On Inverse Limits of Bézout Domains

DAVID E. DOBBS, University of Tennessee, Department of Mathematics, Knoxville, Tennessee 37996-1300

MARCO FONTANA, Università degli Studi “Roma Tre”, Dipartimento di Matematica, 00146 Roma, Italy

ABSTRACT: An example shows that if $A = \varprojlim A_n$ is the inverse limit of an inverse system $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ of Bézout (hence Prüfer) domains A_n, then A need not be a Prüfer (or a Bézout) domain. If, however, each transition map φ_{mn} is surjective, the question whether A must be a Prüfer domain is more subtle. A partial result is given for this context. Enhancement of this result is considered by means of associated inverse systems of CPI-extensions, with applications to Prüfer domains, Bézout domains and locally divided domains.

1 Introduction

This note is a sequel to the work initiated on inverse limits of integral domains in [5]. Because much of [5] had to do with applications to certain infinite-dimensional integral domains called $P^{\infty}VD$s, it was natural to restrict attention in [5] to inverse limits of some special types of inverse systems indexed by $\mathbb{N}$, the set of positive integers. The contexts of several other applications in [5] were motivated by the work in [6] on direct limits of integral domains. As a central result in [6] stated that any direct limit (over a directed index set) of Prüfer domains is a Prüfer domain, it was natural to ask in [5] whether the class of Prüfer domains is stable under inverse limit. In the quasilocal case, there is a complete answer [5, Theorem 2.1 (g)]: the inverse limit of any inverse system of valuation domains (indexed by $\mathbb{N}$) is a valuation domain. For the special type of inverse system emphasized in [5], it was established in [5, Theorem 2.21] that the class of Prüfer domains is stable under inverse limit *for that type* of inverse system. The general question of whether the class of Prüfer domains is stable under inverse limits of *arbitrary* inverse systems indexed by $\mathbb{N}$ was left open in [5]. In this note, we resolve that question.

Sadly, the answer is negative, as Example 2.1 presents an inverse system of Prüfer domains whose index set is $\mathbb{N}$ and whose inverse limit is not a Prüfer domain. From the point of view of category theory, this fact is somewhat palatable, since a nontrivial product of rings is an inverse limit (granted *not* over a directed index set) and is never an integral domain (Prüfer or otherwise). Nevertheless, and more to the point, we notice that the inverse system $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ in Example 2.1 lacks one important ingredient; namely, its transition maps φ_{mn} are not surjective. We thus come to a sharpening of the question: Is the class of Prüfer domains stable under inverse limits of inverse systems which are indexed by $\mathbb{N}$ and which have surjective transition maps? The bulk of this paper studies this question.

The prime ideals P of $A := \varprojlim A_n$ include those of the A_n (assuming surjective φ_{mn}) but we do not know if that is essentially the complete story, as it was in the earlier context

[5, Theorem 2.5 (a)]. (A related problem is that if $B := \varprojlim B_n$ is another inverse limit such that $\mathrm{Spec}(A_n) \cong \mathrm{Spec}(B_n)$ as partially ordered sets for each n, then it need not be the case that $\mathrm{Spec}(A) \cong \mathrm{Spec}(B)$ [11, page 354, lines 1–14; Propositions 2.1 and 3.1]; for a positive partial result in this regard, see [11, Theorem 5.7].) Our methods consider only $P \in \cup\, \mathrm{Spec}(A_n)$ as we seek to determine if A_P is a valuation domain. Theorem 2.3 and Corollary 2.4 provide a positive answer if each A_n is a Bézout domain and each φ_{mn} is surjective when restricted to unit groups. Proposition 2.6 (b) shows that two canonical valuation domains containing A_P are isomorphic and hence, in a sense, equally approximate A_P. One of these canonical extensions of A_P is studied via an associated inverse system in which each A_n is replaced with a suitable CPI-extension (in the sense of [1]) so that each transition map in the new inverse system has kernel a divided prime ideal (in the sense of [2]). The latter inverse system falls under the rubric of [5], thus permitting use of results such as the above-mentioned [5, Theorem 2.21]. For the sake of clarity, some of the "Prüferian" applications in Proposition 2.6 (a) are couched in the more general context of locally divided domains (in the sense of [2], [3]). Finally, Remark 2.7 explains that if the A_n are merely (commutative) rings rather than integral domains, then even in the presence of surjective transition maps, $\mathrm{Spec}(\varprojlim A_n)$ may be much larger than $\cup\, \mathrm{Spec}(A_n)$.

