GENERALIZED GOING-DOWN HOMOMORPHISMS OF COMMUTATIVE RINGS

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ABSTRACT

Sufficient conditions are given for a (unital) homomorphism $f\colon A \to B$ of (commutative) rings to be a chain morphism, in the sense that ${}^a f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ permits the covering of chains of arbitrary cardinality. One such sufficient condition is that f satisfies lying-over, ${}^a f$ be open in the flat (resp., Zariski) topology, and that each reduced fiber of ${}^a f$ be quasilocal (resp., an integral domain). Sufficient conditions are given for f to have the generalized going-down property GGD (that is, "going-down" predicated for chains of arbitrary cardinality). Typical of such sufficient conditions are the following: f is a chain morphism and B is quasilocal treed; f satisfies going-down and either the reduced fibers of ${}^a f$ are integral domains or A is a going-down ring. "Universally going-down" is equivalent to "universally GGD"; in particular, if f is flat, then f satisfies GGD. The universally subtrusive homomorphisms are the same as the universally chain morphisms, and these descend the GGD property.

1 INTRODUCTION

All rings considered below are commutative with identity, and all ring homomorphisms are unital. Adapting the notation in [10, p. 28], we let GU, GD, LO, and

INC denote the going-up, going-down, lying-over, and incomparable properties, respectively, for ring homomorphisms. As in [3], our interests here include the following strengthening of the LO property. A ring homomorphism $f: A \to B$ is called a *chain morphism* if the associated map ${}^a f: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $Q \mapsto f^{-1}(Q)$, permits each chain (of arbitrary cardinality) of prime ideals of A to be "covered" by some chain of prime ideals of B. Theorem 2.3 gives our main sufficient conditions for a ring homomorphism f to be a chain morphism, namely, that f satisfy GU (resp., GD) and LO, with reduced fibers that are quasilocal (resp., integral domains). Theorem 2.8 is essentially a corollary giving sufficient conditions that are couched topologically. (For background on the flat spectral topology, see [8], [4]; for background on the patch, or constructible, topology, see [8], [7].) Proposition 2.2 collects the sufficient conditions for chain morphisms that were established in [3], most notably, that f be injective and integral. Theorem 3.26 presents a significant generalization: any universally subtrusive f (in the sense of [14], for instance, any pure or faithfully flat f) is universally a chain morphism. This result depends on the heart of the paper, Section 3, which develops the theory of the GGD (*generalized going-down*) concept, the property that "going-down" behavior hold for chains of arbitrary cardinality.

Proposition 3.1 states the sufficient condition for a ring homomorphism $f: A \to B$ to satisfy GGD that was obtained in [3]. Numerous other sufficient conditions are given, including that f satisfy GD with reduced fibers that are integral domains (Corollary 3.6); and that B be a quasilocal treed ring with f satisfying either GD or both LO and GU (Corollary 3.4). Theorem 3.9 identifies a context for which GGD and GD are equivalent, namely, where A is a going-down ring (in the sense of [2]) in which each maximal ideal of A contains a unique minimal prime ideal of A. A noteworthy upshot appears in Corollary 3.14: a weak Baer ring A is a going-down ring if and only if $A \hookrightarrow B$ satisfies GD for each overring B of A. Despite the nomenclature, such an assertion fails if A is not a weak Baer ring [2, Examples 1 and 2, pp. 9-12]. Accordingly, since our present focus is on properties of homomorphisms, we intend to devote a subsequent paper to weak Baer going-down rings and related themes.

As a companion for the characterization of universally chain morphisms in Theorem 3.26, we also characterize the universally GGD ring homomorphisms (in Theorem 3.16): they are precisely the universally going-down maps. In particular, each flat ring homomorphism is (universally) GGD. Among other sufficient conditions for a ring homomorphism f to satisfy GGD is that f satisfy GD and ${}^a f$ be injective (Corollary 3.21), in which case ${}^a f$ is a topological immersion (relative to the Zariski topology). Finally, we note a consequence of Theorem 3.26: each universally subtrusive ring homomorphism descends the universally going-down property.

We next describe notational conventions. Unless otherwise specified, maps of the form ${}^a f$ are considered relative to the Zariski topology. As usual, a typical closed set in that topology on $\mathrm{Spec}(A)$ is $V(I) = \{P \in \mathrm{Spec}(A) : P \supseteq I\}$, where I is an ideal of A. We denote the closure of a set X in the Zariski topology by $\overline{X}$, with X^c denoting the closure of X in the patch topology. By a *patch*, we mean a set that is closed in the patch topology. A ring A is *treed* if no maximal ideal of A contains incomparable prime ideals of A. If A is a ring, then $U(A)$ denotes the set of units of A and $tq(A)$ denotes the total quotient ring of A. By an *overring of* a ring A, we mean any A-subalgebra of $tq(A)$; or, more intuitively, any ring B such that $A \subseteq B \subseteq tq(A)$. Finally, $\subset$ denotes proper containment, and $|I|$ denotes the cardinality of the set I.

Background is recalled as needed. Any unexplained material is standard, as in [10], [7], [6].

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As this paper went to press (May, 2002), Kang-Oh have announced a preprint, "Lifting Up a Tree of Prime Ideals to a Going-Up Extension," whose methods, we have determined, can be extended to show that GGD and GD are equivalent for ring homomorphisms $A \to B$ such that both A and B are locally irreducible. In particular, this equivalence holds if A and B are each integral domains.

2 CHAIN MORPHISMS

Let A be a ring and X a subset of $\mathrm{Spec}(A)$. Following [15], we define $\mathcal{U}(X) := \cup\{P : P \in X\}$ and $\mathcal{R}(X) := \cap\{P : P \in X\}$. Observe that if X is a chain, then $\mathcal{U}(X), \mathcal{R}(X) \in \mathrm{Spec}(A)$ [10, Theorem 9]. A chain X is called a *local chain* if X has a (necessarily unique) maximal element. If X is a chain, then $X \cup \{\mathcal{U}(X)\}$ is a local chain; in fact, a chain X is a local chain if and only if $\mathcal{U}(X) \in X$. By reworking the proof of [15, Proposition 2.2], we see that each local chain is quasi-compact in the Zariski topology.

We next introduce the key concepts of this section. Suppose that $f : A \to B$ is a ring homomorphism. Consider $X = \{P_i : i \in I\}$, a subset of $\mathrm{Spec}(A)$. (The notation is generally taken so that $P_i \neq P_j$ whenever $i \neq j$; as a result, $|X| = |I|$.) A subset $Y = \{Q_i : i \in I\}$ of $\mathrm{Spec}(B)$ is said to *cover* (or to *dominate*) X if $f^{-1}(Q_i) = P_i$ for each $i \in I$. (By the notational convention, $Q_i \neq Q_j$ if $i \neq j$, and so $|Y| = |I|$.) We say that f is a *chain morphism* if, for each chain X in $\mathrm{Spec}(A)$, there exists a chain Y in $\mathrm{Spec}(B)$ such that Y covers X. As partial motivation for this definition, we make two observations: each chain Y of $\mathrm{Spec}(B)$ has a subchain that covers the chain $\{P : P = f^{-1}(Q)$ for some $Q \in Y\}$ of $\mathrm{Spec}(A)$; and, by focusing on singleton chains, we see that any chain morphism satisfies LO.

Proposition 2.1 collects some easy but useful facts, and Proposition 2.2 gives examples of chain morphisms that are essentially already known.

PROPOSITION 2.1. *Let $f\colon A \to B$ be a ring homomorphism. Then:*

(a) If a local chain Y in $\mathrm{Spec}(B)$ *covers a subset X of* $\mathrm{Spec}(A)$*, then X is a local chain.*

(b) If a chain Y in $\mathrm{Spec}(B)$ *covers a local chain X in* $\mathrm{Spec}(A)$*, then Y is a local chain and $f^{-1}(U(Y)) = U(X)$.*

(c) If f is a chain morphism and X is a local chain in $\mathrm{Spec}(A)$*, then X is covered by some local chain Y in* $\mathrm{Spec}(B)$ *and $f^{-1}(U(Y)) = U(X)$.*

Proof. (a) By the above observation, X is a chain. If $P \in X$, there exists $Q \in Y$ such that $f^{-1}(Q) = P$, whence $P \subseteq f^{-1}(U(Y)) \in X$. It follows that $U(X) = f^{-1}(U(Y)) \in X$, and so X is a local chain.

