UPPERS TO ZERO AND PRÜFER-LIKE DOMAINS: A SURVEY

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§1. The Genesis

- Let R be a commutative ring and X and indeterminate. A nonzero prime ideal Q of $R[X]$ is called an *upper to zero* if $Q \cap R = (0)$.

Lemma 1

Let D be an integral domain with quotient field K and Q a prime ideal of $D[X]$, then the following are equivalent:

(i) *Q is an upper to zero.*

(ii) *$Q = fK[X] \cap D[X]$, where $f \in K[X]$ is irreducible.*

(iii) *Q is the kernel of a map $D[X] \to D[u] \subseteq L$, $X \mapsto u$, where $u \in L$ and L is an algebraic extension of K, i.e. $Q = (X - u)L[X] \cap D[X]$.*

- Recall also that a prime ideal Q of $R[X]$ is called an *extended prime* if $Q = (Q \cap R)[X]$,
- if $Q \supsetneq (Q \cap R)[X]$, the prime ideal Q is called an *upper to* $\mathfrak{q} := Q \cap R$,
  i.e., in this case, $Q/\mathfrak{q}[X]$ $(\subset R[X]/\mathfrak{q}[X])$ corresponds to an upper to zero in the integral domain $(R/\mathfrak{q})[X]$.

The "upper terminology" in polynomial rings is due to S. McAdam and was introduced in early 1970's.

In 1939 W. Krull gave a famous example of a one-dimensional quasi-local integrally closed domain (D, M) which is not a valuation domain. In this example he showed that the extended ideal $M[X]$ contains an upper to zero Q in such a way $\mathrm{ht}(M[X]) = 2$ and $\dim(D[X]) = 3$.

A. Seidenberg in 1953/54 was inspired by Krull's example for showing that there exist integral domains D with $\dim(D) = n$ and $\dim(D[X]) = n + k$, for each $1 \leq k \leq n + 1$.

I. Kaplansky in 1970 called

- an integral domain D an *S-domain* if uppers to zero avoid extensions of height-one prime ideals; a *strong S-ring* is a ring R such that R/P is an S-domain for each prime ideal P of R.

Kaplansky used this notion to show that, *for a strong S-ring R,* $\dim(R[X]) = \dim(R) + 1$, giving an unified approach to the Krull dimension of polynomial rings with coefficients in Noetherian rings or valuation (Prüfer) domains.

Gilmer and Hoffmann in 1975 gave a characterization of Prüfer domains that can be stated by using uppers to zero. Let us start with a lemma and a definition:

Lemma 2

Let D be an integrally closed domain and P a prime ideal of D. Then D_P is a valuation domain if and only if, for each upper to zero Q of $D[X]$, $Q \not\subseteq P[X]$.

• A (unitary) extension of rings $R \subset S$ is called *a P(rimitive)-extension* if each nonzero element $s \in S$ satisfies a polynomial $g \in R[X]$ one of whose coefficients is a unit of R, i.e., $g(s) = 0$ and $\mathbf{c}_R(g) = R$.

A globalization of the previous Lemma leads to the Gilmer-Hoffmann result:

Theorem 3

Let D be an integrally closed domain with quotient field K. The following are equivalent:

(i) *D is a Prüfer domain.*

(ii) *If Q upper to zero of $D[X]$, $Q \not\subseteq M[X]$, for each $M \in \mathrm{Max}(D)$.*

(iii) *$D \subset K$ is a P(rimitive)-extension.*

Remark 4

Note that the connection of P-extensions to uppers to zero is rather natural, since it is easy to prove that:

The following statements are equivalent:

(i) $R \subset S$ *is a P(rimitive)-extension of integral domains.*

(ii) *for each* $s \in S$, *the upper to zero* $Q_s := (X - s)S[X] \cap R[X]$ *of* $R[X]$ *satisfies* $Q_s \not\subseteq M[X]$ *for each* $M \in \mathrm{Max}(R)$.

I would like to mention now some of the "upper to zero" characterizations of the integrally closed domains.
[I want to emphasize that much is left out: there are various distinguished class of rings that can be characterized using "upper to zero" properties.]

For integrally closed domains we have the following result that appeared in a paper by Querré in 1980.