In addition to the notational conventions indicated above, we mention the following. All rings considered are commutative with identity. If A is a ring, then $U(A)$ denotes the set of units of A, $\mathrm{Spec}(A)$ denotes the set of prime ideals of A and "dimension" refers to the Krull dimension of A. If A is a domain with quotient field K, then an *overring of A* is any ring B such that $A \subseteq B \subseteq K$. Any unexplained material is standard, as in [9], [10].

2 Results

We begin with a negative answer to the naïve question.

Example 2.1. There exists an inverse system $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ such that A_n is a Bézout (hence Prüfer) domain for each $n \in \mathbb{N}$ but $A := \varprojlim A_n$ is not a Prüfer domain (and hence is not a Bézout domain).

Proof. Suppose, for the moment, that there exists an integrally closed integral domain A such that A is not a Prüfer domain and the set of minimal valuation overrings of A is denumerable, say $\{V_i \mid i \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, put $A_n := \cap_{i=1}^{n} V_i$. By [10, Theorem 107], A_n is a Bézout (and, hence, Prüfer) domain for each $n \in \mathbb{N}$. Moreover, $\cap_{n=1}^{\infty} A_n = \cap_{i=1}^{\infty} V_i = A$ since A is integrally closed [9, page 231]. If $m \geq n$ in $\mathbb{N}$, define $\varphi_{mn} : A_m \to A_n$ to be the inclusion map. Then $\{\varphi_{mn} \mid m \geq n\}$ evidently forms an inverse system, but its inverse limit, $\varprojlim A_n = \cap_{n=1}^{\infty} A_n = A$, is not a Prüfer (or a Bézout) domain.

It remains to construct an integral domain A with the properties supposed above. To this end, let k be a countable field, X an indeterminate over k, and $V = k(X) + M$ a valuation domain with maximal ideal $M \neq 0$. Then $A := k + M$ has the desired properties. Indeed, A is integrally closed but not a Prüfer domain, by standard facts about $D + M$ constructions [9, Exercise 11 (2), page 202; Exercise 13 (2), page 286]. Also, the set of minimal valuation overrings of A is in one-to-one correspondence with the set of (minimal) valuation domains W of $k(X)$ contained properly between k and $k(X)$: see [9, Exercise 13 (2), page 203]. Since k is countable, the set of monic irreducible polynomials in $k[X]$ (resp., $k[X^{-1}]$) is denumerable (cf. [10, Exercise 8, page 8]). It is well known that such polynomials serve to classify the valuation domains W in question (cf. [12]) and so the set of such W is denumerable. $\square$

We next fix the **riding assumptions and notation** for the rest of the paper. We assume given an $\mathbb{N}$-indexed inverse system of integral domains A_k, $\{\varphi_{mn} : A_m \to A_n \mid m \geq$

$n\}$, which has the property that each of its transition maps φ_{mn} is surjective. Put

$$A := \varprojlim A_n, \ \Phi_n : A \to A_n \text{ the canonical map, } Q_n := \ker(\Phi_n)$$

and

$$Q_{mn} := \ker(\varphi_{mn}) \text{ for } m \geq n.$$

The next result collects some useful facts. They may be proved as in the corresponding parts of [5, Theorem 2.1, Lemma 2.2 and Proposition 2.4], although the ambient hypotheses for the cited results were more stringent than our current riding assumptions.

Lemma 2.2. *(a)* $A = \{(a_n) \in \prod A_n \mid \varphi_{n+1,n}(a_{n+1}) = a_n$ *for each* $n \in \mathbb{N}\}$.

(b) For each $n \in \mathbb{N}$, Φ_n *is surjective and is the composite of the inclusion map* $A \hookrightarrow \prod A_k$ *and the canonical projection* $\prod A_k \to A_n$.

(c) For each $n \in \mathbb{N}$, $Q_n \in \operatorname{Spec}(A)$ *and* $A/Q_n \cong A_n$.

(d) For each $n \in \mathbb{N}$, $Q_n = \{(a_k) \in A \mid a_k = 0$ *for each* $k \leq n\}$.