(b) Choose $Q \in Y$ such that $f^{-1}(Q) = U(X)$. If $Q \subset Q_1 \in Y$, then $f^{-1}(Q) \subset f^{-1}(Q_1)$, contradicting the fact that $f^{-1}(Q) = U(X) \supseteq f^{-1}(Q_1)$. Thus, $U(Y) = Q \in Y$, and so Y is a local chain. Then $f^{-1}(U(Y)) = U(X)$ by the proof of (a).

(c) Apply (b). □

PROPOSITION 2.2. *(Dobbs [3]) Let $f\colon A \to B$ be a ring homomorphism. Then f is a chain morphism in each of the following four cases:*

(i) f is injective and integral;

(ii) f satisfies LO and GU, and each chain in $\mathrm{Spec}(A)$ *is well-ordered via inclusion;*

(iii) f satisfies LO and GD, and each chain in $\mathrm{Spec}(A)$ *is well-ordered via reverse inclusion;*

(iv) A is Noetherian, and f satisfies LO and either GU or GD.

Proof. (i) was proved in [3, Remark(d)]; (ii) and (iii) follow from what was proved in [3, Theorem] and [3, Remark(a)], respectively, as those proofs, although given for injective f, carry over to the general case; and (iv) follows from (ii) and (iii), since A Noetherian ensures that each chain in $\mathrm{Spec}(A)$ is finite (hence, well-ordered with respect to both inclusion and reverse inclusion). □

As noted above, each chain morphism satisfies LO. Partial converses were given in Proposition 2.2 (ii)-(iv). Before deriving additional partial converses (in Theorems 2.3 and 2.8), we interpret topologically some conditions appearing in those results. Let $f\colon A \to B$ be a ring homomorphism and let $P \in \mathrm{Spec}(A)$. It is well known that ${}^a f^{-1}(P)$, the so-called *topological fiber of P (with respect to f)*, is homeomorphic to $\mathrm{Spec}((A_P/PA_P) \otimes_A B)$ in both the Zariski topology and the flat topology. One calls $(A_P/PA_P) \otimes_A B \cong B_P/PB_P$ *the fiber of f at P*; its associated reduced ring, $B_P/\sqrt{PB_P}$, is called the *reduced fiber (of f at P)*. It is easy to show, via Zorn's Lemma and [10, Theorem 9], that each element of ${}^a f^{-1}(P)$ is contained in some maximal element of ${}^a f^{-1}(P)$ and contains some minimal element of ${}^a f^{-1}(P)$. It follows that ${}^a f^{-1}(P)$ has a unique maximal (resp., unique minimal) element if and only if the reduced fiber of f at P is a quasilocal ring (resp., an integral domain);

that is (cf. [15, Lemme 2.5]), if and only if ${}^a f^{-1}(P)$ is irreducible in the flat (resp., Zariski) topology.

THEOREM 2.3. *Let $f\colon A \to B$ be a ring homomorphism that satisfies at least one of the following two conditions:*

(i) f satisfies GU and each reduced fiber of f is quasilocal;

(ii) f satisfies GD and each reduced fiber of f is an integral domain.

Then:

(a) For each chain $X \subseteq \mathrm{Im}({}^a f)$, there exists a chain Y in $\mathrm{Spec}(B)$ such that Y covers X.

(b) If, in addition, f satisfies LO, then f is a chain morphism.

Proof. It suffices to establish (a). Assume (i) (resp., (ii)). Consider any chain $X = \{P_i : i \in I\}$. By the above comments, we can choose Q_i to be the unique maximal (resp., unique minimal) element of ${}^a f^{-1}(P_i)$. Evidently, $Y := \{Q_i : i \in I\}$ covers X. It remains only to verify that Y is a chain. In fact, if $P_i \subset P_j$, then if follows from GU (resp., GD) and the maximality of Q_j (resp., minimality of Q_i) that $Q_i \subset Q_j$. □

We pause to note additional topological interpretations for some conditions in the statement of Theorem 2.3. Let $f\colon A \to B$ be a ring homomorphism. Then ${}^a f$ is closed in the Zariski (resp., flat) topology if and only if f satisfies GU (resp., GD) [4, Proposition 2.7]. It now seems natural to ask for “open” analogues of the “closed” assertions in Theorem 2.3. We provide such analogues in Theorem 2.8, which is really just a corollary of Theorem 2.3. In order to give an alternate approach to Theorem 2.8, we first develop some topological results. We also take advantage of this opportunity to introduce some deeper results on chains that will be useful in Section 3.

PROPOSITION 2.4. *Let A be a ring and let X be a subset of* $\mathrm{Spec}(A)$. *Then:*

(a) If X is a chain, then its patch closure X^c is also a chain.

(b) X is a chain if and only if there exist a ring homomorphism $A \to V$ and a chain Y in $\mathrm{Spec}(V)$ such that V is a valuation domain and Y covers X.

(c) X is a local chain if and only if there exists a ring homomorphism $f\colon A \to V$ and a local chain Y in $\mathrm{Spec}(V)$ such that V is a valuation domain, Y covers X, and $f^{-1}(U(Y)) = U(X)$.

Proof. (b) The “if” assertion is clear. Conversely, suppose that X is a chain. As in the proof of [3, Remark(d)], there is no harm in replacing A with $A/R(X)$, and so we may suppose that A is an integral domain. The lifting result of Kang-Oh [9, Theorem] provides a valuation domain V containing A and a chain Y in $\mathrm{Spec}(V)$ that covers X.

(a) Let X, V and A be as in the proof of (b). Let f be the composite $A \twoheadrightarrow A/R(X) \hookrightarrow V$; put $Z := \mathrm{Im}({}^a f) \subseteq \mathrm{Spec}(A)$. By definition of the patch (constructible) topology, Z is patch-closed; that is, $Z^c = Z$. Moreover, Z is a chain since V is quasilocal treed. As $X \subseteq Z$, we have $X^c \subseteq Z^c = Z$. Then X^c, being a subset of a chain, is itself a chain.

(c) The "if" assertion follows from Proposition 2.1 (a). The "only if" assertion follows by combining (b) with Proposition 2.1 (b). $\square$

COROLLARY 2.5. *Let A be a ring, X a chain in* $\mathrm{Spec}(A)$, *and* $P \in \mathrm{Spec}(A)$ *such that* $\mathcal{U}(X) \subseteq P$. *Then there exist a ring homomorphism* $f\colon A \to V$, *a chain* Y *in* $\mathrm{Spec}(V)$, *and* $Q \in \mathrm{Spec}(V)$ *such that* V *is a valuation domain,* Y *covers* X, $f^{-1}(Q) = P$, *and* $\mathcal{U}(Y) \subseteq Q$.

Proof. Apply Proposition 2.4 (c) to the local chain $X \cup \{P\}$, to obtain a suitable local chain Z in $\mathrm{Spec}(V)$. It suffices to take $Q := \mathcal{U}(Z)$; and $Y := Z$ (resp., $Z\setminus\{Q\}$) if $P \in X$ (resp., $P \notin X$). $\square$

PROPOSITION 2.6. *Let A be a ring and X a chain in* $\mathrm{Spec}(A)$. *Then:*

(a) If X is a maximal chain in $\mathrm{Spec}(A)$, *then X is stable under unions and intersections and, moreover, X is a patch and a local chain.*

(b) There exists a maximal chain X' in $\mathrm{Spec}(A)$ *such that* $X \subseteq X'$. *For any such* X', *there exist* $P \in \mathrm{Spec}(A)$ *and a minimal valuation overring W of A/P such that* $P \subseteq \mathcal{R}(X)$ *and* $\mathrm{Im}(\mathrm{Spec}(W) \to \mathrm{Spec}(A)) = X'$.

