In this case, for each polynomial $g \in D[X]$, $\boldsymbol{c}(g)$ is an invertible ideal of D and, by choosing a polynomial $u \in K[X]$ such that $\boldsymbol{c}(u) = (\boldsymbol{c}(g))^{-1} := (D : \boldsymbol{c}(g))$, we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and thus $\boldsymbol{c}(g') = D$ (Gauss Lemma).

The fundamental properties of the Kronecker function ring are the following (cf. [63, Chapter II], [26, Theorem 32.6 (for $\star$ equal to the identity star operation)]):

(1) $\mathrm{Kr}(D)$ *is a Bézout domain (i.e. each finite set of elements, not all zero, has a GCD and the GCD can be expressed as linear combination of these elements) and* $D[X] \subseteq \mathrm{Kr}(D) \subseteq K(X)$ *(in particular, the field of rational functions* $K(X)$ *is the quotient field of* $\mathrm{Kr}(D)$*).*

(2) *Let* $a_0, a_1, \ldots, a_n \in D$ *and set* $f := a_0 + a_1 X + \ldots + a_n X^n \in D[X]$*, then:*

$$\begin{aligned}&(a_0, a_1, \ldots, a_n)\mathrm{Kr}(D) = f\mathrm{Kr}(D)\ \ (\textit{thus, } \mathrm{GCD}_{\mathrm{Kr}(D)}(a_0, a_1, \ldots, a_n) = f)\,, \\ &f\mathrm{Kr}(D) \cap K = (a_0, a_1, \ldots, a_n)D = \boldsymbol{c}(f)D\ \ (\textit{hence, } \mathrm{Kr}(D) \cap K = D)\,.\end{aligned}$$

Kronecker's classical theory led to two different major extensions:

• Beginning from 1936 [43], W. Krull generalized the Kronecker function ring to the more general context of *integrally closed domains*, by introducing ideal systems associated to particular star operations: the a.b. (*arithmetisch brauchbar*) star operations.

• Beginning from 1956 [51], M. Nagata investigated, *for an arbitrary integral domain* D, the domain $\{f/g \mid f, g \in D[X] \text{ and } \boldsymbol{c}(g) = D\}$, which coincides with $\mathrm{Kr}(D)$ if (and only if) D is a Prüfer domain [26, Theorem 33.4].

We recall these two major extensions of Kronecker's classical theory, but first we fix the general notation that we use in the sequel.

General notation. Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ represent the set of all nonzero D–submodules of K and $\boldsymbol{F}(D)$ the nonzero fractionary ideals of D (i.e. $E \in \overline{\boldsymbol{F}}(D)$ such that $dE \subseteq D$, for some nonzero element $d \in D$). Finally, let $\boldsymbol{f}(D)$ be the finitely generated D-submodules of K. Obviously:

$$\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)\,.$$

One of the major difficulties for generalizing Kronecker's theory is that Gauss Lemma for the content of polynomials holds for Dedekind domains (or, more generally, for Prüfer domains), but not in general:

Gauss Lemma. *Let* $f, g \in D[X]$, *where* D *is an integral domain. If* D *is a Prüfer domain, then:*

$$\boldsymbol{c}(fg) = \boldsymbol{c}(f)\boldsymbol{c}(g)\,,$$

and conversely. Cf. [26, Corollary 28.5].

For general integral domains, we always have the inclusion of ideals $\boldsymbol{c}(fg) \subseteq \boldsymbol{c}(f)\boldsymbol{c}(g)$. We also have the following result which is weaker than the Gauss Lemma but more widely applicable.

Dedekind–Mertens Lemma. *Let* D *be an integral domain and* $f, g \in D[X]$. *Let* $m := \deg(g)$. *Then*

$$\boldsymbol{c}(f)^m \boldsymbol{c}(fg) = \boldsymbol{c}(f)^{m+1}\boldsymbol{c}(g)\,.$$

Cf. [26, Theorem 28.1].

In order to overcome this obstruction to generalizing the definition of Kronecker's function rings, Krull introduced multiplicative ideal systems having a cancellation property which mirrors Gauss's Lemma. These ideal systems can be defined by what are called now the e.a.b. (*endlich arithmetisch brauchbar*) star operations.

Star operations. A mapping $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $I \mapsto I^\star$, is called *a star operation of* D if, for all $z \in K$, $z \neq 0$ and for all $I, J \in \boldsymbol{F}(D)$, the following properties hold:

$(\star_1)$ $(zD)^\star = zD$, $(zI)^\star = zI^\star$;

($\star_2$) $I \subseteq J \;\Rightarrow\; I^\star \subseteq J^\star$;
($\star_3$) $I \subseteq I^\star$ and $I^{\star\star} := (I^\star)^\star = I^\star$.

An *e.a.b. star operation* on D is a star operation $\star$ such that, for all nonzero finitely generated ideals I, J, H of D:

$$(IJ)^\star \subseteq (IH)^\star \;\Rightarrow\; J^\star \subseteq H^\star .$$

Using these notions, Krull recovers a useful identity for the contents of polynomials:

Gauss–Krull Lemma. *Let $\star$ be an e.a.b. star operation on an integral domain D (this condition implies that D is an integrally closed domain [26, Corollary 32.8]) and let $f,\ g \in D[X]$ then:*

$$\boldsymbol{c}(fg)^\star = \boldsymbol{c}(f)^\star \boldsymbol{c}(g)^\star .$$

Cf. [26, Lemma 32.6].

Remark 1 Krull introduced the concept of a star operation in his first *Beiträge paper* in 1936 [43]. He used the notation " $'$–Operation " ("Strich–Operation") for his generic operation. [In this paper you can find the terminology " $'$–Operation " in footnote 13 and in the title of Section 6, among other places.]

The notation " $*$–operation " ("star–operation") arises from Section 26 of the original version of Gilmer's "Multiplicative Ideal Theory" (1968) [25]. Robert Gilmer gave us this explication ≪ I believe the reason I switched from " $'$–Operation " to " $*$–operation " was because " $'$ " was not so generic at the time: I' was frequently used as the notation for the integral closure of an ideal I, just as D' was used to denote the integral closure of the domain D. (Such notation was used, for example, in both Nagata's Local Rings and in Zariski-Samuel's two volumes.) ≫

Moreover, Krull only considered the concept of an "arithmetisch brauchbar (a.b.) $'$–Operation", not of an e.a.b. operation.
Recall that an *a.b.–operation* is a star operation $\star$ such that, if $I \in \boldsymbol{f}(D)$ and $J, K \in \boldsymbol{F}(D)$ and if $(IJ)^\star \subseteq (IH)^\star$ then $J^\star \subseteq H^\star$.