(e) $Q_1 \supseteq Q_2 \supseteq Q_3 \supseteq \dots$ *and* $\cap Q_n = 0$.

(f) If $r \geq n \in \mathbb{N}$, *then* $Q_{rn} = \Phi_r(Q_n)$, $\Phi_r^{-1}(Q_{rn}) = Q_n$, $\varphi_{r+1,r}$ *restricts to a surjection* $Q_{r+1,n} \to Q_{rn}$, *and* $\varphi_{r+1,r}^{-1}(Q_{rn}) = Q_{r+1,n}$.

(g) If $r \geq n \in \mathbb{N}$, *then* $\varprojlim\{Q_{rn} \mid r \geq n\} = Q_n$ *canonically.*

We turn now to the main question, namely, whether A_n being a Prüfer domain for each n implies that A is a Prüfer domain; i.e., that A_P is a valuation domain for each (without loss of generality) nonzero $P \in \operatorname{Spec}(A)$. Our proofs require the restriction that P contain some Q_ν, a condition that was automatically satisfied by the pullbacks treated in [5]. (See [5, Theorem 2.5 (a)]. We do not know if the riding assumptions of the present paper ensure the $P \supseteq Q_\nu$ condition. See also Remark 2.7.) In view of Example 2.1, it seems natural to focus first on the case in which each A_n is a Bézout domain. For this context, Theorem 2.3 gives a positive conclusion if $\varphi_{n+1,n}(U(A_{n+1})) = U(A_n)$ for each n. (Notice that, since $\varphi_{n+1,n}$ is surjective for each n, the latter condition holds automatically if A_{n+1} is quasilocal, that is a valuation domain, for each n. However, if A_{n+1} is not quasilocal, it need not be the case that $\varphi_{n+1,n}(U(A_{n+1})) = U(A_n)$.) Note that, in contrast with the methods in [5], Theorem 2.3 and Corollary 2.4 avoid the assumption that $Q_{n+1,n}$ is a divided prime ideal of A_{n+1} for each n.

Theorem 2.3. *For each* n, *suppose that* A_n *is a Bézout domain and that* $\varphi_{n+1,n}$ *induces a surjection* $U(A_{n+1}) \to U(A_n)$. *If, in addition,* $P \in \operatorname{Spec}(A)$ *is such that* $P \supseteq Q_\nu$ *for some* ν, *then* A_P *is a valuation domain.*

Proof. It is enough to show that if $\alpha, \gamma \in A_P$, then either $\alpha \in \gamma A_P$ or $\gamma \in \alpha A_P$. Without loss of generality, we may assume that $\alpha, \gamma \in P$. Write $\alpha = (\alpha_n), \gamma = (\gamma_n) \in \prod A_n$. By restricting attention to the (cofinal) set $\{n \in \mathbb{N} \mid n \geq \nu\}$ and relabeling, we may assume that $P \supseteq Q_1$, and so $\alpha_n, \gamma_n \in P_n := \Phi_n(P)$ for each $n \geq 1$. Without loss of generality, $\alpha_n \neq 0$ and $\gamma_n \neq 0$ for all n.

Since A_n is a Bézout domain, it is a GCD-domain (in the sense of [10, page 32]). Let $d_n := \gcd(\alpha_n, \gamma_n)$; in other words, d_n is *a* greatest common divisor of α_n and γ_n in A_n. Then $\alpha_n = d_n\alpha_n'$ and $\gamma_n = d_n\gamma_n'$, where $\alpha_n', \gamma_n' \in A_n$ and $\gcd(\alpha_n', \gamma_n') = 1$. Fix n for the moment. Then, with $\varphi := \varphi_{n+1,n}$, we have the equations

$$\alpha_n = \varphi(\alpha_{n+1}) = \varphi(d_{n+1})\varphi(\alpha_{n+1}') = d_n\alpha_n',$$

$$\gamma_n = \varphi(\gamma_{n+1}) = \varphi(d_{n+1})\varphi(\gamma_{n+1}') = d_n\gamma_n'.$$

Since A_{n+1} is a Bézout domain, $1 = \gcd(\alpha_{n+1}', \gamma_{n+1}')$ is an A_{n+1}-linear combination of α_{n+1}' and γ_{n+1}'. Applying φ, we see that 1 is an A_n-linear combination of $\varphi(\alpha_{n+1}')$ and