Theorem 5

Let D be an integral domain with quotient field K. The following are equivalent:

- **(i)** *D is integrally closed.*
- **(ii)** *If $Q = fK[X] \cap D[X]$ upper to zero of $D[X]$, where $0 \neq f \in K[X]$ is irreducible, then $Q = f(\mathbf{c}_D(f))^{-1}D[X]$.*
- **(iii)** *For nonzero polynomials $f, g \in K[X]$, $\mathbf{c}_D(fg)^v = (\mathbf{c}_D(f)\mathbf{c}_D(g))^v$.*

(Note that Krull showed that (i)⇒(iii) in 1936 and Flanders essentially proved (iii)⇒(i) in 1952 and (i)⇒(ii) is in Gilmer's book (1968).)

Remark 6

Condition (iii) in the previous theorem is a "v-version" of the classical Gauss Lemma on the content formula for polynomials on a Dedekind domain. More precisely:

Gauss-Gilmer-Tsang Theorem *Let D be an integral domain with quotient field K. then D is a Prüfer domain if and only if $\mathbf{c}_D(fg) = \mathbf{c}_D(f)\mathbf{c}_D(g)$, for all nonzero $f, g \in K[X]$.*

In the general situation, we have the following result:

Dedekind-Mertens Lemma: *Let D be an integral domain and $f, g \in D[X]$. Let $m := \deg(g)$, then:*

$$\mathbf{c}_D(f)^m \mathbf{c}_D(fg) = \mathbf{c}_D(f)^{m+1} \mathbf{c}_D(g)\,.$$

§2. Prüfer v-multiplication domains

Since the Prüfer property is not preserved when passing to the polynomial ring, to overcome this inconveniency the idea is to work in a more general class of domains that contains (and "is closed to") Prüfer domains and which is stable when passing to polynomial rings.
This is one of the motivations for considering

- *Prüfer v-multiplication domains* (for short, *PvMD's*), i.e. domains D such that D_Q is a valuation domain for each $Q \in \text{Max}^t(D)$ or, equivalently, each nonzero finitely generated ideal F of D is t-invertible (i.e., $(FF^{-1})^t = D$).

This class of domains was introduced explicitly by M. Griffin (1967) but was also implicitly considered by W. Krull in 1930's and P. Jaffard (1960).

Some characterizations of PvMD's, analogous to characterizations of Prüfer domains that can be deduced from the previous considerations, are given next:

Theorem 7

Let D be an integral domain. The following are equivalent:

(i) *D is a PvMD (resp., Prüfer domain).*

(ii) *D is integrally closed and for each upper to zero Q in $D[X]$, $\mathbf{c}_D(Q)^t = D$ (resp., $\mathbf{c}_D(Q) = D$).*

(iii) *D is integrally closed and each upper to zero Q in $D[X]$ is a maximal t-ideal of $D[X]$ (resp., $Q \not\subseteq M[X]$, for all $M \in \mathrm{Max}(D)$).*

(iv) *$D[X]$ is a PvMD (resp., the Nagata ring $D(X)$ is a Prüfer domain).*

For the "PvMD part", the equivalence (i)$\Leftrightarrow$(ii) is due to Papick (1982, 1983) and, independently and in a different form, to Mott-Zafrullah (1981).

The other equivalences are due to Houston-Malik-Mott (1984).

The fact that
D is a Prüfer domain if and only if $D(X)$ is a Prüfer domain
(more precisely, in this case, $D(X)$ coincides with the Kronecker function ring of D and so is a Bézout domain)
is due to Arnold (1969).

§3. Quasi–Prüfer domains and UMt–domains

Condition (iii) of the previous Theorem 7 (i.e., D is integrally closed and each upper to zero Q in $D[X]$ is a maximal t-ideal of $D[X]$) led to introduce the following notions:

An integral domain D is called

- an *UM t-domain* if each **u**pper to zero in $D[X]$ is a **m**aximal **t**-ideal (of $D[X]$) (Houston-Zafrullah (1989)).
- a *quasi-Prüfer domain* if Q is a prime ideal of $D[X]$ such that $Q \subseteq P[X]$, for some $P \in \mathrm{Spec}(D)$, then $Q = (Q \cap D)[X]$.

The quasi-Prüfer notion was introduced for arbitrary rings (not necessarily integral domains) by Ayache-Cahen-Echi (1996).

