a.b. star operations

vs

e.a.b. star operations

presented by

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Let D be an integral domain with quotient field K.
Let $\boldsymbol{F}(D)$ (respectively, $\boldsymbol{f}(D)$) be the set of all nonzero fractionary ideals (respectively, finitely generated fractionary ideals) of D.

A mapping $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $I \mapsto I^\star$, is called *a star operation of D* if, for all $z \in K$, $z \neq 0$, and for all $I, J \in \boldsymbol{F}(D)$, the following properties hold:

$(\star\star_1)$ $(zD)^\star = zD$, $(zI)^\star = zI^\star$;

$(\star_2)$ $I \subseteq J \Rightarrow I^\star \subseteq J^\star$;

$(\star_3)$ $I \subseteq I^\star$ and $I^{\star\star} := (I^\star)^\star = I^\star$.

W. Krull introduced the concept of a star operation in his first Beiträge paper [K-1936]

In 1994, Okabe and Matsuda introduced the more “flexible” notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the notion of star operation, allowing $D \neq D^\star$.

Let $\overline{\boldsymbol{F}}(D)$ represent the set of all nonzero D–submodules of K (thus, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$).

A mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $z \in K$, $z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

When $D^\star = D$, the map $\star$, restricted to $\boldsymbol{F}(D)$, defines *a star operation of D.*

Let $\star$ be a [semi]star operation on D. If F is in $\boldsymbol{f}(D)$, we say that

F is *$\star$–e.a.b.*
$(FG)^\star \subseteq (FH)^\star \Rightarrow G^\star \subseteq H^\star$, with $G,\ H \in \boldsymbol{f}(D)$,

F is *$\star$–a.b.*
$(FG)^\star \subseteq (FH)^\star \Rightarrow G^\star \subseteq H^\star$, with $G,\ H \in \boldsymbol{F}(D)$
[respectively, with $G,\ H \in \overline{\boldsymbol{F}}(D)$].

The operation $\star$ is said to be *e.a.b.* (respectively, *a.b.*) if each $F \in \boldsymbol{f}(D)$ is $\star$–e.a.b. (respectively, $\star$–a.b.).

An a.b. operation is obviously an e.a.b. operation.

If $\mathcal{W}$ is a given family of valuation overrings overrings of D and if $D = \cap\{W \mid W \in \mathcal{W}\}$ then the star operation $\wedge_{\mathcal{W}}$ defined as follows: for each $E \in \boldsymbol{F}(D)$,

$$E^{\wedge_{\mathcal{W}}} := \cap\{EW \mid W \in \mathcal{W}\}$$

is an a.b. star operation on D.

In particular, if $\mathcal{V}$ is the set of all the valuation overrings of D, then $b_D := \wedge_{\mathcal{V}}$ is called the *b–star operation* on an in integrally closed domain D.

If we do not assume that, given a family of valuation overrings overrings $\mathcal{W}$ of D, $\cap\{W \mid W \in \mathcal{W}\}$ coincides with D then $\wedge_{\mathcal{W}}$ defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\wedge_{\mathcal{W}}} := \cap\{EW \mid W \in \mathcal{W}\}$$

is an a.b. **semi**star operation on D (with $D^{\wedge_{\mathcal{W}}} = \cap\{W \mid W \in \mathcal{W}\}$).

W. Krull in his first Beiträge paper [K-1936] only considered the concept of "arithmetisch brauchbar (a.b.) $\star$–operation" [the original Krull's notation was $'$–Operation], and he did not considered the concept of "endlich arithmetisch brauchbar (e.a.b.) $\star$–operation".

The e.a.b. concept stems from the original version of Gilmer's book (1968). The results of Section 26 show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings.