$\varphi(\gamma'_{n+1})$. Thus, $\gcd(\varphi(\alpha'_{n+1}), \varphi(\gamma'_{n+1})) = 1$. It now follows via [10, Theorem 49 (a)] from the above displayed equations that

$$\gcd(\alpha_n, \gamma_n) = \varphi(d_{n+1}) \gcd(\varphi(\alpha'_{n+1}), \varphi(\gamma'_{n+1})) = \varphi(d_{n+1}).$$

As any two gcds of α_n and γ_n are associates, there exists $u_n \in U(A_n)$ such that $\varphi_{n+1,n}(d_{n+1}) = u_n d_n$.

Since $U(A_1) = \varphi_{21}(U(A_2))$, we may redefine d_2 (to be an associate of the former d_2) so as to ensure that $\varphi_{21}(d_2) = d_1$. (Specifically, replace d_2 with $v_2 d_2$, where $v_2 \in U(A_2)$ satisfies $\varphi_{21}(v_2) = u_1^{-1}$.) Similarly, we may use the hypotheses to redefine $d_3, d_4, \ldots$ so that $\varphi_{n+1,n}(d_{n+1}) = d_n$ for all $n \geq 1$. By *abus de langage*, we keep the above α'_n, γ'_n notation. Then $(\alpha'_n) \in A$, since $\varphi := \varphi_{n+1,n}$ satisfies

$$d_n \varphi(\alpha'_{n+1}) = \varphi(d_{n+1}) \varphi(\alpha'_{n+1}) = \varphi(\alpha_{n+1}) = \alpha_n = d_n \alpha'_n$$

and $d_n \neq 0$. Similarly, $(\gamma'_n) \in A$. Observe that it suffices to show that $(\alpha'_n)A_P$ and $(\gamma'_n)A_P$ are comparable under inclusion, for $\delta := (d_n) \in A$ satisfies $\alpha = \delta(\alpha'_n)$ and $\gamma = \delta(\gamma'_n)$. Thus, we may replace α and γ with (α'_n) and (γ'_n), respectively. In other words, we may assume that $\gcd(\alpha_n, \gamma_n) = 1$ for each n.

We next give two ways to complete the proof. First, recall that $\gcd(\alpha_n, \gamma_n) = 1$ for each n. Hence, $\alpha_n A_n + \gamma_n A_n = A_n$ for each n. Then localizing at P_n yields that

$$(A_n)_{P_n} = \alpha_n (A_n)_{P_n} + \gamma_n (A_n)_{P_n} \subseteq P_n (A_n)_{P_n} \subset (A_n)_{P_n},$$

the desired contradiction.

The following is an alternate way to finish the proof. Since inverse limit preserves monomorphisms, we can view $A \subseteq D := \varprojlim (A_n)_{P_n}$. As A_n is a Prüfer domain, $(A_n)_{P_n}$ is a valuation domain for each n, and so by [5, Theorem 2.1 (g)], D is a valuation domain. Thus, without loss of generality, $\alpha\gamma^{-1} \in D$. In particular, $\xi_n := \alpha_n \gamma_n^{-1} \in (A_n)_{P_n}$ for all n. Hence, $\xi_n = b_n z_n^{-1}$ for some $b_n \in A_n$ and $z_n \in A_n \setminus P_n$. As $\alpha_n \gamma_n^{-1}$ is in "lowest terms" and A_n is a GCD-domain, it follows that $\gamma_n | z_n$ in A_n, whence $z_n \in A_n \gamma_n \subseteq P_n$, the desired contradiction, thus completing the alternate proof. $\square$

For an example illustrating Theorem 2.3, begin with a valuation domain (V, M) having prime spectrum

$$M = P_1 \supset P_2 \supset \cdots \supset P_n \supset P_{n+1} \supset \cdots \supset 0$$

and consider the inverse system defined by $A_n := V/P_n$, with the transition maps $\varphi_{mn} : V/P_m \to V/P_n$ the canonical surjections if $m \geq n$.