Note that:

Proposition 8

Let D be an integral domain. The following are equivalent:

(i) *D is a quasi-Prüfer domain.*

(ii) *for each upper to zero Q in $D[X]$, $Q \not\subseteq M[X]$ for all $M \in \mathrm{Max}(D)$.*

(iii) *the integral closure of D is a Prüfer domain.*

Therefore from Theorem 7 (characterizations of PvMD's) and Proposition 8 (characterizations of quasi-Prüfer), we deduce that:

$$\text{P}v\text{MD} \;\Leftrightarrow\; \text{integrally closed} + \text{UM}t$$

$$\text{Prüfer} \;\Leftrightarrow\; \text{integrally closed} + \text{quasi-Prüfer}$$

It is natural to investigate UMt-domains and quasi-Prüfer domains in order to understand which kind of properties of PvMD's and of Prüfer domains survive when the integrally closed property is missing.

Theorem 9

Let D be an integral domain. The following are equivalent:

- **(i)** *D is a UMt-domain.*
- **(i′)** *$Q \not\subseteq M[X]$, for each upper to zero Q in $D[X]$ and for each $M \in \mathrm{Max}^t(D)$.*
- **(ii)** *If Q is an upper to zero in $D[X]$, then $\mathbf{c}_D(Q)^t = D$.*
- **(iii)** *Each prime ideal of $\mathrm{Na}(D, t)$ is extended from D.*
- **(iv)** *$D[X]$ is a UMt-domain.*

The previous result extends to the non integrally closed case Theorem 7 (providing characterizations of PvMD's).
It is stated in Fontana-Gabelli-Houston (1998) and the proofs are based on techniques due to Houston-Zafrullah (1989).

- Recall that $\mathrm{Na}(D,t) = \mathrm{Na}(D,v) := D[X]_{\mathcal{N}^v}$, where $\mathcal{N}^v := \{g \in D[X] \mid 0 \neq g,\ \mathbf{c}_D(g)^v = D\}$.

The study of this extension of the classical Nagata ring was initiated by Kang (1989).

Condition (iii) in the previous Theorem 9 (i.e., each prime ideal of $\mathrm{Na}(D,t)$ is extended from D) is related to Kang's characterization of a PvMD:

D is a PvMD $\Leftrightarrow$ each ideal of $\mathrm{Na}(D,v)$ is extended from D.

[*D is Prüfer $\Leftrightarrow$ each ideal of $D(X)$ is extended from D* (D.D. Anderson (1976))].

Since a Prüfer domain is a PvMD, it is natural to inquire whether a quasi-Prüfer domain is a UMt domain.
The previous characterizations give a positive answer to this question.
More precisely:

Theorem 10

Let D be an integral domain. The following are equivalent:

- **(i)** *D is a quasi-Prüfer domain.*
- **(ii)** *D is a UM t-domain and* $\mathrm{Max}(D) = \mathrm{Max}^t(D)$.
- **(ii')** *D is a UM t-domain and* $d = w$.
- **(iii)** *Each overring of D is a UM t-domain.*

(i)⇔(ii) is due to Dobbs-Houston-Lucas-Roitman-Zafrullah (1992) and Fontana-Gabelli-Houston (1998) are responsible for (i)⇔(iii).

With respect to condition (iii), note also that
D is a UM t-domain if and only if each t-linked overring is a UM t-domain
(Houston-Zafrullah (1989)).

We conclude this panoramic overview of the "classical" theory, by recalling some local characterizations of the PvMD's and UMt-domains.
It is easy to see that the quasi-Prüfer property, like the Prüfer property, is a local property.

Theorem 11

Let D an integral domain. The following are equivalent:

(i) *D is a UMt-domain (resp., PvMD).*

(ii) *D_M is a quasi-Prüfer domain (resp., Prüfer domain) for all $M \in \mathrm{Max}^t(D)$.*

(iii) *D_M is a UMt-domain (resp., PvMD) and $MD_M \in \mathrm{Max}^t(D_M)$ for all $M \in \mathrm{Max}^t(D)$.*

Note that the "UMt part" is due to Fontana-Gabelli-Houston (1998) (see also Houston-Zafrullah (1989) and Dobbs-Houston-Lucas-Roitman-Zafrullah (1992)).
For the "PvMD part", Griffin (1967) showed (i)⇔(ii).
(i)⇔(iii) follows from results proved by Mott-Zafrullah (1981).

§4. Basic facts on star and semistar operations

Let D be an integral domain with quotient field K.