Robert Gilmer explained to me that $\ll$ I believe I was influenced to recognize this because during the 1966 calendar year in our graduate algebra seminar (Bill Heinzer, Jimmy Arnold, and Jim Brewer, among others, were in that seminar) we had covered Bourbaki's Chapitres 5 and 7 of *Algèbre Commutative*, and the development in Chapter 7 on the v–operation indicated that e.a.b. would be sufficient.$\gg$

Apparently there are no examples in literature of e.a.b. not a.b. star operations.

When $\star := d_D$ the notion of d_D–e.a.b. [respectively, d_D–a.b.] for finitely generated ideal coincides with the notion of *quasi–cancellation ideal* [respectively, *cancellation ideal*] studied by D.D. Anderson and D.F. Anderson [AA-1984].

As a matter of fact, a nonzero ideal I (non necessarily finitely generated) of an integral domain D is called a *cancellation* [respectively, *quasi–cancellation*] *ideal of D* if $(IJ : I) = J$, for each nonzero ideal J of D [respectively, if $(IF : I) = F$, for each nonzero finitely generated ideal F of D].
Obviously, a cancellation ideal is a quasi–cancellation ideal, but in general (for non finitely generated ideals) the converse does not hold (e.g. a maximal ideal of a nondiscrete rank one valuation domain, [AA-1984]).

For a finitely generated ideal, the notion of cancellation ideal coincides with the notion of quasi–cancellation ideal [AA-1984, Corollary 1]. More precisely, by [AA-1984, Lemma 1 and Theorem 1],

Proposition 1 *If I is a nonzero* finitely generated ideal *of D, then the following conditions are equivalent:*

- **(i)** *I is a quasi–cancellation ideal of D;*
- **(ii)** *$IG \subseteq IH$, with G and H non-zero finitely generated ideals of D, implies that $G \subseteq H$;*
- **(iii)** *$IG \subseteq IH$, with G and H non-zero ideals of D, implies that $G \subseteq H$;*
- **(iv)** *I is a cancellation ideal of D;*
- **(v)** *for each prime [maximal] ideal Q of D, ID_Q is an invertible ideal of D_Q;*
- **(vi)** *I is an invertible ideal of D.*

Remark. (1) The notion of quasi-cancellation ideal was introduced in [AA-1984], since in Gilmer's book [G-1972, Exercise 4, p.66] it was erroneously stated that a nonzero ideal I of an integral domain D is a cancellation ideal if and only if $(IF : I) = F$, for each finitely generated ideal F of D (see the above mentioned counter-example).

(2) Kaplansky [Ka-1971] in an unpublished set of notes proved that *a nonzero finitely generated ideal I of a local integral domain D is a cancellation ideal if and only if I is principal.* Therefore Proposition 1 is an extension of Kaplansky's result.

(3) Recall that Jaffard [J-1960] proved that :
I is a (quasi-)cancellation ideal, for each $I \in \boldsymbol{f}(D) \Leftrightarrow D$ is a Prüfer domain,
cf. also Jensen [Je-1963, Theorem 5]. In that paper Jensen [Je-1963, Theorem 6] proved also that:
I is a cancellation ideal, for each $I \in \boldsymbol{F}(D) \Leftrightarrow D$ is an almost Dedekind domain.

Recall also that in [AA-1984, Theorem 7] it is proved that:
I is a quasi cancellation ideal, for each $I \in \boldsymbol{F}(D) \Leftrightarrow D$ is a completely integrally closed Prüfer domain.

(4) Note that, *when D is a Prüfer domain*, it is known [AA-1984, Theorem 2 and Theorem 5] that:
(a) $I \in \boldsymbol{F}(D)$ *is a quasi cancellation ideal* $\Leftrightarrow (I : I) = D$.
(b) $I \in \boldsymbol{F}(D)$ *is a cancellation ideal* $\Leftrightarrow ID_M$ *is principal for each M maximal ideal of D.*

D.D. Anderson and Roitman [AR-1997, Theorem] extended **(b)** outside of the (Prüfer) domain case and proved that, *given a non zero ideal I of an integral domain (respectively, a ring) R:*
I is a cancellation ideal of $R \Leftrightarrow IR_M$ is a (respectively, regular) principal ideal of R_M, for each M maximal ideal of R.