The e.a.b. concept stems from the original version of Gilmer's book [25]. The results of Section 26 show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings.

In this regard, Robert Gilmer gave us this explication ≪ I believe I was influenced to recognize this because during the 1966 calendar year in our graduate algebra seminar (Bill Heinzer, Jimmy Arnold, and Jim Brewer, among others, were in that seminar) we had covered Bourbaki's Chapitres 5 and 7 of *Algèbre Commutative*, and the development in Chapter 7 on the v–operation indicated that e.a.b. would be sufficient.≫

One of the main goals for the classical theory of star operations has been to construct a Kronecker function ring associated to a domain, in a more general context than the original one considered by L. Kronecker in 1882.

More precisely, using star operations, in 1936 W. Krull [43] defined a Kronecker function ring in a more general setting than Kronecker's. (Further references are H. Prüfer (1932) [56], Arnold (1969) [8], Arnold-Brewer (1971) [9], Dobbs-Fontana (1986) [13], D.F. Anderson-Dobbs-Fontana (1987) [6], Okabe-Matsuda (1997) [55].)

Star–Kronecker function ring. Let D be an integrally closed integral domain with quotient field K and let $\star$ be an e.a.b. star operation on D, then:

$$\mathrm{Kr}(D,\star) := \left\{ \frac{f}{g} \mid f,g \in D[X]\,,\ g \neq 0 \ \text{ and } \ \boldsymbol{c}(f)^\star \subseteq \boldsymbol{c}(g)^\star \right\}$$

is an integral domain with quotient field $K(X)$, called *the $\star$–Kronecker function ring of D, having the following properties:*

(1) $\mathrm{Kr}(D,\star)$ *is a Bézout domain and* $D[X] \subseteq \mathrm{Kr}(D,\star) \subseteq K(X)$.
(2) *Let* $a_0, a_1, \ldots, a_n \in D$ *and set* $f := a_0 + a_1 X + \ldots + a_n X^n \in D[X]$, *then:*

$$\begin{aligned}&(a_0, a_1, \ldots, a_n)\mathrm{Kr}(D,\star) = f\mathrm{Kr}(D,\star)\,,\\ &(a_0, a_1, \ldots, a_n)\mathrm{Kr}(D,\star) \cap K = ((a_0, a_1, \ldots, a_n)D)^\star\\ &(\textit{i.e. } f\mathrm{Kr}(D,\star) \cap K = (\boldsymbol{c}(f))^\star)\,.\end{aligned}$$

In particular, $\mathrm{Kr}(D,\star) \cap K = D^\star = D$.

For the proof cf. [26, Theorem 32.7].

Nagata's generalization of the Kronecker function ring. The following construction is possible for any integral domain D and, even, for an arbitrary ring D.

$$\mathrm{Na}(D) := D(X) := \left\{ \frac{f}{g} \mid f,g \in D[X] \ \text{ and } \ \boldsymbol{c}(g) = D \right\},$$

and this ring is called the *Nagata ring of D*. This notion is essentially due to Krull (1943) [44]. Then this ring was studied in Nagata's book (1962) [52, Section 6, page 17], using the notation $D(X)$, and in Samuel's Tata volume (1964) [58, page 27] (where the notation $D(X)_{\mathrm{loc}}$ was used). We introduced the notation $\mathrm{Na}(D)$ that is convenient for generalizations.

In general, $\mathrm{Na}(D)$ is not a Bézout domain. It is not difficult to see that [26, Theorem 33.4] and [1, Theorem 8]

- $\mathrm{Na}(D)$ *is a Bézout domain if and only if D is a Prüfer domain.*
- $\mathrm{Na}(D)$ *coincides with* $\mathrm{Kr}(D)$ *if and only if D is a Prüfer domain.*

- *Every ideal of* $\mathrm{Na}(D)$ *is extended from* D *if and only if* D *is a Prüfer domain.*

The interest in Nagata's ring $D(X)$ is due to the fact that this ring of rational functions has some strong ideal-theoretic properties that D itself need not have, while maintaining a strict relation with the ideal structure of D.

(a) *The map* $P \mapsto PD(X)$ *establishes a 1-1 correspondence between the maximal ideals of* D *and the maximal ideals of* $D(X)$.

(b) *For each ideal* I *of* D,

$$ID(X) \cap D = I\,, \qquad D(X)/ID(X) \cong (D/ID)(X)\,;$$

I is finitely generated if and only if $ID(X)$ is finitely generated.

Among the new properties acquired by $D(X)$ are the following:

(c) *the residue field at each maximal ideal of* $D(X)$ *is infinite;*

(d) *an ideal contained in a finite union of ideals is contained in one of them;*

(e) *each finitely generated locally principal ideal is principal (therefore* $\mathrm{Pic}(D(X)) = 0$*).*

The proofs of the previous results can be found in Arnold (1969) [8] and Gilmer's book [26, Proposition 33.1, Proposition 5.8] (for **(a)**, **(b)**, and **(c)** which is a consequence of **(b)**), Quartararo-Butts (1975) [57] (for **(d)**) and D.D. Anderson (1977) [2, Theorem 2] (for **(e)**).

2 Basic facts on semistar operations

In 1994, Okabe and Matsuda [54] introduced the more flexible notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the notion of star operation, allowing $D \neq D^\star$ (cf. also [53], [48], and [49]).

Semistar operations. A mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of* D if, for all $z \in K$, $z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \ \Rightarrow\ E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

When $D^\star = D$, we say that $\star$ is *a (semi)star operation* of D, since, restricted to $\boldsymbol{F}(D)$ it is *a star operation of* D.

For star operations, the notion of $\star$–ideal leads to the definition of a canonically associated ideal system.

For semistar operations, we need a more general notion, that coincides with the notion of $\star$–ideal, when $\star$ is a (semi)star operation.

- A nonzero (integral) ideal I of D is a *quasi–$\star$–ideal* [respectively, *$\star$–ideal*] if $I^\star \cap D = I$ [respectively, if $I^\star = I$].

• A *quasi–$\star$–prime* [respectively, *$\star$–prime*] of D is a quasi–$\star$–ideal [respectively, an integral $\star$–ideal] of D which is also a prime.

• A *quasi–$\star$–maximal* [respectively, *$\star$–maximal*] of D is a maximal element in the set of all proper quasi–$\star$–ideals [respectively, integral $\star$–ideals] of D.