Let $\overline{\mathfrak{F}}(D)$ represent the set of all nonzero D–submodules of K.

Let $\mathfrak{F}(D)$ represent the nonzero fractional ideals of D
(i.e. $E \in \overline{\mathfrak{F}}(D)$ such that $dE \subseteq D$, for some nonzero element $d \in D$).

Finally, let $\mathbf{f}(D)$ represent the finitely generated D-submodules of K.
Obviously:

$$\mathbf{f}(D) \subseteq \mathfrak{F}(D) \subseteq \overline{\mathfrak{F}}(D).$$

In 1994, Okabe and Matsuda introduced the notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation, allowing $D \neq D^\star$.

- More precisely, a mapping $\star : \overline{\mathfrak{F}}(D) \to \overline{\mathfrak{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathfrak{F}}(D)$, the following properties hold:

  $(\star_1)$ $(zE)^\star = zE^\star$;

  $(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

  $(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

- When $D^\star = D$, we say that $\star$ restricted to $\mathfrak{F}(D)$ defines *a star operation of D* [i.e. $\star : \mathfrak{F}(D) \to \mathfrak{F}(D)$ verifies the properties $(\star_2)$, $(\star_3)$ and

  $(\star\star_1)$ $(zD)^\star = zD$, $(zE)^\star = zE^\star$].

- For a star operation $*$, the notion of *$*$–ideal* (that is, a nonzero ideal $I \subseteq D$, such that $I^* = I$) leads to the definition of a canonically associated ideal system.

  For semistar operations $\star$, we need a more general notion.

- A nonzero (integral) ideal I of D is a *quasi–$\star$–ideal* if $I^\star \cap D = I$.

We designate by *quasi–$\star$–prime* [respectively, *$*$–prime*] of D a quasi–$\star$–ideal [respectively, an integral $*$–ideal] of D which is also a prime ideal.
We designate by *quasi–$\star$–maximal* [respectively, *$*$–maximal*] of D a maximal element in the set of all proper quasi–$\star$–ideals [respectively, integral $*$–ideals] of D.

We denote by $\text{Spec}^*(D)$ [respectively, $\text{Max}^*(D)$, $\text{QSpec}^\star(D)$, $\text{QMax}^\star(D)$] the set of all $*$–primes [respectively, $*$–maximals, quasi–$\star$–primes, quasi–$\star$–maximals] of D.

As in the classical star-operation setting, we associate to a *semistar* operation $\star$ of D a new semistar operation $\star_f$ as follows.

- If $E \in \overline{\mathcal{F}}(D)$ we set:

$$E^{\star_f} := \cup\{F^\star \mid F \subseteq E,\ F \in \mathbf{f}(D)\}\,.$$

  We call $\star_f$ *the semistar operation of finite type of D associated to $\star$*.

- If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type of D*.

Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type.

Lemma 12

Let $\star$ be a non-trivial semistar operation of finite type on D (i.e., $\star = \star_f$). Then:

(1) *Each proper quasi–$\star$–ideal is contained in a quasi–$\star$–maximal.*

(2) *Each quasi–$\star$–maximal is a quasi–$\star$–prime.*

By the previous lemma, when $\star = \star_f$, $\text{QMax}^{\star}(D) \neq \emptyset$.

- Let $\tilde{\star}$ be the operation defined as follows:
$$E^{\tilde{\star}} := \bigcap \{ED_Q \mid Q \in \text{QMax}^{\star_f}(D)\}, \text{ for all } E \in \overline{\mathfrak{F}}(D).$$

The operation $\widetilde{v}$ coincides with the operation w (Wang-McCasland notation).

- *The semistar Nagata ring* is the following straightforward generalization of the classical Nagata ring:

$$\text{Na}(D, \star) := \left\{ \frac{f}{g} \mid f, g \in D[X], \ g \neq 0, \ \mathbf{c}(g)^{\star} = D^{\star} \right\}.$$

Note that, $\text{Na}(D, \star) = \text{Na}(D, \star_f) = \text{Na}(D, \widetilde{\star})$. Therefore, the assumption $\star = \star_f$ is not really restrictive when considering Nagata semistar rings.
If $\star = d$ is the identity (semi)star operation of D, then $\text{Na}(D, d) = D(X)$.