The previous statement was "extended" further on to submodules of the quotient field of an integral domain D by Goeters and Olberding [GO-2000]:
let $E \in \overline{\boldsymbol{F}}(D)$, E is called *a cancellation module for D* if for $G, H \in \overline{\boldsymbol{F}}(D)$:

$$EG \subseteq EH \Rightarrow G \subseteq H.$$

Then [GO-2000, Theorem 2.3]:

E is a cancellation module for D $\Leftrightarrow$
$\Leftrightarrow ED_M$ *is principal, for each* $M \in \text{Max}(D)$
$\Leftrightarrow ED_M$ *is a cancellation module for D_M, for each* $M \in \text{Max}(D)$.

As in the classical star-operation setting, we associate to a *semistar operation* $\star$ of D a new semistar operation $\star_f$ as follows. If $E \in \overline{\boldsymbol{F}}(D)$ we set:

$$E^{\star_f} := \cup\{F^\star \mid F \subseteq E,\, F \in \boldsymbol{f}(D)\}\,.$$

We call $\star_f$ *the semistar operation of finite type of* D associated to $\star$.

If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type of* D. Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type of D.

As we have already remarked, it is obvious that:

a.b. [semi]star operation $\Rightarrow$ e.a.b. [semi]star operation.

Moreover:

Lemma 2 *If $\star = \star_f$, then:*

$\star$ is an e.a.b. [semi]star operation $\Leftrightarrow$ $\star$ is an a.b. [semi]star operation.

Proof. We consider the case of star operations. We need only to prove that if $\star$ is an e.a.b. star operation then $\star$ is an a.b. star operation. Let $I \in \boldsymbol{f}(D)$ and $J, L \in \boldsymbol{F}(D)$. Assume that $(IJ)^\star \subseteq (IL)^\star$. By the assumption, we have $(IJ)^\star = \cup\{H^\star \mid H \in \boldsymbol{f}(D)), \ H \subseteq IJ\} = \cup\{(IF)^\star \mid F \in \boldsymbol{f}(D), \ F \subseteq J\}$ and similarly $(IL)^\star = \cup\{(IG)^\star \mid G \in \boldsymbol{f}(D), \ G \subseteq L\}$. Therefore, for each $F \in \boldsymbol{f}(D), \ F \subseteq J$, we have $IF \subseteq \cup\{(IG)^\star \mid G \in \boldsymbol{f}(D), \ G \subseteq L\}$. Thus we can find $G_1, G_2, ..., G_r$ in $\boldsymbol{f}(D)$ with the property that $G_i \subseteq L$ for $1 \leq i \leq r$, in such a way:

$$(IF)^\star \subseteq (IG_1 \cup IG_2 \cup ... \cup IG_r)^\star \subseteq (I(G_1 \cup G_2 \cup ... \cup G_r))^\star .$$

Since $\star$ is an e.a.b. star operation then $F^\star \subseteq (G_1 \cup G_2 \cup ... \cup G_r)^\star \subseteq \cup\{G^\star \mid G \in \boldsymbol{f}(D), \ G \subseteq L\} = L^{\star_f} = L^\star$ and so $J^\star = J^{\star_f} = \cup\{F^\star \mid F \in \boldsymbol{f}(D), \ F \subseteq J\} \subseteq L^\star$. $\square$

Next result generalizes in the [semi]star setting Lemma 2 (i.e. part of [AA-1984, Theorem]).

Proposition 3 *Let D be an integral domain and let $\star$ be a [semi]star operation on D. If F is in $\boldsymbol{f}(D)$, then:*

F is $\star$–e.a.b. if and only if F is $\star_f$–a.b.

