In memory of Jim Huckaba
for a lifelong friendship

Star and semistar operations in polynomial extensions

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§1. indroduction

One of the first attempt of relating star operations defined on an integral domain D with star operations defined on the polynomial extension $D[X]$ is due to Houston-Malik-Mott [HMM, 1984].

Given a star operation $*$ on $D[X]$, they defined a star operation $*_0$ on D, by setting for all $E \in \mathbf{F}(D)$ (= the set of all nonzero fractional ideals of D)

$$E^{*_0} := (ED[X])^* \cap K\,.$$

They preliminarly observed that

if $*$ is of finite type on $D[X]$ then $*_0$ is of finite type on D, and

$$(E^{*_0}D[X])^* = (ED[X])^* \textit{ for all } E \in \mathbf{F}(D)\,.$$

The following are among the main results obtained in [HMM, 1984].

Theorem 1

With the notation introduced above, assume that $*$ is a star operation of finite type on $D[X]$ and for each $Q \in \mathrm{Spec}^*(D[X])$ either Q is extended (i.e., $Q = (Q \cap D)[X]$) or Q is an upper to zero (i.e., $Q \cap D = (0)$), then

$$D \text{ is a } P{*_0}MD \;\Leftrightarrow\; D[X] \text{ is a } P{*}MD\,.$$

Corollary 2

Assume that D is an integrally closed domain. Then,

$$D \text{ is a } Pv_D MD \;\Leftrightarrow\; D[X] \text{ is a } Pv_{D[X]} MD\,.$$

Note that the previous corollary is a combination of various facts: $* = t_{D[X]} \Rightarrow *_0 = t_D$ (Hedstrom-Houston [HH,1980]); PvMD = PtMD; Theorem 1 (when $D = \overline{D}$ the t-operation verifies the hypothesis of Theorem 1).

• In 2007 in a joint work with G.W. Chang [CF1], we started to study the problem of the possibility of extending in a "canonical way" a semistar (or a star) operation $\star$ defined on D to a semistar (or a star) operation $\star_1$ defined on $D[X]$, having in view, among various questions, a sort of "ascending version" of Theorem 1:

$$D \text{ is a P}\star\text{MD} \;\Leftrightarrow\; D[X] \text{ is a P}\star_1\text{MD}.$$

• At the same time, in 2007 Picozza investigated various problems on semistar Noetherian domains and, in particular, the possibility of a semistar version of Hilbert Basis Theorem: i.e., given a semistar (or a star) operation $\star$ defined on D determine a semistar (or a star) operation $\star'$ defined on $D[X]$ such that

$$D \text{ is } \star\text{–Noetherian} \;\Leftrightarrow\; D[X] \text{ is } \star'\text{–Noetherian}.$$

Picozza motivations were related to the following facts:

- Noetherian = d–Noetherian; Mori = v–Noetherian = t–Noetherian; strong Mori = w–Noetherian.

- D is d_D–Noetherian $\Leftrightarrow$ $D[X]$ is $d_{D[X]}$–Noetherian (Hilbert, 1888)
  D is w_D–Noetherian $\Leftrightarrow$ $D[X]$ is $w_{D[X]}$–Noetherian;
  (F.G. Wang - McCasland, 1999);
  but D is t_D–Noetherian $\not\Leftarrow$ $D[X]$ is $t_{D[X]}$–Noetherian,
  (Roitman, 1990).

Picozza investigated the natural problem: what is the "star-theoretic" reason of the different behaviour of the previous star operations when passing to the polynomial extensions ?

There are several other reasons for investigating the problem of ascending star and semistar operations in polynomial extension (e.g., star (or semistar) Krull dimensions, star (or semistar) class groups, etc.), but I have no time to go more in details with other preliminaries in this talk.

§2. Stable star and semistar operations in polynomial extensions

The problem of ascending in a canonical way a star or a semistar operation to a polynomial domains is not easy in general. We have at the moment a satisfactory solution for stable star or semistar operations of finite type.

However, this case is sufficiently general to lead us to give a complete answer to the problem of ascending for instance the Prüfer star (or, semistar)-multiplication property from a domain D to the polynomial extension $D[X]$.

The starting point is based on a series of results obtained in a joint paper with J. Huckaba (2000), where we established a close connection between stable star or semistar operations and localizing systems of ideals (in the sense of Popescu-Gabriel).