We denote by $\mathrm{Spec}^\star(D)$ [respectively, $\mathrm{Max}^\star(D)$, $\mathrm{QSpec}^\star(D)$, $\mathrm{QMax}^\star(D)$] the set of all $\star$–primes [respectively, $\star$–maximals, quasi–$\star$–primes, quasi–$\star$–maximals] of D.

For example, it is easy to see that, *if $I^\star \neq D^\star$, then $I^\star \cap D$ is a quasi–$\star$–ideal that contains I* (in particular, a $\star$–ideal is a quasi–$\star$–ideal).
Note that:
– when $D = D^\star$ the notions of quasi–$\star$–ideal and $\star$–ideal coincide;
– $I^\star \neq D^\star$ is equivalent to $I^\star \cap D \neq D$.

As in the classical star-operation setting, we associate to a *semistar* operation $\star$ of D a new semistar operation $\star_f$ as follows. If $E \in \overline{\boldsymbol{F}}(D)$ we set:

$$E^{\star_f} := \cup\{F^\star \mid F \subseteq E,\, F \in \boldsymbol{f}(D)\}\,.$$

We call $\star_f$ *the semistar operation of finite type of* D associated to $\star$.

• If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type of* D.
Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is of finite type on D.

The following result is in [21, Lemma 2.3].

Lemma 2 *Let $\star$ be a non-trivial semistar operation of finite type on D. Then*

(1) *Each proper quasi–$\star$–ideal is contained in a quasi–$\star$–maximal.*
(2) *Each quasi–$\star$–maximal is a quasi–$\star$–prime.*
(3) *Set*

$$\Pi^\star := \{P \in \mathrm{Spec}(D) \mid P \neq 0\,, P^\star \cap D \neq D\}\,.$$

Then $\mathrm{QSpec}^\star(D) \subseteq \Pi^\star$ and the set of maximal elements of $\Pi^\star$, denoted by $\Pi^\star_{\max}$, is nonempty and coincides with $\mathrm{QMax}^\star(D)$.

For the sake of simplicity, when $\star = \star_f$, we will denote simply by $\mathcal{M}(\star)$, the nonempty set $\Pi^\star_{\max} = \mathrm{QMax}^\star(D)$.

3 Nagata semistar domain

A generalization of the classical Nagata ring construction was considered by Kang (1987 [39] and 1989 [40]). We further generalize this construction to the semistar setting.

Nagata semistar function ring. Given *any* integral domain D and *any* semistar operation $\star$ on D, we define *the semistar Nagata ring* as follows:

$$\mathrm{Na}(D,\star) := \left\{ \frac{f}{g} \mid f,g \in D[X]\,,\ g \neq 0\,,\ \boldsymbol{c}(g)^\star = D^\star \right\}.$$

Note that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_f)$. Therefore, the assumption $\star = \star_f$ is not really restrictive when considering Nagata semistar rings.

If $\star = d$ is the identity (semi)star operation of D, then:

$$\mathrm{Na}(D,d) = D(X)\,.$$

Some results on *star* Nagata rings proved by Kang in 1989 are generalized to the semistar setting in the following:

Proposition 3 *Let $\star$ be a nontrivial semistar operation of an integral domain D. Set:*

$$N(\star) := N_D(\star) := \{h \in D[X] \mid \boldsymbol{c}(h)^\star = D^\star\}\ .$$

(1) $N(\star) = D[X] \setminus \cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$ *is a saturated multiplicatively closed subset of* $D[X]$ *and* $N(\star) = N(\star_f)$.
(2) $\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$.
(3) $\mathrm{Na}(D,\star) = D[X]_{N(\star)} = \cap\{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.
(4) $\mathcal{M}(\star_f)$ *coincides with the canonical image in* $\mathrm{Spec}(D)$ *of the maximal spectrum of* $\mathrm{Na}(D,\star)$; *i.e.* $\mathcal{M}(\star_f) = \{M \cap D \mid M \in \mathrm{Max}(\mathrm{Na}(D,\star))\}$.

For the proof cf. [21, Theorem 3.1]. From the previous Proposition 3 (4) we have:

Corollary 4 *Let D be an integral domain, then:*
Q is a maximal t–ideal of D $\Leftrightarrow$ $Q = M \cap D$, *for some* $M \in \mathrm{Max}(\mathrm{Na}(D,v))$.

4 The semistar operation associated to $\mathrm{Na}(D,\star)$

We start by recalling some distinguished classes of semistar operations.

• If Δ is a nonempty set of prime ideals of an integral domain D, then the semistar operation $\star_\Delta$ defined on D as follows

$$E^{\star_\Delta} := \cap\{ED_P \mid P \in \Delta\}\,, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D)\,,$$

is called *the spectral semistar operation associated to* Δ.

• A semistar operation $\star$ of an integral domain D is called *a spectral semistar operation* if there exists $\emptyset \neq \Delta \subseteq \mathrm{Spec}(D)$ such that $\star = \star_\Delta$.

• We say that $\star$ *possesses enough primes* or that $\star$ is *a quasi-spectral semistar operation of* D if, for each nonzero ideal I of D such that $I^\star \cap D \neq D$, there exists a quasi–$\star$–prime P of D such that $I \subseteq P$.

• Finally, we say that $\star$ is *a stable semistar operation on* D if

$$(E \cap F)^\star = E^\star \cap F^\star, \quad \text{for all } E,F \in \overline{\boldsymbol{F}}(D)\,.$$

Remark 5 *Mutatis mutandis* the previous notions were considered first in the star settings and, in particular, by D.D. Anderson, D.F. Anderson and S. J. Cook who gave important contributions to the subject [3], [4] and [5]. The general situation was considered among the others by Fontana-Huckaba [17], Fontana-Loper [21] and Halter-Koch [31].

Lemma 6 *Let D be an integral domain and let $\emptyset \neq \Delta \subseteq \mathrm{Spec}(D)$. Then:*

(1) $E^{\star_\Delta} D_P = ED_P$, *for each $E \in \overline{\boldsymbol{F}}(D)$ and for each $P \in \Delta$.*
(2) $(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$, *for all $E, F \in \overline{\boldsymbol{F}}(D)$.*
(3) $P^{\star_\Delta} \cap D = P$, *for each $P \in \Delta$.*
(4) *If I is a nonzero integral ideal of D and $I^{\star_\Delta} \cap D \neq D$, then there exists $P \in \Delta$ such that $I \subseteq P$.*

For the proof cf. [17, Lemma 4.1].