Corollary 2.4. *For each n, suppose that A_n is a Bézout domain and that $\varphi_{n+1,n}$ induces a surjection $U(A_{n+1}) \to U(A_n)$. If, in addition,* $\mathrm{Spec}(A) = \cup\{\mathrm{im}(\mathrm{Spec}(A_n) \to \mathrm{Spec}(A)) \mid n \in \mathbb{N}\}$, *then A is a Prüfer domain.*

Proposition 2.6 studies further the condition that A_P is a valuation domain. First, recall from [1], [7] that if P is a prime ideal of an integral domain R, the *CPI- extension* of R with respect to P is the integral domain given by the following pullback:

$$R(P) := R_P \times_{R_P/PR_P} R/P = R + PR_P.$$

We assume familiarity with the material on $\mathrm{Spec}(R(P))$ in [1], [7]. Note also that PR_P is a divided prime ideal of $R(P)$: cf. [1, Proposition 2.5, Theorem 2.4], [2, Lemma 2.4 (b), (c)].

Suppose that $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ satisfies our riding hypotheses. We proceed to define an inverse system $\{\varphi^*_{mn} : A^*_m \to A^*_n \mid m \geq n \geq 2\}$, called the *associated inverse system of* $\{\varphi_{mn}\}$, which is more tractable. For each $n \geq 2$ in $\mathbb{N}$, let

$$A^*_n := A_n(Q_{n1}) = A_n + Q_{n1}(A_n)_{Q_{n1}}.$$

Define $\varphi^*_{n+1,n} : A^*_{n+1} \to A^*_n$ by

$$\varphi^*_{n+1,n}(a + qz^{-1}) = \varphi_{n+1,n}(a) + \varphi_{n+1,n}(q)\varphi_{n+1,n}(z)^{-1}$$

for all $a \in A_{n+1}$, $q \in Q_{n+1,1}$ and $z \in A_{n+1} \setminus Q_{n+1,1}$. Since Lemma 2.2 (f) ensures that $Q_{n+1,1} = \varphi^{-1}_{n+1,n}(Q_{n1})$, an easy calculation verifies that $\varphi^*_{n+1,n}$ is well defined. Then the inverse system $\{\varphi^*_{mn}\}$ is obtained by defining

$$\varphi^*_{mn} := \varphi^*_{n+1,n} \circ \varphi^*_{n+2,n+1} \circ \cdots \circ \varphi^*_{m,m-1} \; if \; m > n+1 \geq 3.$$

By analogy with the riding notation, we put $A^* := \varprojlim A^*_n$, $Q^*_n := \ker(A^* \to A^*_n)$ and $Q^*_{mn} := \ker(\varphi^*_{mn})$ if $m \geq n \geq 2$.

Lemma 2.5 (a) establishes that, apart from rescaling by using all $n \geq 2$, $\{\varphi^*_{mn}\}$ satisfies our riding hypotheses, and Lemma 2.5 (b) shows that $\{\varphi^*_{mn}\}$ has a desirable property which was assumed for the inverse systems treated in [5].

Lemma 2.5. *Let $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ be an $\mathbb{N}$-indexed inverse system of locally divided integral domains for which φ_{mn} is surjective for each $m \geq n$ in $\mathbb{N}$. Let $\{\varphi^*_{mn} : A^*_m \to A^*_n \mid m \geq n\}$ be the associated inverse system (using the notation introduced above). Then:*

(a) φ^*_{mn} is surjective for each $m \geq n \geq 2$ in $\mathbb{N}$.

(b) $Q^*_{n+1,n}$ is a divided prime ideal of A^*_{n+1} for each $n \geq 2$.

Proof. (a) Without loss of generality, $m = n+1$. Then it is easy to verify the assertion by using the explicit construction of $\varphi^*_{n+1,n}$ given above, since Lemma 2.2 (f) ensures that $\varphi_{n+1,n}$ sends $Q_{n+1,1}$ onto Q_{n1} and $A_{n+1} \setminus Q_{n+1,1}$ onto $A_n \setminus Q_{n1}$.