(c) Let $f\colon A \to V$ *be a ring homomorphism such that V is a valuation domain and* $X \subseteq \mathrm{Im}({}^a f)$. *Then there exist* $P \in \mathrm{Spec}(A)$ *and a minimal valuation overring W of A/P such that* $X \subseteq \mathrm{Im}(\mathrm{Spec}(W) \to \mathrm{Spec}(A))$.

Proof. (a), (b): By the reasoning in [3, pp. 3888-3889], if X is any chain, then $X \cup \{\mathcal{R}(Z) : \phi \neq Z \subseteq X\}$ is a chain. It follows that any maximal chain is stable under intersections. By reasoning similarly with $X \cup \{\mathcal{U}(Z) : \phi \neq Z \subseteq X\}$, we see that any maximal chain is stable under unions. Of course, considering $X \cup \{\mathcal{U}(X)\}$ shows that any maximal chain is a local chain.

It follows easily via Zorn's Lemma that each chain X is contained in a maximal chain. Consider any maximal chain $X' \supseteq X$. By the proof of Proposition 2.4 (a), there exists a valuation overring V of $D := A/\mathcal{R}(X')$ so that the composite ring homomorphism $g\colon A \twoheadrightarrow D \hookrightarrow V$ satisfies $X' \subseteq \mathrm{Im}({}^a g)$. By Zorn's Lemma (cf. [6, p. 231]), V contains a minimal valuation overring W of D. If f denotes the composite $A \twoheadrightarrow D \hookrightarrow W$, then $\mathrm{Im}({}^a g) \subseteq \mathrm{Im}({}^a f)$ since $\mathrm{Spec}(V) \subseteq \mathrm{Spec}(W)$ (cf. [6, Theorem 26.1]). However, $\mathrm{Im}({}^a f)$ is a chain (since W is quasilocal treed), and so $X' = \mathrm{Im}({}^a f)$ by the maximality of X'. Then X' is a patch, by the definition of the patch (constructible) topology. Finally, note that $P := \mathcal{R}(X') \subseteq \mathcal{R}(X)$.

(c) Observe that $P := \ker(f) \in \mathrm{Spec}(A)$. Let k (resp., K) denote the quotient field of A/P (resp., of V). Then the canonical ring inclusion $A/P \hookrightarrow V$ extends to an inclusion of fields, $k \hookrightarrow K$. Since $V \cap k$ is a valuation overring of A/P, another application of Zorn's Lemma produces a minimal valuation overring W of A/P such that $W \subseteq V \cap k$. By hypothesis, $X \subseteq \mathrm{Im}(\mathrm{Spec}(V \cap k) \to \mathrm{Spec}(A))$. It remains only to note that $\mathrm{Spec}(V \cap k) \subseteq \mathrm{Spec}(W)$. $\square$

LEMMA 2.7. *Let $f\colon A \to B$ be a ring homomorphism such that ${}^a f$ is open in the flat topology and each reduced fiber of f is quasilocal. Then a subset X of $\mathrm{Im}({}^a f)$ is irreducible in the flat topology on* Spec*(A) if and only if ${}^a f^{-1}(X)$ is irreducible in the flat topology on* Spec*(B).*

Proof. By the above comments, ${}^a f^{-1}(P)$ is irreducible in the flat topology, for all $P \in \mathrm{Spec}(A)$. Hence, by [7, Proposition 2.1.14, p. 54], we need only verify that ${}^a f$ induces a map ${}^a f^{-1}(X) \to X$ that is continuous, surjective and open in the subspace topology induced by the flat topology. Both "continuous" and "surjective" are clear. As for "open", consider any (flat-)open set U in $\mathrm{Spec}(B)$, and observe that

$$ {}^a f(U \cap {}^a f^{-1}(X)) = {}^a f(U) \cap X. $$

Since the hypothesis ensures that ${}^a f(U)$ is (flat-) open in $\mathrm{Spec}(A)$, the assertion follows. □

THEOREM 2.8. *Let f be a ring homomorphism that satisfies at least one of the following two conditions:*

(i) ${}^a f$ is open in the flat topology and each reduced fiber of f is quasilocal;

(ii) ${}^a f$ is open in the Zariski topology and each reduced fiber of f is an integral domain.

Then:

(a) For every chain $X \subseteq \mathrm{Im}({}^a f)$, there exists a chain Y in $\mathrm{Spec}(B)$ such that Y covers X.

(b) If, in addition, f satisfies LO, then f is a chain morphism.

Proof. (b) is an immediate consequence of (a). As for (a), if (ii) holds, the assertion may be proved exactly as in Theorem 2.3 (ii), since Zariski-open ${}^a f$ entails going-down f [7]. A parallel proof is also available if (i) holds, since flat-open ${}^a f$ entails going-up f (that is, Zariski-closed f) [13, Remarque, p. 2252].

Alternate, more topological proofs are available for Theorem(s 2.3 and) 2.8. We illustrate such methods with another proof for case (i). As in the earlier proof, it suffices to show that if P_i and P_j are distinct elements of a chain in $\mathrm{Im}({}^a f)$ and if Q_i (resp., Q_j) is the maximal element in ${}^a f^{-1}(P_i)$ (resp., in ${}^a f^{-1}(P_j)$), then Q_i and Q_j are comparable under inclusion. As P_i and P_j are comparable and flat-closed sets are stable under generization [4, Lemma 2.1], it follows that $Z := \{P_i, P_j\}$ is irreducible in the flat topology (cf. also [15, Proposition 2.4]). Hence, by Lemma 2.7, $Y := {}^a f^{-1}(Z)$ is also irreducible in the flat topology. As Y is a patch (being the spectral image of $B_{P_i}/P_iB_{P_i} \times B_{P_j}/P_jB_{P_j}$), [15, Lemme 2.5] ensures that Y is directed via inclusion. Thus, Q_i and Q_j are each contained in some prime $Q \in Y$ such that ${}^a f(Q) \in \{P_i, P_j\}$. Without loss of generality, ${}^a f(Q) = P_i$, whence $Q \subseteq Q_i$ by choice of Q_i. Then $Q_i = Q \supseteq Q_j$. □

There are useful algebraic sufficient conditions for the "open" properties in the statement of Theorem 2.8. For instance, if a ring homomorphism f is integral (resp., flat) and of finite presentation, then ${}^a f$ is open in the flat (resp., Zariski) topology, by [13, Proposition 6] (resp., [7, Corollaire 3.9.4(i), p. 254]). We close the section by using this fact to give an application of Theorem 2.8.

COROLLARY 2.9. *Let $P_1, \ldots, P_m \in \mathbb{Z}[X_1, \ldots, X_n]$ be such that $(P_1, \ldots, P_m)$ is a prime ideal in $K[X_1, \ldots, X_n]$ for any field K. Suppose that A is a ring and $f\colon A \to B := A[X_1, \ldots, X_n]/(P_1, \ldots, P_m)$ is such that ${}^a f$ is open in the Zariski topology (for instance, take f to be flat). Then each chain in* $\mathrm{Im}({}^a f)$ *can be covered by a chain in* $\mathrm{Spec}(B)$.

Proof. The hypothesis ensures that each (reduced) fiber of f is an integral domain. Apply Theorem 2.8 (a), using condition (ii). □

A concrete illustration of Corollary 2.9 is provided by $n = 2, m = 1, P_1 = X_1^2 - X_2^3$.

3 GENERALIZED GOING-DOWN

We begin with the key definition of this paper. A ring homomorphism $f\colon A \to B$ is said to satisfy the *generalized going-down* property (GGD) if the following holds: for each local chain X in $\mathrm{Spec}(A)$ and each $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = U(X)$, there exists a local chain Y in $\mathrm{Spec}(B)$ such that $U(Y) = Q$ and Y covers X. Evidently, GGD $\Rightarrow$ GD. We next record the only instance of GGD that has appeared in the literature.

PROPOSITION 3.1. *(Dobbs [3, proof of Remark (a)]). Let A be a ring such that each chain in* $\mathrm{Spec}(A)$ *is well-ordered via reverse inclusion. Then a ring homomorphism $f\colon A \to B$ satisfies GGD if (and only if) f satisfies GD.*

In comparing Propositions 3.1 and 2.2 (iii), one suspects that the notions of GGD and chain morphism are closely related. Proposition 3.2 states some evident connections, with less evident connections in the subsequent results. Of course, the two concepts are logically independent: if f is an injective integral ring homomorphism that does not satisfy GD, then f is a chain morphism that does not satisfy GGD; and if S is a multiplicatively closed subset of a ring A such that S contains a nonunit of A, then the canonical map $A \to A_S$ satisfies GGD but (as it fails to have LO) is not a chain morphism.