Let D be an integral domain with quotient field K.
Let $\overline{\mathbf{F}}(D)$ be the set of all nonzero D-submodules of K, $\mathbf{F}(D)$ the set of all nonzero fractional ideals of D, and $\mathbf{f}(D)$ the set of all nonzero finitely generated D–submodules of K.
Then, obviously $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D)$.

Some definitions

• A *semistar operation* $\star$ on an integral domain D is *stable* if distributes over finite intersections (i.e., $(E_1 \cap E_2)^\star = E_1^\star \cap E_2^\star$ for all $E_1, E_2 \in \overline{\mathbf{F}}(D)$).

• A *semistar operation of finite type* $\star$ is an operation such that $E^\star = \bigcup\{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\}$ for all $E \in \overline{\mathbf{F}}(D)$.

• A *localizing system of ideals* $\mathcal{F}$ of an integral domain is a set of ideals verifying the following properties:

- $I \in \mathcal{F}$ and $I \subseteq J \Rightarrow J \in \mathcal{F}$
- $I \in \mathcal{F}$ and $(J :_D iD) \in \mathcal{F}$ for all $i \in I \Rightarrow J \in \mathcal{F}$.

• A *localizing system $\mathcal{F}$ of finite type* is a localizing system $\mathcal{F}$ such that for each $I \in \mathcal{F}$ there exists a nonzero finitely generated ideal $J \in \mathcal{F}$ with $J \subseteq I$.

In a joint paper with J. Huckaba we have established a bridge between semistar operations and localizing systems. More precisely:

Theorem Fontana-Huckaba, 2000

• If $\mathcal{F}$ is a localizing system on D, then $\star_{\mathcal{F}}$ defined as follows $E^{\star_{\mathcal{F}}} := E_{\mathcal{F}} := \bigcup\{(E : I) \mid I \in \mathcal{F}\}$, for $E \in \overline{\mathbf{F}}(D)$, is a stable semistar operation on D.

• If $\mathcal{F}$ is a localizing system of finite type, then $\star_{\mathcal{F}}$ is a (stable) semistar operation of finite type.

• If $\star$ is a semistar operation [of finite type] on D, then $\mathcal{F}^{\star} := \{I \text{ ideal of } D \mid I^{\star} = D^{\star}\}$ is a localizing system [of finite type] of D.

• The mapping $\mathcal{F} \mapsto \star_{\mathcal{F}}$ establishes a one-to-one correspondence between the localizing systems of finite type on D and the stable semistar operations of finite type on D.

Note that related results, in the star-operation setting, were also obtained by D.D. Anderson and Cook [AC] in 2000 and in the monoid setting by F. Halter-Koch in [H-K] in 2001.

§3. Some results on stable semistar operations and polynomial extensions.

Recall that to a given semistar operation $\star$ on an integral domain D we can associate canonically a *semistar operation of finite type* $\star_f$ and a *stable semistar operation of finite type* $\widetilde{\star}$ on D

$$E^{\star_f} := \bigcup \{F^\star \mid F \in \mathbf{f}(D),\ F \subseteq E\},$$
$$E^{\widetilde{\star}} := \bigcap \{ED_P \mid P \in \mathrm{QMax}^{\star_f}(D)\},$$

where $\mathrm{QMax}^{\star_f}(D)$ is the set of all *quasi-$\star_f$-maximal* ideals of D (we say that a nonzero ideal I of D is a *quasi-$\star_f$-ideal* if $I^{\star_f} \cap D = I$). If we set

$\mathcal{N}^\star := \{0 \neq g \in D[X] \mid \mathbf{c}_D(g)^\star = D^\star\}$ and
$\mathrm{Na}(D, \star) := \{f/g \mid f \in D[X],\ g \in \mathcal{N}^\star\}$, then it is known that:

$$\mathrm{Na}(D, \star) = \bigcap \{D_P(X) \mid P \in \mathrm{QMax}^{\star_f}(D)\}\ ,$$
$$E^{\widetilde{\star}} = E\mathrm{Na}(D, star) \cap K \text{ for all } E \in \overline{\mathbf{F}}(D)\,.$$

and *$\star$ is a stable semistar operation of finite type if and only if $\star = \widetilde{\star}$*.