Lemma 7 *Let $\star$ be a nontrivial semistar operation of an integral domain D. Then:*

(1) *$\star$ is spectral if and only if $\star$ is quasi-spectral and stable.*
(2) *Assume that $\star = \star_f$. Then $\star$ is quasi-spectral and $\mathcal{M}(\star) \neq \emptyset$.*

For the proof cf. [17, Theorem 4.12] and [21, Lemma 2.5].

Theorem 8 *Let $\star$ be a nontrivial semistar operation and let $E \in \overline{\boldsymbol{F}}(D)$. Set*

$$\tilde{\star} := (\star_f)_{sp} := \star_{\mathcal{M}(\star_f)}.$$

[$\tilde{\star}$ is called the spectral semistar operation associated to $\star$.] Then:

(1) $E^{\tilde{\star}} = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$ *[and $E^{\star_f} = \cap\{E^{\star_f} D_Q \mid Q \in \mathcal{M}(\star_f)\}$].*
(2) $\tilde{\star} \leq \star_f$.
(3) $E\mathrm{Na}(D, \star) = \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$, *thus:* $E\mathrm{Na}(D, \star) \cap K = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$.
(4) $E^{\tilde{\star}} = E\mathrm{Na}(D, \star) \cap K$.

For the proof cf. [21, Proposition 3.4].

Proposition 3 (4) assures that, when a maximal ideal of $\mathrm{Na}(D, \star)$ is contracted to D, the result is exactly a prime ideal in $\mathcal{M}(\star_f)$. This result can be reversed. Moreover, the semistar operation $\tilde{\star}$ generates the same Nagata ring as $\star$.

Corollary 9 *Let $\star$, $\star_1$, $\star_2$ be semistar operations of an integral domain D. Then:*

(1) $\mathrm{Max}(\mathrm{Na}(D, \star)) = \{QD_Q(X) \cap \mathrm{Na}(D, \star) \mid Q \in \mathcal{M}(\star_f)\}$.
(2) $(\tilde{\star})_f = \tilde{\star} = \tilde{\tilde{\star}}$.
(3) $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.
(4) $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star})$.
(5) $\star_1 \leq \star_2 \Rightarrow \mathrm{Na}(D, \star_1) \subseteq \mathrm{Na}(D, \star_2) \Leftrightarrow \widetilde{\star_1} \leq \widetilde{\star_2}$.

For the proof cf. [21, Corollary 3.5 and Theorem 3.8].

Remark 10 Note that, when $\star$ is the (semi)star v–operation, then the (semi)star operation $\tilde{v}$ coincides with *the (semi)star operation* w defined as follows:

$$E^{w} := \cup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^{v} = D\},$$

for each $E \in \overline{\boldsymbol{F}}(D)$, cf. [17, page 182]. This (semi)star operation was considered by J. Hedstrom and E. Houston in 1980 under the name of F_{∞}–operation [34].

Later, starting in 1997, this operation was studied by Wang Fanggui and R. McCasland under the name of w–operation [61] (cf. also [59], [60] and [62]). (Unfortunately, the same notation is also used for the star a.b. operations defined by a family of valuation overrings [26, page 398] and the two notions are not related, in general.) Note also that the notion of w–ideal coincides with the notion of semi-divisorial ideal considered by S. Glaz and W. Vasconcelos in 1977 [27].

Finally, in 2000, for each (semi)star operation $\star$, D.D. Anderson and S.J. Cook [5] considered the $\star_w$–operation which can be defined as follows:

$$E^{\star_w} := \cup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^{\star} = D\},$$

for each $E \in \overline{\boldsymbol{F}}(D)$. From their theory (and from the results by Hedstrom and Houston) it follows that:

$$\star_w = \tilde{\star}.$$

The relation between $\tilde{\star}$ and the localizing systems of ideals (in the sense of Gabriel and Popescu) was established by M. Fontana and J. Huckaba in 2000 [17].

5 The Kronecker function ring in a general setting

The problem of the construction of a Kronecker function ring for general integral domains was considered indipendently by F. Halter-Koch (2003) [32] and Fontana-Loper (2001, 2003) [19], [21].

Halter-Koch's approach is axiomatic and makes use of the theory of finitary ideal systems (star operations of finite type) [30]. He also establishes a connection with Krull's theory of Kronecker function rings and introduces the Kronecker function rings for integral domains with an ideal system which does not necessarily verify the cancellation property (e.a.b.). Fontana-Loper's treatment is based on the Okabe-Matsuda's theory of semistar operations [19], [21], [55], and [47].

Halter-Koch [32] gives the following abstract definition which does not rely on semistar operations or valuation overrings.

***K*–function ring.** Let K be a field, R a subring of $K(X)$ and $D := R \cap K$. If

(Kr.1) $X \in \boldsymbol{\mathcal{U}}(R)$ (i.e. X is a unit in R);

(Kr.2) $f(0) \in fR$ for each $f \in K[X]$;
then R is called *a K–function ring of D.*

Using only these two axioms, he proved that R "behaves as a Kronecker function ring":

Theorem 11 *Let R be a K–function ring of $D = R \cap K$, then:*

(1) *R is a Bézout domain with quotient field $K(X)$.*
(2) *D is integrally closed in K.*
(3) *For each polynomial $f := a_0 + a_1X + \ldots + a_nX^n \in K[X]$, we have $(a_0, a_1, \ldots, a_n)R = fR$.*

For the proof cf. [32, Theorem 2.2].

Our next goal is to describe Fontana-Loper's approach and to illustrate the relation with Halter-Koch's K–function rings.

Semistar Kronecker function ring. If $\star$ is *any* semistar operation of *any* integral domain D, then we define *the Kronecker function ring of D with respect to the semistar operation $\star$* by:

$$\mathrm{Kr}(D,\star) := \{f/g \mid f, g \in D[X],\ g \neq 0, \text{ and there exists } h \in D[X] \setminus \{0\} \text{ with } (\boldsymbol{c}(f)\boldsymbol{c}(h))^\star \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^\star\}.$$

At this point, we need some preliminaries in order:
— to show that this construction leads to a natural extension of the classical Kronecker function ring,
— to investigate the connections between the semistar Kronecker function ring $\mathrm{Kr}(D,\star)$ and the axiomatically defined K-function ring,
— to show that $\mathrm{Kr}(D,\star)$ defines a new semistar operation on D, behaving with respect $\mathrm{Kr}(D,\star)$ in a similar way to $\tilde{\star}$ with respect to $\mathrm{Na}(D,\star)$.

We start by recalling that it is possible to associate to an arbitrary semistar operation an e.a.b. semistar operation.