PROPOSITION 3.2. *Let $f\colon A \to B$ be a ring homomorphism. Then:*

(a) If f satisfies LO and GGD, then f is a chain morphism.

(b) If ${}^a f$ is injective and f is a chain morphism, then f satisfies GGD.

PROPOSITION 3.3. *If B is a quasilocal treed ring and $f\colon A \to B$ is a chain morphism, then f satisfies GGD.*

Proof. Consider a local chain $X = \{P_i : i \in I\}$ in $\mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = U(X)$. Since f is a chain morphism, Proposition 2.1 (c) provides a local chain $Y = \{Q_i : i \in I\}$ in $\mathrm{Spec}(B)$ that covers X, with $f^{-1}(U(Y)) = U(X)$. Choose (the unique) $j \in I$ such that $P_j = U(X)$. Then $Q_j = U(Y)$. If $Q_j = Q$, then Y is the desired local chain Z in $\mathrm{Spec}(B)$ such that $U(Z) = Q$ and Z covers X. If $Q_j \subset Q$, then $Z := (Y \setminus \{Q_j\}) \cup \{Q\}$ suffices. Since B is quasilocal treed, there is only one remaining case, namely, $Q \subset Q_j$. For this case, it suffices to take $Z := \{Q_i \cap Q : i \in I\}$. □

COROLLARY 3.4. *Let B be a quasilocal treed ring. Let $f\colon A \to B$ be a ring homomorphism that satisfies either GD or both LO and GU. Then f satisfies GGD.*

Proof. If $P \in \mathrm{Im}({}^a f)$, then B quasilocal treed implies that ${}^a f^{-1}(P)$ has a unique maximal element and a unique minimal element; that is, each reduced fiber of f is a quasilocal integral domain. The conclusion therefore follows by combining Theorem 2.3 and the proof of Proposition 3.3. □

By reworking the proof of Proposition 3.3, we next find two companion results. Just as Corollary 3.4 issued from combining Proposition 3.3 with Theorem 2.3, one can produce additional applications by combining part (a) or part (b) of Corollary 3.5 with Theorem 2.3. We leave such formulations to the reader.

COROLLARY 3.5. *Let $f\colon A \to B$ be a chain morphism that satisfies at least one of the following two conditions:*

(i) B is treed and each reduced fiber of f is quasilocal;

(ii) Each (Zariski-) irreducible component of $\mathrm{Spec}(B)$ *is a chain (via inclusion) and each reduced fiber of f is an integral domain.*

Then f satisfies GGD.

Proof. We proceed to rework the proof of Proposition 3.3. It suffices to verify that Q_j and Q are comparable via inclusion. In case (i), this follows since B is treed and Q_j, Q are each contained in (any maximal ideal of B that contains) the unique maximal element of ${}^a f^{-1}(P_j)$. An essentially “dual” proof is available if (ii) holds. Indeed, Q_j, Q each contain the unique minimal element I of ${}^a f^{-1}(P_j)$. Using Zorn’s Lemma, choose a minimal prime ideal N of B such that $N \subseteq I$ [10, Theorem 10]. Then Q_j, Q are each in the (Zariski-) irreducible set $V(N)$, which is a chain by hypothesis, whence Q_j and Q are comparable. □

COROLLARY 3.6. *Let $f\colon A \to B$ be a ring homomorphism such that each reduced fiber of f is an integral domain and f satisfies GD. Then f satisfies GGD.*

Proof. Once again, we rework the proof of Proposition 3.3. Even if f is not a chain morphism, the requisite chain $Y = \{Q_i\}$ is provided by Theorem 2.3 (with emphasis on its condition (ii)). By taking Q_i to be the unique minimal element of ${}^a f^{-1}(P_i)$, we are assured that $Q_j \subseteq Q$, and so $Z := (Y \setminus \{Q_j\}) \cup \{Q\}$ suffices. □

REMARK 3.7. (a) Recall from [4, pp. 567-568] that there is a "weak going down" concept that can be used to characterize the flat topology. In a different vein, we can also use an ostensibly "weaker" property to characterize GGD. Indeed, it is not difficult to show that a ring homomorphism $f: A \to B$ satisfies GGD if and only if the following holds: for each chain X in $\mathrm{Spec}(A)$, each $P \in \mathrm{Spec}(A)$ such that $\mathcal{U}(X) \subseteq P$, and each $Q \in {}^a f^{-1}(P)$, there exists a chain Y in $\mathrm{Spec}(B)$ such that $\mathcal{U}(Y) \subseteq Q$ and Y covers X.

(b) On the other hand, a related property that is ostensibly "stronger" than GGD may actually be stronger. For instance, consider the following property, say (*), that a ring homomorphism $f: A \to B$ can satisfy: for each chain X in $\mathrm{Spec}(A)$, with $P := \mathcal{U}(X)$, and each $Q \in {}^a f^{-1}(P)$, there exists a chain Y in $\mathrm{Spec}(B)$ such that $\mathcal{U}(Y) = Q$ and Y covers X. It is straightforward to verify that property (*) implies GGD. However, unlike the situation in (a), the converse is false. In other words, GGD fails to imply property (*). To see this, take f to be an inclusion map $A \to B$, where B is a valuation domain with prime spectrum $0 = Q_0 \subset Q_1 \subset \ldots \subset Q_n \subset \ldots \subset Q' \subset Q$, such that B/Q' is a $K-$algebra for some field K, and define A to be the pullback $B \times_{B/Q'} K$. Put $P := Q \cap A (= Q' \cap A)$. Then, by a standard gluing argument, with $X := \mathrm{Spec}(A) \setminus \{P\}$, one checks that f fails to satisfy property (*), for the only chain Y in $\mathrm{Spec}(B)$ that covers X is $Y = \mathrm{Spec}(B) \setminus \{Q', Q\}$, with $\mathcal{U}(Y) = Q' \neq Q$.

Next, we collect some elementary but useful facts indicating that *GGD* behaves rather similarly to known behavior of GD.

PROPOSITION 3.8. *(a) Let $f: A \to B$ and $g: B \to C$ be ring homomorphisms. If f and g each satisfies GGD, so does $g \circ f$. If g satisfies LO and $g \circ f$ satisfies GGD, then f satisfies GGD.*
(b) If f is a ring homomorphism, then the following seven conditions are equivalent:

(1) f satisfies GGD;
(2) $f_S: A_S \to B_S := B \otimes_A A_S$ satisfies GGD for each multiplicatively closed subset S of A;
(3) $f_P: A_P \to B_P := B \otimes_A A_P$ satisfies GGD for each $P \in \mathrm{Spec}(A)$;
(4) $A_P \to B_Q$ satisfies GGD for each $Q \in \mathrm{Spec}(B)$ and $P := f^{-1}(Q)$;
(5) $A/I \to B/IB$ satisfies GGD for each ideal I of A;
(6) $A/P \to B/PB$ satisfies GGD for each minimal prime ideal P of A;
(7) f_{red} satisfies GGD.

(c) Let $f_i: A_i \to B_i\, (i = 1, \ldots, n)$ be finitely many ring homomorphisms. Then the induced map $A_1 \times \cdots \times A_n \to B_1 \times \cdots \times B_n$ satisfies GGD if and only if f_i satisfies GGD for each i. If $A_1 = \ldots = A_n =: A$, then the induced map $A \to B_1 \times \cdots \times B_n$ satisfies GGD if and only if f_i satisfies GGD for each i.

A principal theme of this section is that the classic sources of going-down homomorphisms (namely, going-down domains and flat maps) give rise to GGD behavior. We pursue this point somewhat more generally in Theorems 3.9 and 3.16 after giving some background material and applications.