Finally, it is not difficult to show that:

$$E^{\tilde{\star}} = E\text{Na}(D, \star) \cap K, \text{ for all } E \in \overline{\mathfrak{F}}(D).$$

§5. Prüfer $\star$–multiplication domains

- Let $\star$ be a semistar operation on an integral domain D. We recall that a *Prüfer $\star$–multiplication domain* (for short, a *P$\star$MD*) is an integral domain such that, for each $F \in \mathbf{f}(D)$, then: $(FF^{-1})^{\star_f} = D^{\star_f}$ $(= D^{\star})$ (i.e., each F is $\star_f$–*invertible*).

Clearly:

$$\text{(Prüfer domain =) P}d\text{MD} \quad \rightarrow \quad \text{P}\star\text{MD} \quad \dashrightarrow \quad \text{P}v\text{MD}$$

This general approach gives new insight also to the classical cases.

As a matter of fact, besides a generalization of some of the classical results and characterizations of the Prüfer [v–multiplication] domains (cf. Griffin (1967), Arnold-Brewer (1971), Mott-Zafrullah (1981), Zafrullah (1984) and Kang (1989)), new results have been obtained for P$\star$MD's that are also new for the special classical case of a PvMD's.

Theorem 13

Let D be an integral domain with quotient field K, X an indeterminate over K and $\star$ a semistar operation on D. The following are equivalent:

(i) *D is a P$\star$MD.*

(ii) $\mathrm{Na}(D,\star)$ *is a Prüfer domain.*

(iii) $\mathbf{c}_D(fg)^{\widetilde{\star}} = (\mathbf{c}_D(f)\mathbf{c}_D(g))^{\widetilde{\star}}$ *for all* $0 \neq f, g \in K[X]$.

(iv) $\star_f$ *is stable and e.a.b. (i.e., $\tilde{\star} = \star_a$).*

In particular, D is a P$\star$MD if and only if it is a P$\tilde{\star}$MD.

(i)⇔(iv) is due to Fontana-Jara-Santos (2003) and D.F. Anderson-Fontana-Zafrullah (2007) have recently showed that (i)⇔(iii).

We have the following application of the previous result:

Corollary 14

An integral domain D is a PvMD if and only if $\mathbf{c}_D(fg)^w = (\mathbf{c}_D(f)\mathbf{c}_D(g))^w$ for all $0 \neq f, g \in K[X]$.

This corollary on the one hand gives a nice general characterization of PvMD's, and on the other hand it establishes the "superiority" of the w–operation over the t–operation.

Since $F^t = F^v$ for each finitely generated nonzero ideal F, we know already (Theorem 5):

$$\begin{aligned} D \text{ is integrally closed } &\Leftrightarrow \quad \mathbf{c}_D(fg)^v = (\mathbf{c}_D(f)\mathbf{c}_D(g))^v \text{ for all } 0 \neq f, g \in K[X] \\ &\Leftrightarrow \quad \mathbf{c}_D(fg)^t = (\mathbf{c}_D(f)\mathbf{c}_D(g))^t \text{ for all } 0 \neq f, g \in K[X]. \end{aligned}$$

In other words, for characterizing PvMD's, the w–operation can do what the t–operation cannot do.

Corollary 14 will appear in a paper by D.F. Anderson-Fontana-Zafrullah (2007). A similar result has been recently announced by G.W. Chang.

Given a finite type star operation $*$, P$*$MD's were introduced by Houston-Malik-Mott (1984).

Note that for any star operation $*$, a $*$-invertible $*$-ideal is a v-invertible v-ideal and so P$*$MD's are particular PvMD's.
That is *in the star operation $*$ setting:*

$$(PdMD =)\ \text{Prüfer} \quad \rightarrow \quad P{*}MD \quad \rightarrow \quad PvMD.$$

The first examples of PvMD's <u>not</u> P$*$MD's for some *nontrivial* $*$ operation were given by Fontana-Jara-Santos (2003) using the following characterization.
(Note that for $* = t$ or $* = w$ it is well known that P$*$MD = PvMD.)