• Given any semistar operation $\star$ of D, we can define an e.a.b. semistar operation of finite type $\star_a$ of D, called *the e.a.b. semistar operation associated to $\star$,* as follows for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$F^{\star_a} := \cup\{((FH)^\star : H^\star) \mid H \in \boldsymbol{f}(D)\},$$
$$E^{\star_a} := \cup\{F^{\star_a} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}.$$

The previous construction is essentially due to P. Jaffard (1960) [38, Chapitre II, §2] and F. Halter-Koch (1997, 1998) [29, Section 6], [30, Chapter 19] (cf. also Lorenzen (1939) [46] and Aubert (1983) [10]).

Obviously $(\star_{_f})_a = \star_a$. Note that (for instance [19, Proposition 4.3 and 4.5]):
– *when $\star = \star_{_f}$, then $\star$ is e.a.b. if and only if $\star = \star_a$.*
– *$D^{\star_a}$ is integrally closed and contains the integral closure of D.*

When $\star = v$, then D^{v_a} coincides with *the pseudo-integral closure of* D introduced by D.F. Anderson, Houston and Zafrullah (1992) [7].

Remark 12 In the classical context of *star* operations, $\star_a$ is expected to be a star operation too and for this reason is defined on the "star closure" of D (or, on an integral domain which is "star closed"), cf. Okabe-Matsuda (1992) [53], Halter-Koch (1997, 1998, 2003) [29], [30], [32].
More precisely (even if $\star$ is a semistar operation), we call *the* $\star$*–closure of* D:

$$D^{\mathrm{cl}^\star} := \cup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\}\,.$$

It is easy to see that $D^{\mathrm{cl}^\star}$ *is an integrally closed overring of* D and D is said $\star$*–closed* if $D = D^{\mathrm{cl}^\star}$.
We can now define a new (semi)star operation on D if $D = D^{\mathrm{cl}^\star}$ (or, in general, a semistar operation on D), $\mathrm{cl}^\star$ by setting for each $F \in \boldsymbol{f}(D)$, for each $E \in \overline{\boldsymbol{F}}(D)$:

$$\begin{aligned} F^{\mathrm{cl}^\star} &:= \cup\{((H^\star : H^\star)F)^\star \mid H \in \boldsymbol{f}(D)\}\,,\\ E^{\mathrm{cl}^\star} &:= \cup\{F^{\mathrm{cl}^\star} \mid F \subseteq E\,,\ F \in \boldsymbol{f}(D)\}\,. \end{aligned}$$

If we set $\overline{\star} := \mathrm{cl}^\star$, it is not difficult to see that $D^{\mathrm{cl}^{\overline{\star}}} = D^{\mathrm{cl}^\star}$ (and that it coincides with $D^{\star_a}$) and $D^{\mathrm{cl}^\star}$ contains the "classical" integral closure of D. Moreover (as semistar operations on D):

$$\star_{_f} \leq \mathrm{cl}^\star \leq \star_a\,, \quad (\star_{_f})_a = (\mathrm{cl}^\star)_a = (\star_a)_a = \star_a\,.$$

We now turn our attention to the valuation overrings. The notion that we recall next is due to P. Jaffard (1960) [38, page 46] (cf. also Halter-Koch (1997) [30, Chapters 15 and 18]).

- For a domain D and a semistar operation $\star$ on D, we say that a valuation overring V of D is *a* $\star$*–valuation overring of* D provided $F^\star \subseteq FV$, for each $F \in \boldsymbol{f}(D)$.

Note that, by definition the $\star$–valuation overrings coincide with the $\star_f$–valuation overrings.

Proposition 13 *Let* D *be a domain and let* $\star$ *be a semistar operation on* D.

(1) *The* $\star$*–valuation overrings also coincide with the* $\star_a$*–valuation overrings.*
(2) $D^{\mathrm{cl}^\star} = \cap\{V \mid V$ *is a* $\star$*–valuation overring of* $D\}$.
(3) *A valuation overring* V *of* D *is a* $\tilde{\star}$*–valuation overring of* D *if and only if* V *is an overring of* D_P, *for some* $P \in \mathcal{M}(\star_f)$.

For the proof cf. for instance [20, Proposition 3.2, 3.3 and Corollary 3.6] and [21, Theorem 3.9].

Theorem 14 *Let $\star$ be a semistar operation of an integral domain D with quotient field K. Then:*

(1) $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$.
(2) *V is a $\star$–valuation overring of D if and only if $V(X)$ is a valuation overring of $\mathrm{Kr}(D,\star)$.*
The map $W \mapsto W \cap K$ establishes a bijection between the set of all valuation overrings of $\mathrm{Kr}(D,\star)$ and the set of all the $\star$–valuation overrings of D.
(3) $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_f) = \mathrm{Kr}(D,\star_a) = \cap\{V(X) \mid V$ *is a $\star$–valuation overring of $D\}$ is a Bézout domain with quotient field $K(X)$.*
(4) $E^{\star_a} = E\mathrm{Kr}(D,\star) \cap K = \cap\{EV \mid V$ *is a $\star$–valuation overring of $D\}$, for each $E \in \overline{\boldsymbol{F}}(D)$.*
(5) $R := \mathrm{Kr}(D,\star)$ *is a K–function ring of $R \cap K = D^{\star_a}$ (in the sense of Halter-Koch's axiomatic definition).*

For the proof cf. [19, Theorem 3.11], [20, Theorem 3.5], [21, Proposition 4.1].

6 Some relations between $\mathrm{Na}(D,\star)$, $\mathrm{Kr}(D,\star)$, and the semistar operations $\widetilde{\star}$, $\star_a$

An elementary first question to ask is whether the two semistar operations $\tilde{\star}$ and $\star_a$ are actually the same - or usually the same - or rarely the same.

Proposition 13 indicates that for a semistar operation $\star$ on a domain D, the $\tilde{\star}$–valuation overrings of D are all the valuation overrings of the localizations of D at the primes in $\mathcal{M}(\star_f)$. On the other hand, we know from Theorem 14 that the $\star_a$–valuation overrings (or, equivalently, the $\star$–valuation overrings) of D correspond exactly to the valuation overrings of the Kronecker function ring $\mathrm{Kr}(D,\star)$. In particular, each $\star_a$–valuation overring is also a $\tilde{\star}$–valuation overring.

It is easy to imagine that these two collections of valuation domains can frequently be different and, even when the two collections of valuation domains coincide, it may happen that $\tilde{\star} \neq \star_a$. Fontana-Loper [21] gives some examples which illustrate the different situations that can occur.