(b) Since $Q_{n+1,n} \subseteq Q_{n+1,1}$, a direct calculation using the above explicit construction of $\varphi^*_{n+1,n}$ shows that

$$Q^*_{n+1,n} := \ker(\varphi^*_{n+1,n}) = Q_{n+1,n}(A_{n+1})_{Q_{n+1,1}}.$$

The assertion is a consequence of the following useful fact: if $P \subseteq Q$ are prime ideals of an integral domain R such that R_Q is a divided domain, then PR_Q is a divided prime ideal of $R(Q) := R + QR_Q$. (Apply this fact to $R = A_{n+1}$, $P = Q_{n+1,n}$, and $Q = Q_{n+1,1}$.) To prove the above "useful fact", note by an easy calculation that one has to show that $PR_P = PR_Q$, and so an appeal to the proof of a characterization of locally divided domains [3, Theorem 2.4] completes the argument. □

Proposition 2.6. *Let $\{\varphi_{mn} : A_m \to A_n \mid m \geq n\}$ satisfy the riding hypotheses, with $A := \varprojlim A_n$. Let $\{\varphi^*_{mn} : A^*_m \to A^*_n \mid m \geq n\}$ be the associated inverse system, with $A^* := \varprojlim A^*_n$. Then:*

(a) Let $\mathcal{C}$ be a class of integral domains. If $A_n \in \mathcal{C}$ for each $n \in \mathbb{N}$, then $A^* \in \mathcal{C}$ in each of the following cases: $\mathcal{C}$ is the class of all (i) Prüfer domains, (ii) Bézout domains, (iii) divided domains, (iv) locally divided domains.

(b) Suppose that A_n is a locally divided domain for each n (for instance, repeat the hypotheses in (a).) Let $P \in \operatorname{Spec}(A)$ with $P \supseteq Q_1$; take $P_n := \Phi_n(P)$. Put $B := \varprojlim A_n(P_n)$. Then $\mathcal{P} := \varprojlim P_n(A_n)_{P_n} \in \operatorname{Spec}(B)$. Moreover, the canonical injection $A_P \to B_{\mathcal{P}}$ is an isomorphism if and only if the canonical injection $A_P \to \varprojlim (A_n)_{P_n}$ is an isomorphism. Indeed, $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ are isomorphic as A_P-algebras.

Proof. (a) Note that $A^*_n \in \mathcal{C}$ for each $n \geq 2$. Indeed, for (i) and (ii), this holds since each overring of a Prüfer (resp., Bézout) domain is a Prüfer (resp., Bézout) domain [9], while for (iii) and (iv), the proof of [4, Proposition 2.12] combines with [2, Lemma 2.2 (a), (c)] to ensure that the class of divided (resp., locally divided) domains is stable for CPI-extensions. Let $\{\varphi^*_{mn} : A^*_m \to A^*_n\}$ be the associated inverse system, with $A^* := \varprojlim A^*_n$.

The strategy is now to apply appropriate results of [5] to $\{\varphi^*_{mn}\}$. To be able to do so, we must verify that $\{\varphi^*_{mn}\}$ satisfies the riding assumptions of [5]. In view of Lemma 2.5, it follows from [5, Remark 2.24] that we need only verify that A^*_2 is not a field and $Q^*_{n+1,n} \neq 0$ for all $n \geq 2$.

If A^*_2 is a field, then by cofinality, we can delete the index $2 \in \mathbb{N}$. If the concern persists, then by cofinality, we may assume that $A^*_{n+1} = A_{n+1}(Q_{n+1,1})$ is a field for *each* $n \in \mathbb{N}$, whence $Q_{n+1,1} = 0$ and $\varphi_{n+1,1}$ is an isomorphism for each $n \in \mathbb{N}$. In that case, $A \cong A_1 \in \mathcal{C}$ and so, since $A^*_n \cong A_n$ for each n, $A^* \cong \varprojlim A_n = A \in \mathcal{C}$.

Similarly, if passing to cofinal index sets does not remove concerns about $Q^*_{n+1,n}$, then we may assume that $Q^*_{n+1,n} = 0$ for *each* $n \in \mathbb{N}$. By Lemma 2.5, it follows that $A^* \cong A^*_2 \in \mathcal{C}$.

We may now apply the results of [5] to $\{\varphi^*_{mn}\}$ as follows: for (i), use [5, Theorem 2.21]; for (ii), use [5, Corollary 2.23]; for (iii), use [5, Corollary 2.17 (a)]; and for (iv), use [5, Corollary 2.17 (b)].