Proof. Since the notion of $\star$–e.a.b. coincides with the notion of $\star_f$–e.a.b., it remains to show that if F is $\star$–e.a.b. then F is $\star_f$–a.b.. Let $G, H \in \overline{\boldsymbol{F}}(D)$ and assume that $(FG)^{\star_f} \subseteq (FH)^{\star_f}$, then arguing as in Lemma 2, for each $G' \in \boldsymbol{f}(D)$, with $G' \subseteq G$, we can find a $H'_{G'} \in \boldsymbol{f}(D)$, with $H'_{G'} \subseteq H$, in such a way that $(FG')^\star \subseteq (FH'_{G'})^\star$. Since F is $\star$–e.a.b., then $(G')^\star \subseteq (H'_{G'})^\star$ and so $G^{\star_f} = \cup\{(G')^\star \mid G' \in \boldsymbol{f}(D)\,,\ G' \subseteq G\} \subseteq \cup\{(H'_{G'})^\star \mid G' \in \boldsymbol{f}(D)\,,\ G' \subseteq G\} \subseteq \cup\{(H')^\star \mid H' \in \boldsymbol{f}(D)\,,\ H' \subseteq H\} = H^{\star_f}$. $\square$

Some of the characterizations given in [AA-1984] for quasi–cancellation and cancellation ideals, can be generalized in the [semi]star setting.

Lemma 4 *Let $F \in \boldsymbol{f}(D)$ and let $\star$ be a [semi]star operation on D. F is $\star$–e.a.b. (respectively, $\star$–a.b.) if and only if $((FH)^\star : F^\star) = H^\star$, for each $H \in \boldsymbol{f}(D)$ (respectively, for each $H \in \boldsymbol{F}(D)$ [$H \in \overline{\boldsymbol{F}}(D)$ in the semistar case]) .*

(Note that $((FH)^\star : F^\star) = ((FH)^\star : F) = H^\star$, so the previous equivalences can be stated in a formally slightly different way.)

Proof. We consider only the a.b. case in the star setting (since $\star$–e.a.b. coincides with $\star_{_f}$–a.b. and $((FH)^\star : F^\star) = H^\star$, coincides with $((FH)^{\star_f} : F^{\star_f}) = H^{\star_f}$, when $F, H \in \boldsymbol{f}(D)$)). The "if" part: it is easy to see that, F is $\star$–a.b. if and only if $(FG)^\star = (FH)^\star$, with $G,\ H \in \boldsymbol{F}(D)$, implies that $G^\star = H^\star$. Then,

$$(FG)^\star = (FH)^\star \ \Rightarrow \ ((FG)^\star : F^\star) = ((FH)^\star : F^\star)\ .$$

The conclusion now is a straightforward consequence of the assumption.

The "only if" part: given $H \in \boldsymbol{F}(D)$, clearly $H^\star \subseteq ((FH)^\star : F^\star)$. Conversely, note that $F\left((FH)^\star : F^\star\right) \subseteq (FH)^\star$, and so we have $\left(F\left((FH)^\star : F^\star\right)\right)^\star \subseteq (FH)^\star$. Therefore, by the assumption, $((FH)^\star : F^\star)^\star \subseteq H^\star$. □

It is known that *if $\star$ is an e.a.b. star operation on an integral domain D, then there exists an a.b. star operation $\ast$ on D such that $\star|_{\boldsymbol{f}(D)} = \ast|_{\boldsymbol{f}(D)}$* [G-1972, Corollary 32.13].
This property holds also in the semistar setting.

More precisely, if we consider the Kronecker function ring (associated to [semi]star operation $\star$), $\mathrm{Kr}(D, \star)$, then the a.b. [semi]star operation $\ast$ can be defined as follows: for each $E \in \boldsymbol{F}(D)$ [$E \in \overline{\boldsymbol{F}}(D)$, in the semistar setting],

$$E^{\ast} := E\mathrm{Kr}(D, \star) \cap K \,.$$

This operation can described in various equivalent ways.