Given a *stable semistar operation of finite type* $\star$ *on an integral domain* D, the problem that we want to study is how to define in a canonical way a *stable operation of finite type* $\star_1$ *on* $D[X]$ such that $(\star_1)_0 = \star$ and, as an application, we want to show that

$$D \text{ is a P}\star\text{MD} \Leftrightarrow D[X] \text{ is a P}\star_1\text{MD}.$$

It is important to note that, without loss of generality, we can consider the case of *stable operations of finite type*, since Fontana-Jara-Santos [FJS, Theorem 31] in 2003, giving a characterization of P$\star$MD's, have observed that the notions of P$\star$MD and P$\widetilde{\star}$MD coincide.

Note that a similar result, in the star-operation setting, was obtained by D.D. Anderson and Cook [AC].

Note that, to a multiplicative subset $\mathcal{S}$ of $D[X]$, we can associate the semistar operation $*_{\mathcal{S}}$ on $D[X]$ defined by
$A^{*_{\mathcal{S}}} := A_{\mathcal{S}} = \bigcup\{(A : J) \mid J \text{ ideal of } D[X],\ J \cap \mathcal{S} \neq \emptyset\} = AD[X]_{\mathcal{S}}$, for all $A \in \overline{\mathbf{F}}(D[X])$.

Chang and Fontana [CF1] investigated the map
$E \mapsto ED[X]_{\mathcal{S}} \cap K$ $(=: E^{\circlearrowleft_{\mathcal{S}}})$, defined for all $E \in \overline{\mathbf{F}}(D)$, showing that

the previous map gives rise to a semistar operation $\star$ $(:= \circlearrowleft_{\mathcal{S}})$ on D, such that
- *$D^\star = R := D[X]_{\mathcal{S}} \cap K$ is t-linked to $(D, \star)$*
(i.e., for each nonzero finitely generated ideal I of D, $I^\star = D^\star$ implies $(IR)^{t_R} = R$ or, equivalently, $R = R^{\widetilde{\star}}$),
- the operation $\star$ $(= \circlearrowleft_{\mathcal{S}})$ on D coincides with $(*_{\mathcal{S}})_0$.

Clearly, if $\overline{\mathcal{S}} := D[X] \setminus \bigcup\{Q \mid Q \in \mathrm{Spec}(D[X]) \text{ and } Q \cap \mathcal{S} = \emptyset\}$ is the saturation of the multiplicative set $\mathcal{S}$, then $*_{\mathcal{S}} = *_{\overline{\mathcal{S}}}$ and so, in particular, $\circlearrowleft_{\mathcal{S}} = \circlearrowleft_{\overline{\mathcal{S}}}$.

In order to deepen the knowledge of the semistar operation $\circlearrowleft_{\mathcal{S}}$, we need a definition of a stronger version of saturation. Set:

$$\mathcal{S}^\sharp := D[X] \setminus \bigcup \{P[X] \mid P \in \mathrm{Spec}(D) \text{ and } P[X] \cap \mathcal{S} = \emptyset\}.$$

It is clear that $\mathcal{S}^\sharp$ is a saturated multiplicative set of $D[X]$ and that $\mathcal{S}^\sharp$ contains the saturation of $\mathcal{S}$, i.e. $\mathcal{S}^\sharp \supseteq \overline{\mathcal{S}} \supseteq \mathcal{S}$.
We call *$\mathcal{S}^\sharp$ the extended saturation of $\mathcal{S}$* in $D[X]$ and a multiplicative set $\mathcal{S}$ of $D[X]$ is called *extended saturated* if $\mathcal{S} = \mathcal{S}^\sharp$.

Clearly, in general, $*_{\mathcal{S}^\sharp} \geq *_{\mathcal{S}}$ $(= *_{\overline{\mathcal{S}}})$. However, it can be shown that $(*_{\mathcal{S}^\sharp})_0 = (*_{\mathcal{S}})_0$ [CF1, Theorem 2.1(c)].

Lemma Chang-Fontana, 2007

(a) $\circlearrowleft_{\mathcal{S}}$ is stable and of finite type, i.e., $\circlearrowleft_{\mathcal{S}} = \widetilde{\circlearrowleft_{\mathcal{S}}}$.