Recall from [1] and [5] that an integral domain A is called a *going-down domain* if $A \subseteq B$ satisfies GD for each overring B of A. The most natural examples of going-down domains are arbitrary valuation domains and the integral domains of (Krull) dimension at most 1. As in [2], a ring A is called a *going-down ring* if A/P is a going-down domain for each (equivalently, each minimal) prime ideal P of A. Any integral domain is a going-down ring if and only if it is a going-down domain [2, Remark (a), p. 4]; any ring of dimension at most 1 is a going-down ring [2, Proposition 2.1 (c)]; a finite ring product $A_1 \times \cdots \times A_n$ is a going-down ring if and only if each A_i is a going-down ring [2, Proposition 2.1 (b)]; but there exists a going-down ring A and an overring B of A such that $A \subseteq B$ does not satisfy GD [2, Example 1, p. 9]. Adapting terminology from [12], we say that a ring homomorphism $f \colon A \to B$ is a *min morphism* if $f^{-1}(Q)$ is a minimal prime ideal of A for each minimal prime ideal Q of B. It is evident that if a ring homomorphism f satisfies GD, then f is a min morphism. In the theory of Krull domains, an example of min morphisms is proved by the classical condition of *pas d'éclatement*, PDE (also known as *no blowing up*, NBU). Finally, recall that a ring A is said to be *locally irreducible* if each maximal ideal of A contains a unique minimal prime ideal of A.

THEOREM 3.9. *Let A be a locally irreducible ring and a going-down ring and let $f \colon A \to B$ be a ring homomorphism. Then the following conditions are equivalent:*

(1) f is a min morphism;

(2) f satisfies GD;

(3) f satisfies GGD.

Proof. By the above comments, $(3) \Rightarrow (2) \Rightarrow (1)$. It remains to show that if f is a min morphism, X a local chain in $\mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = U(X)$, then there exists a local chain Y in $\mathrm{Spec}(B)$ such that $U(Y) = Q$ and Y covers X. By [10, Theorem 10], Q contains some minimal prime ideal J of B. Since f is a min morphism, $I := J \cap A$ is a minimal prime ideal of A. Of course, $I \subseteq Q \cap A = U(X)$ and so, since A is locally irreducible, I is the only minimal prime ideal of A that is contained in $U(X)$. As each $P \in X$ contains a minimal prime ideal of A, it follows that $I \subseteq P$, whence $I \subseteq R(X)$. There is no harm in replacing f with $A/I \hookrightarrow B/J$. Hence, without loss of generality, $A \subseteq B$ are integral domains and A is a going-down domain (cf. [2, Proposition 2.1 (b) and Remark (a), p. 4]). Choose a valuation overring (V,N) of B such that $N \cap B = Q$ (cf. [10, Theorem 56]). Of course, V is quasilocal and treed. Moreover, $A \subseteq V$ satisfies GD since A is a going-down domain. Hence, by Corollary 3.4, there exists a local chain $Z = \{Q_i\}$ in $\mathrm{Spec}(V)$ such that Z covers X. Then, by Proposition 2.1 (b), $Y := \{Q_i \cap B\}$ has the desired properties. □

Recall that $A \subseteq B$ need not satisfy GD when B is an overring of a going-down ring A [2, Example 1, p. 9]. By avoiding one feature of that example, we have the following pleasant consequence.

COROLLARY 3.10. *If $f\colon A \to B$ is an injective ring homomorphism such that A is a going-down ring and B has a unique minimal prime ideal, then f satisfies GGD.*

Proof. If P is a minimal prime ideal of A, then [10, Exercise 1, p. 41] ensures that $f^{-1}(Q) = P$ for some prime ideal Q of B. By [10, Theorem 10], we can take Q to be the unique minimal prime ideal Q_0 of B. Hence, P is uniquely determined as $f^{-1}(Q_0)$; that is, A has a unique minimal prime ideal and f is a min morphism. In particular, A is locally irreducible. An application of Theorem 3.9 completes the proof. $\square$

COROLLARY 3.11. *Let $f\colon A \to B$ be a ring homomorphism such that A is a going-down ring. Then f satisfies GGD if and only if f satisfies GD.*

Proof. The "only if" assertion is valid even without the hypothesis on A. Conversely, suppose that f satisfies GD. It follows that if $P \in \mathrm{Spec}(A)$, then the induced map $g\colon A/P \to B/PB$ is a min morphism (by the proof of [10, Exercise 37, p. 44]). As A/P is a going-down ring [2, Proposition 2.1 (b)], Theorem 3.9 yields that g satisfies GGD. By Proposition 3.8 (b), so does f. $\square$

Corollary 3.12 isolates the most important instance of Corollaries 3.10 and 3.11. This result was actually established in the proof of Theorem 3.9.

COROLLARY 3.12. *If $A \subseteq B$ are integral domains and A is a going-down domain, then $A \hookrightarrow B$ satisfies GGD.*

Corollary 3.13 will present a more concrete application of Theorem 3.9 in the context of rings with nontrivial zero-divisors. Recall that a ring A is called a *weak Baer ring* if, for each $a \in A$, the annihilator of A is generated by an idempotent; that is, $\{b \in A : ba = 0\} = Ae$ for some $e = e^2 \in A$. Among many known characterizations is the following: A is a weak Baer ring if and only if A is a (necessarily reduced) locally irreducible ring such that $tq(A)$ is von Neumann regular. An example of a weak Baer ring that is a going-down ring but not an integral domain is provided by any finite product $A_1 \times \cdots \times A_n$ where each A_i is a weak Baer ring and a going-down ring (for instance, a going-down domain) and $n \geq 2$; to see this, recall that the class of weak Baer rings (resp., going-down rings) is stable under arbitrary (resp., finite) products [12, p. 28] (resp., [2, Proposition 2.1 (b)]).

COROLLARY 3.13. *Let $f\colon A \to B$ be a ring homomorphism such that A is a weak Baer ring and a going-down ring. Then f satisfies GGD if and only if f is a min morphism.*

Proof. Apply Theorem 3.9. $\square$

COROLLARY 3.14. *(a) If A is a ring and B is an overring of A, then $A \hookrightarrow B$ is a min morphism.*
(b) Let A be a weak Baer ring. Then the following conditions are equivalent:

(1) $A \hookrightarrow B$ satisfies GD for each overring B of A;

(2) $A \hookrightarrow B$ satisfies GGD for each overring B of A;

(3) A is a going-down ring.

Proof. (a) Let B be an overring of A; that is, $A \subseteq B \subseteq T := tq(A)$. Let P be a minimal prime ideal of B. Then there exists a minimal prime ideal Q of T such that $Q \cap B = P$ (by [10, Exercise 1, p. 41 and Theorem 10]). As T is a ring of fractions of A, it follows that T is A-flat, so that $A \hookrightarrow T$ satisfies GD (cf. [10, Exercise 37, p. 44]), whence $P \cap A = Q \cap A$ is a minimal prime ideal of A, as desired.

(b) Since weak Baer rings are locally irreducible, (a) combines with Theorem 3.9 to yield that $(3) \Rightarrow (2)$. As $(2) \Rightarrow (1)$ trivially, it remains only to prove that $(1) \Rightarrow (3)$. Suppose (1). By [2, Proposition 2.1 (b)], it suffices to establish that if $P \in \text{Spec}(R)$, then A_P is a going-down ring. The hypothesis on A ensures that A_P is an integral domain, since A is reduced and locally irreducible. Therefore, by a characterization of going-down domains (cf. [1], [5]), it is enough to show that $A_P \hookrightarrow E$ satisfies GD for each overring E of A_P. Now, since T is von Neumann regular, [16, Proposition 1.4(2)] gives an identification $tq(A_P) \cong T_P$, whence $E = B_P$ for some suitable overring B of A. Then $A_P \hookrightarrow E$ inherits GD from $A \hookrightarrow B$, to complete the proof. □

Recall from [1, Proposition 3.2] and [5, Theorem 1] that in order to determine whether a given integral domain A is a going-down domain, it suffices to verify that GD is satisfied by all inclusions $A \hookrightarrow V$ for which V is a valuation domain. In this spirit, we next provide characterizations of the “universally chain morphism” and “universally GGD” properties. Theorems 3.26 and 3.16 establish that these properties are equivalent to “universally subtrusive” and “universally going-down,” respectively.