(b) As $P \supseteq Q_1 \supseteq Q_n = \ker(\Phi_n)$, we have $P_n \in \mathrm{Spec}(A_n)$ for each n. As $P = \Phi_n^{-1}(P_n)$, we infer a canonical ring homomorphism $\alpha : A_P \to D := \varprojlim (A_n)_{P_n}$. It is straightforward to use the construction of α to verify that α is an injection. We next sketch how to rework the construction of the "associated inverse system" to produce B.

We produce an inverse system $\{\psi_{mn} : B_m \to B_n \mid m \geq n \geq 2\}$ as follows. For each $n \in \mathbb{N}$, consider the CPI-extension $B_n := A_n(P_n) = A_n + P_n(A_n)_{P_n}$. As $\varphi^{-1}_{n+1,n}(P_n) = P_{n+1}$ (as a consequence of Lemma 2.2 (f), (g)), we can mimic the construction of $\varphi^*_{n+1,n}$ to produce a surjective ring homomorphism $\psi_{n+1,n} : B_{n+1} \to B_n$ and, hence, the required surjection $\psi_{mn} : B_m \to B_n$ by composition if $m > n+1 \geq 3$. We show that the methods of [5] apply, more or less, in studying $B := \varprojlim B_n$.

Observe that the kernel of $\psi_{n+1,n}$ is $Q_{n+1,n}(A_{n+1})_{P_{n+1}}$. Since the hypothesis in (b) ensures that $(A_{n+1})_{P_{n+1}}$ is a divided domain, reasoning as in the proof of Lemma 2.5 (b) shows that $\ker(\psi_{n+1,n})$ is a divided prime ideal of B_{n+1}. There are two ways that the methods of [5] might not apply: either each such $\psi_{n+1,n}$ is an isomorphism or each B_n is a field. In the first case, all the canonical maps in question are isomorphisms, since A_P, $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ all canonically identify with $(A_1)_{P_1}$ in this case. In the second case, each $P_n = 0$ by the standard theory of CPI-extensions, whence the inverse systems defining A and B are essentially the same, with A_P, $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ all canonically identified with the quotient field of A_1 in this case. Thus, we can assume henceforth that the inverse system $\{\psi_{mn}\}$ satisfies the riding assumptions in [5].

View $\mathcal{P} := \varprojlim P_n(A_n)_{P_n}$ canonically inside $\varprojlim B_n = B$. It is straightforward to use the condition $\varphi^{-1}_{n+1,n}(P_n) = P_{n+1}$ to verify that $\mathcal{P} \in \mathrm{Spec}(B)$. (The same conclusion holds in the two cases noted above, for then $\mathcal{P} \cong P_1(A_1)_{P_1}$ and $B \cong B_1$.) Therefore, by [5, Proposition 2.15 (d)], the canonical ring homomorphism $\beta : B_{\mathcal{P}} \to E := \varprojlim (B_n)_{P_n(A_n)_{P_n}}$ is an isomorphism. Moreover, there is an isomorphism $\gamma : D \to E$ because one has compatible isomorphisms $(A_n)_{P_n} \to (B_n)_{P_n(A_n)_{P_n}}$ at every level. To finish the proof of (b), it suffices to find a ring homomorphism $\delta : A_P \to B_{\mathcal{P}}$ such that $\beta \circ \delta = \gamma \circ \alpha : A_P \to E$.

By composing the inclusions $A \to B$ and $B \to B_{\mathcal{P}}$, one obtains an injection $f : A \to B_{\mathcal{P}}$. We claim that $f(A \setminus P) \subseteq B \setminus \mathcal{P}$. Indeed, if $a = (a_n) \in A \cap \mathcal{P}$, then $a_n \in P_n(A_n)_{P_n} \cap A_n = P_n$ for each n, whence $a \in \varprojlim P_n = P$, thus proving the claim. The universal mapping property of localization produces a unique ring homomorphism $\delta : A_P \to B_{\mathcal{P}}$ that extends f, and a routine calculation verifies that $\beta \circ \delta = \gamma \circ \alpha$, to complete the proof. $\square$

In the context of Proposition 2.6 (b), suppose that A_n is a Prüfer (hence, locally divided) domain for each n. Then both $B_{\mathcal{P}}$ and $\varprojlim (A_n)_{P_n}$ are valuation domains, by [5, Theorem 2.21 and Theorem 2.1 (g)]. (In the two degenerate cases noted above, the assertion about $B_{\mathcal{P}}$ follows since $B \cong B_1$ is Prüfer in these cases.) Thus, we come to the main point of Proposition 2.6 (b): these two standard ways to produce a valuation domain containing A_P are isomorphic, and A_P coincides with the first of these valuation domains if and only if A_P coincides with the second.