For a star operation $*$

$$P{*}MD \quad \Leftrightarrow \quad PvMD \ \text{ and } \ *_f = t \ (\text{or, equivalently, } \widetilde{*} = t).$$

We want to present here briefly a more recent class of examples of PvMD's not P$*$MD's, for some $*$ operation, constructed by D.F. Anderson-Fontana-Zafrullah (2007).
For this purpose we use the following general result concerning the semistar operation case:

Proposition 15

Let D be an integral domain and let $\star$ be a semistar operation of finite type induced by a family $\mathcal{T}$ of flat overrings of D, i.e., $E^\star := \bigcap\{ET \mid T \in \mathcal{T}\}$, for all nonzero D–submodules E of K. Then

$$D \text{ is a P}{\star}\text{MD} \quad \Leftrightarrow \quad \mathbf{c}_T(fg) = \mathbf{c}_T(f)\mathbf{c}_T(g)$$

for all $0 \neq f, g \in K[X]$ (i.e., T is a Prüfer domain) and all $T \in \mathcal{T}$.

From the previous result we obtain an explicit example:

Example 1

Let D be a PvMD and suppose that D has nonzero nonunits $x_1, x_2, \ldots, x_n$ with $(x_1, x_2, \ldots, x_n)^v = D$, $n \geq 2$, and D_{x_i} is not a Prüfer domain for some i. Let $*$ be the operation defined by the finite family (of flat overrings of D) $\{D_{x_i} \mid 1 \leq i \leq n\}$, i.e.,

$$E^* := \bigcap_{1 \leq i \leq n} ED_{x_i}\,, \quad \text{for all } E \in \mathfrak{F}(D)\,.$$

Then $*$ is a star operation of finite type on D such that D is not a P$*$MD.

The key fact is that $(x_1, x_2, \ldots, x_n)^v = D$ if and only if $D = \bigcap_{i=1}^n D_{x_i}$ as observed by Zafrullah (1988).

For instance, consider the 2-dimensional Krull domain (hence, PvMD) $D := K[X, Y]$ and consider $M := (X, Y)$.
Clearly, $M^v = (X, Y)^v = D$ and D_X and D_Y are non Prüfer overrings of D.

Final remarks: recent developments ...

Using the semistar operation setting, we have studied in a unified frame (i.e., P$\star$MD's) Prüfer domains and PvMD's.

It is natural to try to do the same with the notions of quasi-Prüfer domain and UMt-domain.

The notion of quasi-Prüfer domain can be rather naturally extended to the semistar operation setting. But, there are some difficulties to do the same for the UMt-domain property.

The reason is due to the fact that the UMt-domains are domains such that each upper to zero in $D[X]$ is a maximal $t_{D[X]}$-ideal.

There is no immediate extension to the semistar setting of the previous property, since in the general case we do not have the possibility to work at the same time with a semistar operation (like the t-operation) defined "natively" both on D and on $D[X]$.

... recent developments ...

At this point it is natural to formulate the following question.

Question

Given a semistar operation of finite type $\star$ on D, is it possible to define in a canonical way a semistar operation of finite type $\star_{D[X]}$ on $D[X]$, such that D is a $\star$-quasi-Prüfer domain if and only if each upper to zero in $D[X]$ is a quasi-$\star_{D[X]}$-maximal ideal ?

In a joint work (preprint) with Chang, we have given a positive answer to the previous question in case of stable semistar operations, which is enough for introducing UM$\star$-domains and developing a theory along the lines of the theory of P$\star$MD's.

... recent developments ... UM$*$-domains and integral closure

M. Zafrullah (2000) posed the following problem:

- *is the integral closure of a UMt-domain a PvMD?*

It can be shown that the semistar operation setting gives new insight to this problem, leading to an "appropriate" positive answer to it (Chang-Fontana (2007)).

A related question is the following:

- *if the integral closure $\overline{D}$ of an integral domain D is a PvMD what can be said about the UMt-ness of D?*

An answer to the second question was recently given by Chang-Zafrullah (2006) where they provide an example of a non-UMt domain with the integral closure which is a PvMD.

§6. $\star$–quasi–Prüfer domains

The notions of quasi-Prüfer domain and UMt-domain can be naturally extended to the semistar operation setting and studied in a unified frame, as we did with the notion of P$\star$MD (obtaining a unified frame for Prüfer domains and PvMD's).

More precisely, given a semistar operation $\star$ on an integral domain D, in a joint paper with Chang, we introduce in a natural way the notion of $\star$-quasi-Prüfer domains.

- We say that an integral domain D is a *$\star$-quasi-Prüfer domain* if the following property holds:
  if Q is a prime ideal in $D[X]$ and $Q \subseteq P[X]$, for some $P \in \mathrm{QSpec}^{\star}(D)$ (i.e., $P = P^{\star} \cap D$), then $Q = (Q \cap D)[X]$.