PROPOSITION 3.15. *Let $f: A \to B$ be a ring homomorphism. Then the following conditions are equivalent:*

(1) f is universally GGD (resp., is a universally chain morphism), in the sense that the induced map $D \to D \otimes_A B$ satisfies GGD (resp., is a chain morphism) for all ring homomorphisms $A \to D$;

(2) The induced map $V \to V \otimes_A B$ satisfies GGD (resp., is a chain morphism) for all ring homomorphisms $A \to V$ for which V is a valuation domain.

Proof. We treat the assertion about “universally GGD” first. Of course, $(1) \Rightarrow (2)$ trivially. Assume (2). It suffices to show that f satisfies GGD. (Indeed, given ring homomorphisms $A \to D \to V$, observe the canonical isomorphism $V \otimes_D (D \otimes_A B) \cong V \otimes_A B$.) Consider a local chain X in $\text{Spec}(A)$ and $Q \in \text{Spec}(B)$ such that $f^{-1}(Q) = U(X) =: P$. Our task is to produce a local chain Y in $\text{Spec}(B)$ such that $U(Y) = Q$ and Y covers X. By Proposition 2.4 (c), there exists a ring homomorphism $g\colon A \to V$ and a local chain W in $\operatorname{Spec}(V)$ such that V is a valuation domain, W covers X, and $g^{-1}(U(W)) = P$. Put $E := V \otimes_A B$. In the category of affine schemes, we have $\operatorname{Spec}(E) \cong \operatorname{Spec}(V) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B)$. Therefore, by a property of pullbacks of schemes [7, Corollaire 3.2.7.1(i), p. 235], there exists $J \in \operatorname{Spec}(E)$ such that J lies over $U(W)$ (in $\operatorname{Spec}(V)$) and J lies over Q (in $\operatorname{Spec}(B)$). Moreover, by hypothesis, the induced map $V \to E$ satisfies GGD. Therefore, there exists a local chain Z in $\operatorname{Spec}(E)$ such that $U(Z) = J$ and Z covers W. By applying ${}^a(B \to V \otimes_A B)$ to the elements of Z, we obtain the elements of a chain Y with the desired properties. To prove the assertion about a "universally chain morphism," adapt the above proof, replacing the appeal to Proposition 2.4 (c) with a citation of Proposition 2.4 (b). $\square$

THEOREM 3.16. *A ring homomorphism $f\colon A \to B$ is universally GGD if and only if f is universally going-down.*

Proof. As GGD $\Rightarrow$ GD, the "only if" assertion is immediate. For the converse, suppose that f is universally going-down. Consider any ring homomorphism $A \to V$ for which V is a valuation domain. By the hypothesis on f, the induced map $h\colon V \to V \otimes_A B$ satisfies GD. Since V is a going-down ring, it follows from Corollary 3.11 (also from Theorem 3.9) that h satisfies GGD. An application of Proposition 3.14 yields that f is universally GGD, as desired. $\square$

As noted prior to Proposition 3.2, the structure map of any ring of fractions $A \to A_S$ satisfies GGD. We next obtain a substantial generalization of this fact.

COROLLARY 3.17. *Each flat ring homomorphism satisfies (universally) GGD.*

Proof. Each flat ring homomorphism is universally going-down (cf. [10, Exercise 37, p. 44]). Apply Theorem 3.16. $\square$

Proposition 3.18 will present another class of ring homomorphisms satisfying GGD. First, we adapt some terminology introduced in [11, p. 123]. A ring homomorphism $f\colon A \to B$ is called a *prime morphism* if the following condition is satisfied: if $f(a)b \in PB$ where $a \in A, b \in B$ and $P \in \operatorname{Spec}(A)$, then either $a \in P$ or $b \in PB$; equivalently, if B/PB is a torsion-free A/P-module for each $P \in \operatorname{Spec}(A)$. In general,

$$f \text{ is a flat ring-homomorphism} \Rightarrow f \text{ is a prime morphism } \Rightarrow f \text{ satisfies GD}$$

(cf. [11, Proposition 2], [10, Exercise 37, p. 44]). An interesting example is provided by $g\colon A \to A[T]/(pT) =: B$, where p is a prime integer, $A := \mathbb{Z}/p^2\mathbb{Z}$ and T is an indeterminate. Indeed, g is not flat (since $(p+p^2\mathbb{Z}) \otimes (T+pT)$ is a nonzero element of the kernel of $pA \otimes_A B \to B$), g is a prime morphism, and g satisfies GGD by Theorem 3.9. Finally, recall from [7, p. 145] that a *normal ring* is, by definition, a ring A such that A_P is an integrally closed integral domain for each $P \in \operatorname{Spec}(A)$.

PROPOSITION 3.18. *If A is a normal ring and a prime morphism $f\colon A \to B$ is integral, then f satisfies GGD.*

Proof. By Proposition 3.8 (b), it suffices to show that if P is any minimal prime ideal of A, then the induced map $g\colon A/P \to B/PB$ satisfies GGD. Observe that A/P is an integrally closed integral domain since A is a normal ring [7, p. 145]; B/PB is a torsion-free A/P-module since f is a prime morphism; and g is integral. Accordingly, by Seydi's generalization of the classical Going-down Theorem [19], g is universally (Zariski-) open. It follows that g is universally going-down [7, Corollaire 3.9.4 (i), p. 254] and, hence, by Theorem 3.16, that g is (universally) GGD. $\square$

We say that a ring homomorphism $f\colon A \to B$ is *prime-producing* if, for each $P \in \mathrm{Spec}(A)$, either $PB \in \mathrm{Spec}(B)$ or $PB = B$. Examples of prime-producing maps f include the structure maps of arbitrary rings of fractions $A \to A_S$ and the weak content maps of Rush [17]. It is evident that if a prime-producing map f satisfies LO, then f is a prime morphism and, hence, satisfies GD. A generalization of this fact will be given in Proposition 3.19. First, it is convenient to say that a ring homomorphism $f\colon A \to B$ satisfies the CNI property (so dubbed because it is a sort of "dual" of the INC property) if the following condition is satisfied: whenever $P \subseteq Q$ are prime ideals of A such that $PB = QB \neq B$, then $P = Q$. It is clear that if f satisfies LO, then f satisfies CNI (for then $f^{-1}(\mathfrak{p}B) = \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}(A)$).

PROPOSITION 3.19. *If a ring homomorphism $f\colon A \to B$ is prime-producing and satisfies CNI, then f satisfies GGD.*

Proof. Consider a local chain $X = \{P_i : i \in I\}$ in $\mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = U(X) = P_j$. We seek a local chain Y in $\mathrm{Spec}(B)$ such that $U(Y) = Q$ and Y covers X. Now, for each $i \in I$, we have $P_iB \subseteq P_jB \subseteq Q \subset B$. Hence, $P_iB \in \mathrm{Spec}(B)$, since f is prime-producing. Moreover, the CNI property ensures that P_i coincides with $Q_i := f^{-1}(P_iB)$, since $P_i \subseteq Q_i$ and $P_iB = Q_iB$. It therefore suffices to take $Y := \{P_iB : i \in I, i \neq j\} \cup \{Q\}$. $\square$

Proposition 3.2 (b) illustrated that GGD-theoretic consequences can ensue in the presence of a ring homomorphism f for which ${}^af$ is injective. We next pursue this theme by enhancing the set-theoretic restriction with a topological one. Specifically, we say that a continuous function $f\colon X \to Y$ of topological spaces is a *topological immersion* if the induced map $X \to f(X)$ is a homeomorphism (that is, injective and either open or closed). It is straightforward to verify that a continuous map $f\colon X \to Y$ is a topological immersion if and only if f is injective and $f^{-1}(\overline{f(Z)}) = \overline{Z}$ for each subset Z of X. Our main interest here concerns ring homomorphisms $f\colon A \to B$ for which ${}^af\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a topological immersion (relative to the Zariski topology); in such a case, we also call f a *topological immersion*. There are many ring-theoretic characterizations of such f. A particularly useful characterization is given next.