(b) The extended saturation $\mathcal{S}^\sharp$ of $\mathcal{S}$ coincides with $\mathcal{N}^{\circlearrowleft_{\mathcal{S}}} := \{g \in D[X] \mid g \neq 0 \text{ and } \mathbf{c}_D(g)^{\circlearrowleft_{\mathcal{S}}} = D^{\circlearrowleft_{\mathcal{S}}}\}$ and $\circlearrowleft_{\mathcal{S}} = \circlearrowleft_{\mathcal{S}^\sharp}$.

(c) If $\mathcal{S}$ is extended saturated, then $\mathrm{Na}(D, \circlearrowleft_{\mathcal{S}}) = D[X]_{\mathcal{S}}$.

(d) The map $\mathcal{S} \mapsto \circlearrowleft_{\mathcal{S}}$ establishes a 1-1 correspondence between the extended saturated multiplicative subsets of $D[X]$ [resp., extended saturated multiplicative subsets of $D[X]$ contained in $\mathcal{N}^{v_D}$] and the set of the stable semistar [resp., star] operations of finite type on D.

Let D be an integral domain with quotient field K, let X, Y be two indeterminates over D and let $\star$ be a semistar operation on D. Set

$D_1 := D[X]$, $K_1 := K(X)$ and take the following subset of $\mathrm{Spec}(D_1)$:

$$\mathbf{\Delta}_1^\star := \{Q_1 \in \mathrm{Spec}(D_1) \mid Q_1 \cap D = (0) \text{ or } Q_1 = (Q_1 \cap D)[X] \text{ and } (Q_1 \cap D)^{\star_f} \subsetneq D^\star\}\,.$$

Set $\mathcal{S}_1^\star := \mathcal{S}(\mathbf{\Delta}_1^\star) := D_1[Y] \setminus (\bigcup\{Q_1[Y] \mid Q_1 \in \mathbf{\Delta}_1^\star\})$.

Using the previous lemma, in the next theorem and in the subsequent corollary we give a satistactory answer to the question stated above.

The motivation for the above definition of $\mathcal{S}_1^\star$ (or $\mathbf{\Delta}_1^\star$) is related to a characterization of P$\star$MD's in terms of $\star_f$–quasi Prüfer domains (i.e., domains D such that if $Q \in \mathrm{Spec}(D)$ and $Q \subseteq P[X]$, with $P \in \mathrm{QMax}^{\star_f}(D)$ then $Q = (Q \cap D)[X]$), given in a second paper joint with Chang [CF2].

Theorem

With the previous notation, set

$$A^{\circlearrowleft_{\mathcal{S}_1^\star}} := A[Y]_{\mathcal{S}_1^\star} \cap K_1\,, \quad \text{for all } A \in \overline{\mathbf{F}}(D_1).$$

(a) The mapping $[\star] := \circlearrowleft_{\mathcal{S}_1^\star} : \overline{\mathbf{F}}(D[X]) \to \overline{\mathbf{F}}(D[X])$, $A \mapsto A^{\circlearrowleft_{\mathcal{S}_1^\star}}$ is a stable semistar operation of finite type on $D[X]$, i.e., $\widetilde{[\star]} = [\star]$. Moreover, if $\star$ is a star operation on D, then $[\star]$ is a star operation on $D[X]$.

(b) $[\widetilde{\star}] = [\star_f] = [\star]$.

(c) $(ED[X])^{[\star]} \cap K = ED_1[Y]_{\mathcal{S}_1^\star} \cap K = E^{\widetilde{\star}}$ for all $E \in \overline{\mathbf{F}}(D)$, i.e., $[\star]_0 = \widetilde{\star}$.

(d) $(ED[X])^{[\star]} = E^{\widetilde{\star}}D[X]$, for all $E \in \overline{\mathbf{F}}(D)$.

(e) $[w_D] = [t_D] = [v_D] = \widetilde{v_{D[X]}} = w_{D[X]}$.

Corollary

Let $\star$ be a semistar operation on an integral domain D and let $[\star]$ be the stable semistar operation of finite type on $D[X]$ canonically associated to $\star$ as in the previous theorem. Then,

$$D \text{ is a P}\star\text{MD if and only if } D[X] \text{ is a P}[\star]\text{MD.}$$

Note that it is also true that if $\star$ is a stable semistar operation of finite type on D (i.e., $\star = \widetilde{\star}$), then

$$D \text{ is } \star\text{–Noetherian} \Leftrightarrow D[X] \text{ is } [\star]\text{–Noetherian.}$$