Remark 2.7. It is well known (cf. [6]) that if $\{B_i\}$ is a *directed* system of (commutative) rings indexed by a directed index set, then $\operatorname{Spec}(\varinjlim B_i) \cong \varprojlim \operatorname{Spec}(B_i)$. Accordingly, it may seem reasonable to speculate that if $\{D_n\}$ is an *inverse* system of rings which is indexed by $\mathbb{N}$ and has surjective transition maps, then there should be a close connection between $\operatorname{Spec}(\varprojlim D_n)$ and $\varinjlim \operatorname{Spec}(D_n)$. If each D_n is an integral domain, this is indeed so for certain natural inverse systems: see [5, Theorem 2.5 (a)]. However, the following example shows that the situation can be more complicated if the D_n are not integral domains. In this example, each D_n is a principal ideal ring.

Let $\{k_i \mid i \in \mathbb{N}\}$ be any sequence of fields. For each $n \in \mathbb{N}$, put $D_n := \prod_{i=1}^{n} k_i$. If $r \geq n$ in $\mathbb{N}$, let $\varphi_{rn} : D_r \to D_n$ denote the canonical projection map; of course, each φ_{rn} is surjective. Moreover, $\varinjlim \operatorname{Spec}(D_n)$ is countable, since it can be viewed as a union of a countable chain of finite sets. However, $\{\varphi_{rn} \mid r \geq n\}$ leads to $D := \varprojlim D_n$ which is such that $\operatorname{Spec}(D)$ is not countable. Indeed, $D \cong \prod_{i=1}^{\infty} k_i$ canonically, and so $\operatorname{Spec}(D)$ is the Stone-Čech compactification of $\mathbb{N}$ when $\mathbb{N}$ is endowed with the discrete topology. (The "Stone-Čech" part of the preceding assertion seems to be folklore. In case $k_i = \mathbb{R}$ for all i, this piece of folklore follows from [8, items 7.10 and 7.11, page 105].) We conclude from this example that care must be taken if one attempts to extend the work in [5] and this note to $\mathbb{N}$-indexed inverse systems having surjective transition maps for arbitrary (commutative) rings.

Acknowledgement: Both authors were supported in part by a NATO Collaborative Research Grant. The first author was also supported in part by a University of Tennessee Faculty Development Award and a Visiting Professorship funded by the Istituto Nazionale di Alta Matematica. Dobbs thanks the Università degli Studi "Roma Tre" for the warm hospitality accorded to him during his visits.

References

[1] M. Boisen, Jr. and P. B. Sheldon, *CPI-extensions: overrings of integral domains with special prime spectrums*, Can. J. Math. **29** (1977), 722–737.

[2] D. E. Dobbs, *Divided rings and going-down*, Pacific J. Math. **67** (1976), 353–363.

[3] D. E. Dobbs, *On locally divided integral domains and CPI-overrings*, Internat. J. Math. & Math. Sci. **4** (1981), 119–135.

[4] D. E. Dobbs, On Henselian pullbacks, *Factorizations in Integral Domains*, Lecture Notes Pure Appl. Math., Marcel Dekker **189** (1997), 317–326.

[5] D. E. Dobbs and M. Fontana, *Inverse limits of integral domains arising from iterated Nagata composition*, Math. Scand. **88** (2001), 17–40.

[6] D. E. Dobbs, M. Fontana and I. J. Papick, *Direct limits and going-down*, Comm. Math. Univ. St. Pauli **31** (1982), 129–135.

[7] M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. **123** (1980), 331–355.

[8] L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, 1960.

[9] R. Gilmer, Multiplicative Ideal Theory, Dekker, 1972.

[10] I. Kaplansky, Commutative Rings, Revised Edition, Univ. Chicago Press, 1974.

[11] C. Rotthaus and S. Wiegand, *Direct limits of two-dimensional prime spectra*, Contemp. Math. **171** (1994), 353–384.

[12] E. Weiss, Algebraic Number Theory, Unabridged Edition, Dover, 1998.