PROPOSITION 3.20. *Let* $f\colon A \to B$ *be a ring homomorphism. Then:*
(a) The following two conditions are equivalent:

(1) If Q_1 *and* Q_2 *are prime ideals of* B *such that* $f^{-1}(Q_1) \subseteq f^{-1}(Q_2)$, *then* $Q_1 \subseteq Q_2$;

(2) f *is a topological immersion.*

(b) Suppose that the equivalent conditions in (a) hold and that a subset Y *of* $\mathrm{Spec}(B)$ *covers a subset* X *of* $\mathrm{Spec}(A)$. *Then* Y *is a chain (resp., local chain) if and only if* X *is a chain (resp., local chain).*

Proof. (a) $(2) \Rightarrow (1)$: Consider $Q_1, Q_2 \in \mathrm{Spec}(B)$, with $f^{-1}(Q_1) \subseteq f^{-1}(Q_2)$. Then, by the definition of the Zariski topology and the above characterization of topological immersions, we have

$$Q_2 \in {}^a f^{-1}({}^a f(Q_2)) \subseteq {}^a f^{-1}(\overline{{}^a f(Q_1)}) = \overline{Q_1};$$

that is, $Q_1 \subseteq Q_2$.

$(1) \Rightarrow (2)$: Assume (1). If ${}^a f(Q_1) = {}^a f(Q_2)$, then (1) yields that $Q_1 \subseteq Q_2$ and $Q_2 \subseteq Q_1$. Therefore, ${}^a f$ is injective. It remains to prove that if F is a (Zariski-) closed subset of $\mathrm{Spec}(B)$, then $G := {}^a f(F)$ is (Zariski-) closed in $\mathrm{Im}({}^a f)$. We shall show, in fact, that $G = \overline{G} \cap \mathrm{Im}({}^a f)$. One conclusion is obvious. For the reverse inclusion, consider $P \in \overline{G} \cap \mathrm{Im}({}^a f)$; pick $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = P$. Now, observe that F is a patch (since $\mathrm{Im}(\mathrm{Spec}(B/J) \to \mathrm{Spec}(B)) = V(J)$ for each ideal J of B), and hence so is its spectral image, G. Thus, $\overline{G}$ is the union of the specializations of the points of G [7, Corollaire 7.3.2, p. 339]. In particular, $\mathfrak{p} \subseteq P$ for some $\mathfrak{p} \in G$. Pick $\mathfrak{q} \in F$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Using (1), we infer that $\mathfrak{q} \subseteq Q$, whence $Q \in F$, since Zariski-closed sets are stable under specialization. Therefore, $P = {}^a f(Q) \in {}^a f(F) = G$, as desired.

(b) In view of Proposition 2.1 (a),(b), it remains only to show that if $X =: \{P_i\}$ is a chain, then so is $Y =: \{Q_i\}$. As $f^{-1}(Q_i) = P_i$ for each i, the conclusion follows from condition (1) in (a). $\square$

We next mention two families of examples of ring homomorphisms that induce/are topological immersions; the verifications follow most readily by checking condition (1) in Proposition 3.20. The first family consists of the flat epimorphisms (that is, the flat maps $A \to B$ such that the induced multiplication map $B \otimes_A B \to B$ is an isomorphism). In particular, the structure map of any ring of fractions $A \to A_S$ is a topological immersion. The second family consists of the ring homomorphisms $f\colon A \to B$ with the following property: for each $b \in B$, there exists $a \in A$ and $u \in U(B)$ such that $b = f(a)u$. Besides rings of fractions, this second family includes all surjective ring homomorphisms and, for each field k and analytic indeterminate T, the inclusion map $k[T] \hookrightarrow k[[T]]$.

COROLLARY 3.21. *Let* $f\colon A \to B$ *be a ring homomorphism. Then the following conditions are equivalent:*

(1) ${}^a f$ *is injective and* f *satisfies GD;*

(2) f is a topological immersion and satisfies GGD.

Proof. $(2) \Rightarrow (1)$ trivially. Conversely, assume (1). One then readily verifies condition (1) in Proposition 3.20, and so f is a topological immersion. Next, to verify that f satisfies GGD, consider a local chain $X = \{P_i\}$ in $\mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = U(X)$. For each i, take Q_i to be the unique element of ${}^a f^{-1}(P_i)$. It follows from (1) that $P_i \subseteq P_j$ entails $Q_i \subseteq Q_j$. Accordingly, $Y := \{Q_i\}$ is a local chain in $\mathrm{Spec}(B)$ such that $U(Y) = Q$ and Y covers X, as desired. $\square$

COROLLARY 3.22. *Let $f\colon A \to B$ be a ring homomorphism such that ${}^a f$ is a topological immersion with closed image. Then the induced inclusion of rings $A/\ker(f) \hookrightarrow B$ satisfies GGD.*

Proof. Put $I := \ker(f)$. We begin with a fact that depends only on f being a ring homomorphism, namely, that $\overline{\mathrm{Im}({}^a f)} = V(I)$. (To fashion a proof, recall that minimal prime ideals of a base ring are lain over from any ring extension [10, Exercise 1, p. 41] and Zariski-closed sets are stable under specialization.) Under the given assumptions, it follows that $\mathrm{Im}({}^a f) = V(I)$.

Our task is to show that if X is a local chain in $\mathrm{Spec}(A/I)$ and $Q \in \mathrm{Spec}(B)$ lies over $U(X)$, then there exists a local chain Y in $\mathrm{Spec}(B)$ such that $U(Y) = Q$ and Y covers X. Of course, X induces a local chain Z in $\mathrm{Spec}(A)$ such that $Z \subseteq V(I)$ and Q lies over $U(Z)$. We shall show that $Y := {}^a f^{-1}(Z)$ has the asserted properties. Indeed, since ${}^a f$ is a topological immersion, it follows via condition (1) in Proposition 3.20 that Y is a chain. Moreover, Y is a local chain, with $U(Y) = Q$. Now, ${}^a f(Y) = Z \cap \mathrm{Im}({}^a f) = Z \cap V(I) = Z$. Finally, Y covers X since $\mathrm{Spec}(B) \to \mathrm{Spec}(A/I)$ is an injection. $\square$

COROLLARY 3.23. *Let f be a ring homomorphism. Then:*

(a) If f is an injection and ${}^a f$ is a topological immersion with closed image, then f satisfies GGD.

(b) If f is an injection satisfying GU and ${}^a f$ is an injection, then f satisfies GGD.

(c) Suppose that for all $Q \in \mathrm{Spec}(B)$ and $P := f^{-1}(Q)$, the induced map $A_P \to B_Q$ is an injection whose corresponding map $\mathrm{Spec}(B_Q) \to \mathrm{Spec}(A_P)$ is a topological immersion with closed image. Then f satisfies GGD.

Proof. (a) is immediate from Corollary 3.22; (b) admits a simple direct proof but can also be obtained as a corollary of (a); to prove (c), combine (a) and Proposition 3.8 (b). $\square$

For applications of the next result, it is useful to have examples of ring homomorphisms $g\colon A \to D$ that are universally topological immersions. Among these, we mention flat epimorphic g, surjective g, and g such that ${}^a g$ is a universal homeomorphism.