It is clear from the definition that *d-quasi-Prüfer domains coincide with quasi-Prüfer domains.*

Since a quasi-$\star$-ideal is also a quasi-$\star_f$-ideal, it is clear that $\star_f$-quasi-Prüfer implies $\star$-quasi-Prüfer.

Recall that, by standard arguments, every quasi-$\star_f$-ideal is contained in a quasi-$\star_f$-maximal ideal and each quasi-$\star_f$-maximal ideal is a prime ideal. Therefore, the set of quasi-$\star_f$-prime ideals $\mathrm{QSpec}^{\star_f}(D)$ is always nonempty.

On the other hand $\mathrm{QSpec}^{\star}(D)$ can be empty when $\star \neq \star_f$ and in this case the notion of $\star$-quasi-Prüfer domain looses interest.
It is possible to give explicit examples of $\star$-quasi-Prüfer domains that are not $\star_f$-quasi-Prüfer domains.

Therefore, from now on, we will consider only the finite case setting, i.e. we will consider $\star_f$-quasi-Prüfer domains.

Theorem 16

Let $\star$ be a semistar operation of finite type on an integral domain D. Then the following statements are equivalent.

(1) *D is a $\star$-quasi-Prüfer domain.*

(2) *$D \subseteq K$ is a $\star$-primitive extension (i.e., $0 \neq z \in K$ satisfies a polynomial $g \in D[X]$ with $\mathbf{c}_D(g)^\star = D^\star$).*

(3) *Each overring R of D is a $(\star)_\iota$-quasi-Prüfer domain, where $\iota : D \hookrightarrow R$ is the canonical embedding.*

(4) $\mathrm{Na}(D, \star)$ *is a quasi-Prüfer domain (i.e., the integral closure of* $\mathrm{Na}(D, \star)$ *is a Prüfer domain).*

(5) *D_P is a quasi-Prüfer domain, for each quasi-$\star$-maximal ideal (or, quasi-$\star$-prime ideal) P of D.*

In case of *a star operation* $\ast$, we have more precise results.

Corollary 17

When the star operation $\ast$ is the t operation, we have:

$$t\text{-quasi Prüfer domain} \quad \Leftrightarrow \quad \text{UM}\,t\text{-domain}$$

Corollary 18

Let $\ast$ be a star operation of finite type on an integral domain D. The following are equivalent:

(i) *D is a $\ast$-quasi-Prüfer domain.*

(ii) *D is a a UM t-domain and each $\ast$-maximal ideal of D is a t-ideal.*

(ii') *D is a UM t-domain and $\widetilde{\ast} = w$.*

§7. $*$-quasi-Prüfer domain and integral closure

M. Zafrullah (2000) posed the following problem:

- *is the integral closure of a UMt-domain a PvMD?*

We will show next how the semistar operation setting gives new insight to this problem, leading to an "appropriate" answer to it.

A related question is the following:

- *if the integral closure $\overline{D}$ of an integral domain D is a PvMD what can be said about the UMt-ness of D?*

An answer to the second question was recently given by Chang-Zafrullah (2006) where they provide an example of a non-UMt domain with the integral closure which is a PvMD.

Let $\star$ be a semistar operation of finite type on an integral domain D. Recall that *a P$\star$MD D is characterized by the fact that D_P is a valuation domain for each quasi-$\star$-maximal ideal P of D* (Fontana-Jara-Santos (2003), Houston-Malik-Mott (1984)).
Thus, since a valuation domain is trivially quasi-Prüfer, *a P$\star$MD is a $\star$-quasi-Prüfer domain* by Theorem 16 ((1) $\Leftrightarrow$ (5)).
This fact generalizes the well known property that *a PvMD is a UMt-domain* (Theorem 7).

However, when $\star$ is a semistar operation a P$\star$MD need not be integrally closed, while being a PvMD is equivalent to being an integrally closed UMt domain (Houston-Zafrullah (1989)).

The next proposition gives an appropriate generalization of the previous result to the case of semistar operations.

Proposition 19

Let $\star = \star_f$ be a semistar operation on an integral domain D. Then the following statements are equivalent.