It is possible to prove positive statements about the relationship between $\widetilde{(\text{-})}$ and $(\text{-})_a$ under certain conditions [21, Proposition 5.4 and Remark 5.5]. However, we limit ourselves to stating a result that generalizes the fundamental result that is at the basis of Krull's theory of Kronecker function rings, i.e. $\mathrm{Na}(D) = \mathrm{Na}(D,d) = \mathrm{Kr}(D,b) = \mathrm{Kr}(D)$ if and only if D is a Prüfer domain, cf. for instance [26, Theorem 33.4].

We recall the following definition, which generalizes the classical notion of Prüfer domain.

Prüfer semistar multiplication domain. Let $\star$ be a semistar operation on an integral domain D. A *Prüfer $\star$–multiplication domain* (for short, a

P⋆MD) is an integral domain D such that $(FF^{-1})^{\star_f} = D^{\star_f}$ $(= D^{\star})$ (i.e., each F is $\star_f$*–invertible*) for each $F \in \boldsymbol{f}(D)$

Some of the statements of the following theorem, due to Fontana-Jara-Santos (2003) [18] generalize some of the classical characterizations of the Prüfer v–multiplication domains (for short, PvMD).

Theorem 15 *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent:*

(i) *D is a P⋆MD.*
(ii) $\mathrm{Na}(D,\star)$ *is a Prüfer domain.*
(iii) $\mathrm{Na}(D,\star) = \mathrm{Kr}(D,\star)$.
(iv) $\tilde{\star} = \star_a$.
(v) $\star_f$ *is stable and e.a.b..*

In particular, D is a P⋆MD if and only if it is a P$\tilde{\star}$MD.

For the proof cf. [18, Theorem 3.1 and Remark 3.1].

The following gives the converse of the implication PvMD $\Rightarrow$ PwMD proved by Wang-McCasland (1999) [62, Section 2, page 160], cf. also D.D. Anderson-Cook (2000) [5, Theorem 2.18].

Corollary 16 *Let D be an integral domain. The following are equivalent:*

(i) *D is a PvMD.*
(ii) $\mathrm{Na}(D,t) = \mathrm{Kr}(D,t)$.
(iii) $w := \tilde{v} = v_a$.
(iv) *t is stable and e.a.b..*

In particular, D is a PvMD if and only if it is a PwMD.

For the proof cf. [18, Corollary 3.1].

In the star setting the relation between the P⋆MDs and the PvMDs is described by the following:

Corollary 17 *Let D be an integral domain and $\star$ a* star *operation on D.*

D is a P⋆MD $\Leftrightarrow$ D is a PvMD and $t = \widetilde{\star}$ (or, equivalently, $t = \star_f$).

For the proof cf. [18, Proposition 3.4].

Remark 18 The PvMDs were studied by Griffin in 1967 [28] under the name of v–multiplication domains, cf. also [42] and [38]. Relevant contributions to the subject were given among the others by Arnold-Brewer (1971) [9], Heinzer-Ohm (1973) [35], Mott-Zafrullah (1981) [50], Zafrullah (1984) [64], Houston (1986) [36], Kang (1989) [40], Dobbs-Houston-Lucas-Zafrullah (1989) [14] and El Baghdadi (2002) [16].

For $\star$ a star operation, P⋆MDs were considered by Houston-Malik-Mott in 1984 [37], introducing a unified setting for studying Krull domains, Prüfer domains and PvMDs. This class of domains was also investigated by Garcia-Jara-Santos (1999) [24] and Halter-Koch (2003) [33] These papers led naturally to the study of the Prüfer semistar multiplication domains initiated in 2003 by Fontana-Jara-Santos [18].

Related to this study are the questions on the invertibility property of ideals and modules especially in the star and semistar setting, cf. the survey paper by Zafrullah [65], Chang-Park (2003) [11] and Fontana-Picozza (2005) [**?**].

7 Intersections of local Nagata domains

Given a semistar operation $\star$ on D, the integral domains $\mathrm{Na}(D,\star)$ and $\mathrm{Kr}(D,\star)$ (and the related semistar operations $\tilde{\star}$ and $\star_a$) have in many regards a similar behaviour. The following natural question is the starting point of a recent paper by M. Fontana and K.A. Loper [22]:

Is it possible to find an integral domain of rational functions, denoted by $\mathrm{KN}(D,\star)$ (obtained as an intersection of local Nagata domains associated to any semistar operation $\star$), *such that:*
- $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}(D,\star) \subseteq \mathrm{Kr}(D,\star)$ *;*
- $\mathrm{KN}(D,\star)$ *generalizes at the same time* $\mathrm{Na}(D,\star)$ *and* $\mathrm{Kr}(D,\star)$ *and coincides with* $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star})$ *or* $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_a)$*, when the semistar operation (of finite type)* $\star$ *assumes the extreme values of the interval* $\tilde{\star} \leq \star \leq \star_a$ *?*

In order to present the anwer to the previous question we need to settle some terminology:

- If F is in $\boldsymbol{f}(D)$, we say that F is $\star$*–e.a.b.* [respectively, $\star$*–a.b.*] if $(FG)^\star \subseteq (FH)^\star$, with $G,\ H \in \boldsymbol{f}(D)$ [respectively, $G,\ H \in \overline{\boldsymbol{F}}(D)$], implies that $G^\star \subseteq H^\star$.

Lemma 19 *Let $\star$ be a semistar operation on an integral domain D, let $F \in \boldsymbol{f}(D)$ be $\star_{_f}$–invertible and let (L,N) be a local $\star$–overring of D. Then FL is a principal fractional ideal of L.*

For the proof cf. [22, Corollary 4.3].

Note that, in general, $\star$–(e.)a.b. does not imply $\star$–invertible, even for finite type semistar operations. However, it is possible to show that, for finite type stable semistar operations $\star$ (i.e. when $\star = \tilde{\star}$), the notions of $\star$–e.a.b., $\star$–a.b. and $\star$–invertible coincide [22, Proposition 5.3(2)].

Our next goal is to generalize Lemma 19 to the case of $\star$–e.a.b. ideals.

Semistar monolocalities. Let $\star$ be a semistar operation on an integral domain D. *A $\star$–monolocality of* D is a local overring L of D such that:

– FL is a principal fractionary ideal of L, for each $\star$–e.a.b. $F \in \boldsymbol{f}(D)$;
– $L = L^{\star_f}$.