It is possible to associate to any semistar operation $\star$ of D an (e.)a.b. semistar operation of finite type $\star_a$ of D, called *the (e.)a.b. semistar operation associated to* $\star$, defined as follows for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$\begin{aligned} F^{\star_a} &:= \cup\{((FH)^\star : H^\star) \mid H \in \boldsymbol{f}(D)\}, \\ E^{\star_a} &:= \cup\{F^{\star_a} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}. \end{aligned}$$

The previous construction, in the star setting, is essentially due to P. Jaffard [J-1960] and F. Halter-Koch [HK-1997], [HK-1998] .

Obviously $(\star_f)_a = \star_a$. Note that:

– *when* $\star = \star_f$, *then* $\star$ *is (e.)a.b. if and only if* $\star = \star_a$.

– $D^{\star_a}$ *is integrally closed and contains the integral closure of* D *in* K .

When $\star = v$, then D^{v_a} coincides with *the pseudo-integral closure of* D introduced by D.F. Anderson, Houston and Zafrullah [AHZ-1991].

We now turn our attention to the valuation overrings. The notion that we recall next is due to P. Jaffard [J-1960] (cf. also Halter-Koch [HK-1997]).

For a domain D and a semistar operation $\star$ on D, we say that a valuation overring V of D is *a $\star$–valuation overring of D* provided $F^\star \subseteq FV$, for each $F \in \boldsymbol{f}(D)$. Note that, by definition the $\star$–valuation overrings coincide with the $\star_f$–valuation overrings.

Theorem 5 *Let $\star$ be a semistar operation of an integral domain D with quotient field K. Then:*

(1) *The $\star$–valuation overrings also coincide with $\star_a$–valuation overrings of D.*

(2) *V is a $\star$–valuation overring of D if and only if $V(X)$ is a valuation overring of* Kr$(D, \star)$.
The map $W \mapsto W \cap K$ establishes a bijection between the set of all valuation overrings of Kr$(D, \star)$ *and the set of all the $\star$–valuation overrings of D.*

(3) $E^{\star_a} = E\text{Kr}(D, \star) \cap K = \cap\{EV \mid V \text{ is a } \star\text{–valuation overring of } D\}$ $(=: E^{\wedge_{\mathcal{V}(\star)}})$, *for each $E \in \overline{\boldsymbol{F}}(D)$ (where $\boldsymbol{\mathcal{V}}(\star) := \{V \mid V$ is a $\star$–valuation overring of $D\}$).*

□

Example 6 *Example of an e.a.b. star operation that is not an a.b. star operation.*

Let k be a field, let $X_1, X_2, X_n, ...$ be an infinite set of indeterminates over k and let $N := (X_1, X_2, X_n, ...)k[X_1, X_2, X_n, ...]$. Clearly, N is a maximal ideal in $k[X_1, X_2, X_n, ...]$. Set $D := k[X_1, X_2, X_n, ...]_N$, let $M := ND$ be the maximal ideal of the local domain D and let K be the quotient field of D.

Note that D is an UFD and consider $\boldsymbol{\mathcal{W}}$ the set of all the rank one valuation overrings of D. Let $\wedge_{\boldsymbol{\mathcal{W}}}$ be the star a.b. operation on D defined by $\boldsymbol{\mathcal{W}}$. It is wellknown that the t–operation t_D on D is an a.b. star operation, since $t_D|_{\boldsymbol{f}(D)} = \wedge_{\boldsymbol{\mathcal{W}}}|_{\boldsymbol{f}(D)}$ [G-1972, Proposition 44.13].