COROLLARY 3.24. *Let* $f\colon A \to B$ *be a ring homomorphism such that* ${}^a f$ *is injective and* f *satisfies GD. Let* $g\colon A \to D$ *be a ring homomorphism that is universally a topological immersion. Then the induced ring homomorphism* $h\colon D \to D \otimes_A B$ *satisfies GGD.*

Proof. Put $E := D \otimes_A B$. Our task is to show that if X is a local chain in $\mathrm{Spec}(D)$ and $Q \in \mathrm{Spec}(E)$ satisfies $h^{-1}(Q) = U(X)$, then there exists a local chain Y in $\mathrm{Spec}(E)$ such that $U(Y) = Q$ and Y covers X. As ${}^a g$ is injective, it follows from Proposition 2.1 (a),(b) that $W := {}^a g(X)$ is a local chain in $\mathrm{Spec}(A)$ such that $g^{-1}(U(X)) = U(W)$. Now, since Corollary 3.21 ensures that f satisfies GGD, there exists a local chain Z in $\mathrm{Spec}(B)$ such that $f^{-1}(U(Z)) = U(W)$ and Z covers W. Next, since X and Z have the same index set, we can use a result on pullbacks of schemes [7, Corollaire 3.2.7.1(i), p. 235] to produce the individual elements of a subset Y of $\mathrm{Spec}(E)$ such that Y covers X (relative to h) and Y covers Z (relative to the canonical ring homomorphism $j\colon B \to E$). As the hypothesis on g ensures that j is a topological immersion, Proposition 3.20 (b) yields that Y is a local chain. Finally, we shall show that $U(Y) = Q$. By Proposition 2.1 (b), $j^{-1}(U(Y)) = U(Z)$. Therefore,

$$
\begin{aligned}
&{}^a(j \circ f)(U(Y)) = f^{-1}(j^{-1}(U(Y))) = f^{-1}(U(Z)) = \\
&U(W) = g^{-1}(U(X)) = g^{-1}(h^{-1}(Q)) = {}^a(h \circ g)(Q) = {}^a(j \circ f)(Q).
\end{aligned}
$$

Since ${}^a(j \circ f) = {}^a f \circ {}^a j$ is a composite of injections, $U(Y) = Q$. $\square$

By analogy with the earlier definition of "chain morphism", we say that a ring homomorphism $f\colon A \to B$ is a *2-chain morphism* (or, as in [14, p. 528], *subtrusive*) if the following condition is satisfied: for all prime ideals $P_1 \subseteq P_2$ of A, there exist prime ideals $Q_1 \subseteq Q_2$ of B such that $f^{-1}(Q_i) = P_i$ for $i = 1, 2$. It is easy to see that any ring homomorphism f that satisfies LO and either GU or GD must be a 2-chain morphism. As noted in [14, p. 538], examples of universally 2-chain morphisms include the ring homomorphisms f that are pure; the f that satisfy LO and are universally going-down; and the f that satisfy LO and are integral. For us, the most important examples of universally 2-chain morphisms are special cases of the last two classes just mentioned, namely, the faithfully flat ring homomorphisms and (thanks to a result on pullbacks of schemes [7, Corollaire 3.2.7.1(i), p. 235] and the Lying-over Theorem [10, Theorem 44]) the injective integral ring homomorphisms.

Before stating a useful characterization of universally 2-chain morphisms, we recall the following definitions. If $f\colon A \to B$ is a ring homomorphism, the *torsion ideal of* f is $T(f) := \{b \in B :$ there exists a non-zero-divisor $c \in A$ such that $cb = 0\}$; and f is called *torsion-free* if $T(f) = 0$.

PROPOSITION 3.25. *(Picavet [14, Théorème 37(a), p. 556 and Proposition 16, p. 543]) Let* $f\colon A \to B$ *be a ring homomorphism. Then the following conditions are equivalent:*

(1) If $A \to V$ is a ring homomorphism for which V is a valuation domain and the induced map $V \to V \otimes_A B =: E$ has torsion ideal T, then the induced ring homomorphism $V \to E/T$ is faithfully flat;

(2) f is a universally 2-chain morphism.

Observe that LO is a universal property (as can be seen via [7, Corollaire 3.2.7.1(i), p. 235]); and, of course, so is "integral". Accordingly, the proof of our motivating result Proposition 2.2 (i) in [3, Remark (d)] actually establishes that any integral ring homomorphism that satisfies LO (for instance, any injective integral map) must be a universally chain morphism. We next present a substantial generalization of this fact.

THEOREM 3.26. *A ring homomorphism $f: A \to B$ is a universally chain morphism if and only if f is a universally 2-chain morphism.*

Proof. Any chain morphism is a 2-chain morphism, and so the "only if" assertion is trivial. For the converse, it suffices to show that if f is a universally 2-chain morphism, then f is a chain morphism. Our task is to show that if X is a chain in $\mathrm{Spec}(A)$, then there exists a chain Y in $\mathrm{Spec}(B)$ such that Y covers X. By Proposition 2.4 (b), we find a valuation domain V and a ring homomorphism $g: A \to V$ such that some chain W in $\mathrm{Spec}(V)$ covers X. Put $E := V \otimes_A B$. By Proposition 3.25, the induced ring homomorphism $h: V \to E/T$ is faithfully flat, where T denotes the torsion ideal of the canonical map $V \to E$. Accordingly, by Corollary 3.17, h satisfies GGD; and, being faithfully flat, h also satisfies LO. Therefore, by Proposition 3.2 (a), h is a chain morphism. In particular, some chain Z in $\mathrm{Spec}(E/T)$ covers W. If j denotes the composite $B \to E \to E/T$, it follows from the fact that $h \circ g = j \circ f$ and the functoriality of Spec that $Y := {}^a j(Z)$ covers X, as desired. $\square$

COROLLARY 3.27. *Universally (2-) chain morphisms descend both GGD and GD. More precisely: if $f: A \to B$ is a ring homomorphism and $g: A \to D$ is a universally (2-) chain morphism such that the induced map $h: D \to D \otimes_A B =: E$ satisfies GGD (resp., GD), then f satisfies GGD (resp., GD).*

Proof. We give a proof for the "GGD" assertion, as it carries over for the "GD" assertion. Consider a local chain X in $\mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = U(X)$. Since g is a chain morphism, there exists a chain Z in $\mathrm{Spec}(D)$ such that Z covers X. By Proposition 2.1 (b), Z is a local chain and $g^{-1}(U(Z)) = U(X)$. As $U(Z)$ and Q each lie over $U(X)$, the oft-used fact about pullbacks of schemes [7, Corollaire 3.2.7.1(i), p. 235] supplies $J \in \mathrm{Spec}(E)$ such that J lies over $U(Z)$ in $\mathrm{Spec}(D)$ and J lies over Q in $\mathrm{Spec}(B)$. Since h satisfies GGD, there exists a local chain W in $\mathrm{Spec}(E)$ such that $U(W) = J$ and W covers Z. If j denotes the canonical ring homomorphism $B \to E$, then the chain $Y := {}^a j(W)$ covers X. Moreover, by Proposition 2.1 (b), Y is a local chain satisfying $Q = j^{-1}(J) = j^{-1}(U(W)) = U(Y)$. Therefore, f satisfies GGD. $\square$

COROLLARY 3.28. *Universally (2-) chain morphisms descend universally going-down (universally GGD).*

Proof. It follows from Corollary 3.27 via standard tensor product identities that any universally (2-) chain morphism descends universally GGD. An application of Theorem 3.16 permits the "universally going-down" formulation. □

COROLLARY 3.29. *Let $f: A \to B$ be a ring homomorphism, and let $a_1, \ldots a_n$ be finitely many elements of A such that $(a_1, \ldots, a_n) = A$. Then f satisfies GGD if and only if the induced ring homomorphism $f_i : A_{a_i} \to B_{a_i}$ satisfies GGD for all $i = 1, \ldots, n$.*

Proof. The "only if" assertion is immediate from Proposition 3.8 (b). For the converse, assume that each f_i satisfies GGD. By Proposition 3.8 (c), so does the induced map $\prod A_{a_i} \to \prod B_{a_i}$. Of course, $\prod B_{a_i} \cong (\prod A_{a_i}) \otimes_A B$; and $A \to \prod A_{a_i}$ is faithfully flat, hence a universally 2-chain morphism. Hence, by Corollary 3.27, f satisfies GGD. □

REMARK 3.30. In view of the diversity of contexts identified above which give sufficient conditions for GGD, one might well ask if any traditional construction can produce a ring A supporting a ring homomorphism $f: A \to B$ that satisfies GD but not GGD. In this regard, one could consider $A = C(X)$, the ring of continuous real-valued functions defined on a topological space X. However, such A cannot support f with the above properties. Indeed, any ring of the form $C(X)$ is a real closed ring, in the sense of Schwartz [18]. By [18, Propositions 1.4 and 1.5], it follows that any real closed ring is a locally irreducible ring and a going-down ring. Thus, if A is a real closed ring (for instance, a ring of the form $C(X)$) and a ring-homomorphism $f: A \to B$ satisfies GD, then each of Theorem 3.9, Corollary 3.10, Corollary 3.11 implies that f satisfies GGD.

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