Obviously, each $\star$–valuation overring is a $\star$–monolocality. It is not hard to prove that, for each $Q \in \mathcal{M}(\star_f)$, D_Q is a $\tilde{\star}$–monolocality [22, Proposition 5.3(1)]. Set:

$$\begin{aligned} \boldsymbol{\mathcal{L}}(D,\star) &:= \{L \mid L \text{ is a } \star\text{–monolocality of } D\} \\ \mathrm{KN}(D,\star) &:= \cap\{L(X) \mid L \in \boldsymbol{\mathcal{L}}(D,\star)\}\,. \end{aligned}$$

We are now in condition to state some among the main results proved in Fontana-Loper (2005) [22].

Theorem 20 *Let $\star$ be a semistar operation on an integral domain D with quotient field K.*

(1) $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}(D,\star) \subseteq \mathrm{Kr}(D,\star))$.

(2) $\mathrm{KN}(D,\star) := \{f/g \in K(X) \mid f, g \in D[X],\ g \neq 0,$ *such that* $\boldsymbol{c}(g)$ *is* $\star$–e.a.b. *and* $\boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^\star\}$.

(3) *For each maximal ideal $\mathfrak{m}$ of $\mathrm{KN}(D,\star)$, set $L(\mathfrak{m}) := \mathrm{KN}(D,\star)_{\mathfrak{m}} \cap K$. Then:*

$L(\mathfrak{m})$ is a $\star$–monolocality of D (with maximal ideal $\mathfrak{M} := \mathfrak{m}\mathrm{KN}(D,\star)_{\mathfrak{m}} \cap L(\mathfrak{m})$),

$\mathrm{KN}(D,\star)_{\mathfrak{m}}$ coincides with the Nagata ring $L(\mathfrak{m})(X)$ and $\mathfrak{m}$ coincides with $\mathfrak{M}(X) \cap \mathrm{KN}(D,\star)$.

(4) *Every $\star$–monolocality of an integral domain D contains a minimal $\star$–monolocality of D. If we denote by $\boldsymbol{\mathcal{L}}(D,\star)_{min}$ the set of all the minimal $\star$–monolocalities of D, then*
$\boldsymbol{\mathcal{L}}(D,\star)_{min} = \{L(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}(D,\star))\}$ *and, obviously,*
$\mathrm{KN}(D,\star) = \cap\{L(X) \mid L \in \boldsymbol{\mathcal{L}}(D,\star)_{min}\}$.

(5) *For each $J := (a_0,\ a_1,\ \ldots,\ a_n)D \in \boldsymbol{f}(D)$, with $J \subseteq D$ and J $\star$–e.a.b., let $g := a_0 + a_1X + \ \ldots\ + a_nX^n \in D[X]$, then:*

$$J\mathrm{KN}(D,\star) = J^\star\mathrm{KN}(D,\star) = g\mathrm{KN}(D,\star)\,.$$

(6) *Let $\star_\ell$ and $\wedge_{\boldsymbol{\mathcal{L}}}$ be the semistar operations of D defined as follows, for each $E \in \overline{\boldsymbol{F}}(D)$,*

$$\begin{aligned} E^{\star_\ell} &:= E\mathrm{KN}(D,\star) \cap K\,, \\ E^{\wedge_{\boldsymbol{\mathcal{L}}}} &:= \cap\{EL \mid L \in \boldsymbol{\mathcal{L}}(D,\star)\}\,. \end{aligned}$$

Then:

$$\widetilde{\star} \leq \star_\ell = \wedge_{\boldsymbol{\mathcal{L}}} \leq \star_a\,.$$

(7) $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\widetilde{\star}) = \mathrm{KN}(D,\widetilde{\star})$ *and* $\mathrm{KN}(D,\star_a) = \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D,\star)$.

For the proof cf. [22, Theorem 5.11 ((1),(4),(5),(6), and (7)), Proposition 6.3, and Corollary 6.4 ((1) and (2))].

References

1. D. D. Anderson, *Multiplication ideals, multiplication rings, and the ring $R(X)$,* Canad. J. Math. **28** (1976), 760–768.
2. D. D. Anderson, *Some remarks on the ring $R(X)$,* Comm. Math. Univ. St. Pauli **26** (1977), 137–140.
3. D. D. Anderson, *Star operations induced by overrings,* Comm. Algebra **28** (1988), 2535–2553.
4. D. D. Anderson and D. F. Anderson, *Examples of star operations on integral domains,* Comm. Algebra **18** (1990), 1621–1643.
5. D. D. Anderson and S. J. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), 2461–2475.
6. D. F. Anderson, D. E. Dobbs and M. Fontana, *When is a Bézout domain a Kronecker function ring ?* C.R. Math. Rep. Acad. Sci. Canada **9** (1987), 25–30.
7. D. F. Anderson, E. G. Houston and M. Zafrullah, *Pseudo-integrality*, Canad. Math. Bull. **34** (1991), 15–22.
8. J. Arnold, *On the ideal theory of the Kronecker function ring and the domain $D(X)$*, Canad. J. Math. **21** (1969), 558–563.
9. J. Arnold and J. Brewer, *Kronecker function rings and flat $D[X]$–modules*, Proc. Amer. Math. Soc. **27** (1971), 483–485.
10. K. E. Aubert, *Divisors of finite type,* Ann. Mat. Pura Appl. **133** (1983), 327–361.
11. G. W. Chang and J. Park, *Star-invertible ideals of integral domains,* Boll. Un. Mat. Ital. **6-B** (2003), 141–150.
12. R. Dedekind, *Supplement XI* to *Vorlesungen über Zahlentheorie* von P.G. Lejeune Dirichlet. Gesammelte mathematische Werke (R. Fricke, E. Noether and O. Ore, Editors) Vol. 3, 1-222. Vieweg, 1930-1932.
13. D. E. Dobbs and M. Fontana, *Kronecker function rings and abstract Riemann surfaces,* J. Algebra **99** (1986), 263–284.
14. D. Dobbs, E. Houston, T. Lucas and M. Zafrullah, *t-linked overrings and Prüfer v-multiplication domains,* Comm. Algebra **17** (1989), 2835–2852.
15. H.M. Edwards, *Divisor Theory,* Birkhäuser, 1990.
16. S. El Baghdadi, *On a class of Prüfer v-multiplication domains,* Comm. Algebra **30** (2002), 3723-3742.
17. M. Fontana and J. Huckaba, *Localizing systems and semistar operations,* in “Non Noetherian Commutative Ring Theory” (S. Chapman and S. Glaz, Editors), Kluwer Academic Publishers, Dordrecht, 2000, Chapter 8, 169–187.
18. M. Fontana, P. Jara and E. Santos, *Prüfer $\star$–multiplication domains and semistar operations,* J. Algebra Appl. **2** (2003), 21–50.
19. M. Fontana and K. A. Loper, *Kronecker function rings: a general approach,* in “Ideal Theoretic Methods in Commutative Algebra” (D.D. Anderson and I.J. Papick, Editors), M. Dekker Lecture Notes Pure Appl. Math. **220** (2001), 189–205.
20. M. Fontana and K. A. Loper, *A Krull-type theorem for the semistar integral closure of an integral domain,* ASJE Theme Issue “Commutative Algebra” **26** (2001), 89–95.
21. M. Fontana and K. A. Loper, *Nagata rings, Kronecker function rings and related semistar operations,* Comm. Algebra **31** (2003), 4775–4805.