We consider the following subset of fractionary ideals of D:

$$\mathcal{J} := \{xF^{t_D}, yM, zM^2 \mid x, y, z \in K \setminus \{0\}, F \in \boldsymbol{f}(D)\}.$$

By using [G-1972, Proposition 32.4], it is not difficult to verify that the set $\mathcal{J}$ defines on D a star operation $\star$ as follows, for each $E \in \boldsymbol{F}(D)$:

$$E^\star := \cap\{J \mid J \in \mathcal{J}, \ J \supseteq E\}.$$

Clearly $\star|_{\boldsymbol{f}(D)} = t_D|_{\boldsymbol{f}(D)}$ and so $\star$ is an e.a.b. operation on D, since t_D is an a.b. (and hence an e.a.b.) star operation on D. Note that $(X_1, X_2)M \subset M^2$ and $(M^2)^\star = M^2$.

We claim that:

$$((X_1, X_2)M)^\star = ((X_1, X_2))^{t_D} \cap M^2 = M^2 = ((X_1, X_2)M^2)^\star.$$

As a matter of fact, if $(X_1, X_2)M \subseteq G^{t_D}$ for some $G \in \boldsymbol{f}(D)$, then we have $((X_1, X_2)D)^{t_D} M^{t_D} \subseteq G^{t_D}$, with $((X_1, X_2)D)^{t_D} = M^{t_D} = D$, since X_1 and X_2 are coprime in D and so $(X_1, X_2)D$ is not contained in any proper principal ideal of D . Therefore $(X_1, X_2)M$ is not contained in any nontrivial ideal of the type xF^{t_D} $(= G^{t_D}) \in \mathcal{J}$.
A similar argument shows that $(X_1, X_2)M$ is neither contained in any ideal of the type $yM, zM^2 \in \mathcal{J}$, with y and z nonzero and non unit in D, and thus the only nontrivial ideals of $\mathcal{J}$ containing $(X_1, X_2)M$ are M^2 and M, hence $((X_1, X_2)M)^\star = M^2$.
Similarly, it can be shown that $((X_1, X_2)M^2)^\star = M^2$.

From the claim, if $\star$ would be an a.b. star operation then we would deduce that $M^\star$ $(= M)$ equal to $(M^2)^\star$ $(= M^2)$, which is not the case.

References

[AA-1984] D. D. Anderson and D. F. Anderson, *Some remarks on cancellation ideals,* Math. Japonica **6** (1984), 879–886.

[AR-1997] D. D. Anderson and M. Roitman, *A characterization of cancellation ideals,* Proc. Am. math. soc. **125** (1997), 2853–2854.

[AHZ-1991] D. F. Anderson, E. G. Houston and M. Zafrullah, *Pseudo-integrality*, Canad. Math. Bull. **34** (1991), 15–22.

[FL-2001] M. Fontana and K. A. Loper, *Kronecker function rings: a general approach,* in "Ideal Theoretic Methods in Commutative Algebra" (D.D. Anderson and I.J. Papick, Eds.), M. Dekker Lecture Notes Pure Appl. Math. **220** (2001), 189–205.

[FP-2005] M. Fontana and G. Picozza, *Semistar invertibility on integral domains,* Algebra Colloquium (to appear).

[G-1972] R. Gilmer, Multiplicative Ideal Theory, M. Dekker, New York, 1972.

[GO-2000] H. P. Goeters and B. Olberding, *On the multiplicative properties of submodules of the quotient field of an integral domain,* **26** (2000), 241–254.

[HK-1997] F. Halter-Koch, *Generalized integral closures,* in "Factorization in Integral Domains" (D.D. Anderson, Ed.). M. Dekker Lecture Notes Pure Appl. Math. **187** (1997), 349–358.

[HK-1998] F. Halter-Koch, Ideal Systems: An Introduction to Multiplicative Ideal Theory, M. Dekker, New York, 1998.

[J-1960] P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1960.

[Je-1963] C. H. Jensen, *On characterizations of Prüfer domains,* Math. Scand. **13** (1963), 90–98.

[Ka-1971] I. Kaplansky, Topics in commutative ring theory. Unpublished notes. Chicago, 1971.

[K-1936] W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche,* I - II. Math. Z. **41** (1936), 545–577; 665–679.

[OM-1994] A. Okabe and R. Matsuda, *Semistar operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.