(i) *D is a P$\star$MD.*

(ii) *D is a $\star$-quasi-Prüfer domain and D_P is integrally closed for all $P \in \mathrm{QMax}^{\star}(D)$ (= the set of all the quasi-$\star$-maximal ideals of D).*

(iii) *D is a $\star$-quasi-Prüfer domain and $D^{\widetilde{\star}} := \bigcap\{D_P \mid P \in \mathrm{QMax}^{\star}(D)\}$ is integrally closed.*

When $\star = *$ is a star operation then clearly also $\widetilde{*}$ is a star operation and so $D^{\widetilde{*}} = D$. Therefore, from the previous proposition we deduce:

Corollary 20

Let $* = *_f$ be a star operation on an integral domain D. Then:

$$D \text{ is a P}{*}\text{MD} \quad \Leftrightarrow \quad D \text{ is an integrally closed } *\text{-quasi-Prüfer domain.}$$

An essentially equivalent characterization of a P$*$MD was given by Houston-Malik-Mott (1984).

From the previous considerations (in the star operation case $\ast = \ast_f$), we have a very similar behaviour when we loose the integrally closed assumption, as summarized in the following diagram

$$
\begin{array}{ccc}
\text{Prüfer domain } (= \text{P}d\text{MD}) & \longrightarrow & (d\text{-})\text{quasi-Prüfer domain} \\
\big\downarrow & & \big\downarrow \\
\text{P}\ast\text{MD} & \longrightarrow & \ast\text{-quasi-Prüfer domain} \\
{\scriptstyle \ast = w}\big\uparrow\big\downarrow & & \big\downarrow\big\uparrow{\scriptstyle \ast = w} \\
\text{P}v\text{MD } (= \text{P}t\text{MD}) & \longrightarrow & \text{UM}t\ (= t\text{-quasi-Prüfer}) \text{ domain}
\end{array}
$$

Next result gives an answer to Zafrullah's problem, using the semistar operation setting.

Corollary 21

Let D be an integral domain. Set $\widetilde{D} := (\overline{D})^w$ and let $\widetilde{\iota} : D \hookrightarrow \widetilde{D}$ be the canonical embedding. The following statements are equivalent:

- **(i)** *D is a UMt-domain.*
- **(ii)** *$\widetilde{D}$ is a PvMD and $(w)_{\widetilde{\iota}}$ coincides with the w-operation of $\widetilde{D}$.*
- **(iii)** *$\widetilde{D}$ is a $P(w)_{\widetilde{\iota}}MD$.*

Therefore, for a negative answer to the problem of whether the integral closure of a UMt-domain is a PvMD we need examples of integral domains D such that the integral closure $\overline{D}$ is not t-linked to D (i.e., $\overline{D} \subsetneq (\overline{D})^w = \widetilde{D}$).

This is not an easy task, even in a general situation. Note that the integral closure $\overline{D}$ is t-linked to D if D is one-dimensional or if D is quasi-coherent (e.g., D is Noetherian) (Dobbs-Houston-Lucas-Zafrullah (1989)).

A first class of examples of integral domains of dimension ≥ 3 such that the integral closure $\overline{D}$ is not t-linked to D was given by Dobbs-Houston-Lucas-Roitman-Zafrullah (1992).

The 2-dimensional case was left open in that paper.

A first example in dimension two of an integral domain such that the integral closure $\overline{D}$ is not t-linked to D was given by Dumitrescu (2001), using the $A + XB[X]$ constructions.

Another example of this type was given by Chang-Fontana (2007).

This is *an example of a quasi-local strong Mori non Noetherian 2-dimensional UMt-domain D such that $\overline{D}$ is not t-linked to D, but still $\overline{D}$ is a $PvMD$.*

§8. $\star$-quasi-Prüfer domains and the "Upper Maximal" property

For $\star = v$, we have already observed that *t-quasi-Prüfer domains coincide with UMt-domains, i.e., domains such that each upper to zero in $D[X]$ is a maximal $t_{D[X]}$-ideal.*

There is no immediate extension to the semistar setting of the previous characterization, since in the general case we do not have the possibility to work at the same time with a semistar operation (like the t-operation) defined "natively" both on D and on $D[X]$.

At this point it is natural to formulate the following question.

Question

Given a semistar operation of finite type $\star$ on D, is it possible to define in a canonical way a semistar operation of finite type $\star_{D[X]}$ on $D[X]$, such that D is a $\star$-quasi-Prüfer domain if and only if each upper to zero in $D[X]$ is a quasi-$\star_{D[X]}$-maximal ideal ?