22. M. Fontana and K. A. Loper, *A generalization of Kronecker function rings and Nagata rings,* Preprint.
23. Marco Fontana and Giampaolo Picozza. *Semistar invertibility on integral domains*, Algebra Colloq., **12** (2005), 645–664.
24. J. M. García, P. Jara and E. Santos, *Prüfer $\star$–multiplication domains and torsion theories,* Comm. Algebra **27** (1999), 1275–1295.
25. R. Gilmer, *Multiplicative Ideal Theory (Part I and II),* Queen's Papers in Pure and Applied Mathematics, Queen's University, Kingston, Ontario, Canada, 1968.
26. R. Gilmer, *Multiplicative Ideal Theory,* M. Dekker, New York, 1972.
27. S. Glaz and W. Vasconcelos, *Flat ideals, II*, Manuscripta Math. **22** (1977), 325–341.
28. M. Griffin, *Some results on v–multiplication rings*, Canad. J. Math. **19** (1967), 710–721.
29. F. Halter-Koch, *Generalized integral closures,* in "Factorization in Integral Domains" (D.D. Anderson, Editor), M. Dekker Lecture Notes Pure Appl. Math. **187** (1997), 349–358.
30. F. Halter-Koch, *Ideal Systems: An Introduction to Multiplicative Ideal Theory,* M. Dekker, New York, 1998.
31. F. Halter-Koch, *Localizing systems, module systems and semistar operations,* J. Algebra **238** (2001), 723–761.
32. F. Halter-Koch, *Kronecker function rings and generalized integral closures*, Comm. Algebra **31** (2003), 45–59.
33. F. Halter-Koch, *Characterization of Prüfer multiplication monoids and domains by means of spectral module systems,* Monatsh. Math. **139** (2003), 19-31.
34. J. R. Hedstrom and E. G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), 37–44.
35. W. Heinzer and J. Ohm, *An essential ring which is not a v-multiplication ring,* Can. J. Math. **25** (1973), 856–861.
36. E. Houston, *On divisorial prime ideals in Prüfer v-multiplication domains,* J. Pure Appl. Algebra **42** (1986), 55–62.
37. E. G. Houston, S. B. Malik, and J. L. Mott, *Characterization of $\star$–multiplication domains*, Canad. Math. Bull. **27** (1984), 48–52.
38. P. Jaffard, *Les Systèmes d'Idéaux,* Dunod, Paris, 1960.
39. B. G. Kang, *$\star$–operations on integral domains*, Ph.D. Dissertation, Univ. Iowa 1987.
40. B. G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$,* J. Algebra **123** (1989), 151–170.
41. L. Kronecker, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen,* J. Reine Angew. Math., **92** (1882), 1–122; *Werke* **2**, 237–387 (K. Hensel, Editor, 5 Volumes published from 1895 to 1930, Teubner, Leipzig) Reprint, Chelsea 1968.
42. W. Krull, *Idealtheorie,* Springer-Verlag, Berlin, 1935 (II edition, 1968).
43. W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche,* I - II. Math. Z. **41** (1936), 545–577; 665–679.
44. W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche,* VIII. Math. Z. **48** (1942/43), 533–552.
45. E. E. Kummer, *Collected papers,* (A. Weil, Editor), Springer 1975.

46. P. Lorenzen, *Abstrakte Begründung der multiplikativen Idealtheorie,* Math. Z. **45** (1939), 533–553.
47. R. Matsuda, *Kronecker function rings of semistar operations on rings*, Algebra Colloquium **5** (1998), 241–254.
48. R. Matsuda and I. Sato, *Note on star operations and semistar operations*, Bull. Fac. Sci. Ibaraki Univ. Ser. A **28** (1996), 5–22.
49. R. Matsuda and T. Sugatani, *Semistar operations on integral domains*, II, Math. J. Toyama Univ. **18** (1995), 155–161.
50. J. L. Mott and M. Zafrullah, *On Prüfer v–multiplication domains,* Manuscripta Math. **35** (1981), 1–26.
51. M. Nagata, *A general theory of algebraic geometry over Dedekind domains, I,* Amer. J. Math. **78** (1956), 78–116.
52. M. Nagata, *Local rings*, Interscience, New York, 1962.
53. A. Okabe and R. Matsuda, *Star operations and generalized integral closures,* Bull. Fac. Sci. Ibaraki Univ. Ser. A **24** (1992), 7–13.
54. A. Okabe and R. Matsuda, *Semistar operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.
55. A. Okabe and R. Matsuda, *Kronecker function rings of semistar operations*, Tsukuba J. Math., **21** (1997), 529-540.
56. H. Prüfer, *Untersuchungen über Teilbarkeitseigenschaften in Körpern,* Journ. Reine Angew. Math. **168** (1932), 1–36.
57. P. Quartaro jr. and H. S. Butts, *Finite unions of ideals and modules*, Proc. Amer. Math. Soc. **52** (1975), 91–96.
58. P. Samuel, *Lectures on unique factorization domains*, Tata Institute, Bombay, 1964.
59. Fanggui Wang, *On w–projective modules and w–flat modules*, Algebra Colloquium **4** (1997), 111–120.
60. Fanggui Wang, *On UMT-domains and w–integral dependence*, Preprint.
61. Fanggui Wang and R. L. McCasland, *On w–modules over strong Mori domains*, Comm. Algebra **25** (1997), 1285–1306.
62. Fanggui Wang and R. L. McCasland, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), 155–165.
63. H. Weyl, *Algebraic Theory of Numbers,* Princeton University Press, Princeton, 1940.
64. M. Zafrullah, *Some polynomial characterizations of Prüfer v–multiplication domains,* J. Pure Appl. Algebra **32** (1984), 231–237.
65. M. Zafrullah, *Putting t-invertibility to use,* in "Non Noetherian Commutative Ring Theory" (S. Chapman and S. Glaz, Editors), Kluwer Academic Publishers, Dordrecht, 2000, 429–457.