Algebra Colloquium **12** : 4 (2005) 645–664

Algebra Colloquium

Semistar Invertibility on Integral Domains

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Received 10 July 2003
Communicated by Quanshui Wu

Abstract. A natural development of the recent work about semistar operations leads to investigate the concept of "semistar invertibility". Here, we show the existence of a "theoretical obstruction" for extending many results, proved for star-invertibility, to the semistar case. For this reason, we introduce two distinct notions of invertibility in the semistar setting (called $\star$-invertibility and quasi-$\star$-invertibility), we discuss the motivations of these "two levels" of invertibility and we extend, accordingly, many classical results proved for the d-, v-, t- and w-invertibility.

2000 Mathematics Subject Classification: 13A15, 13C13, 13F15, 13G05

Keywords: star operation, divisorial ideal, Nagata ring, invertible ideal, v-operation, content of a polynomial

1 Introduction and Background Results

The notions of t-invertibility, v-invertibility and w-invertibility, that generalize the classical concept of (d-)invertibility (these definitions will be recalled in Section 2), have been introduced in the recent years for a better understanding of the multiplicative (ideal) structure of integral domains. In particular, t-invertibility has a key role for extending the notion of class group from Krull domains to general integral domains (cf. [8–10] and the survey paper [7]). An interesting chart of a large set of various t-, v-, d-invertibility based characterizations of several classes of integral domains can be found at the end of [4]; some motivations for introducing the w-invertibility and the first properties showing the "good" behaviour of this notion can be found in [47] (cf. also [30]). The concept of star operation (or equivalently, ideal system, cf. the books by Jaffard [33], Gilmer [24] and Halter-Koch [27]) provides an abstract setting for approaching these different aspects of invertibility. A recent paper by Zafrullah [51] gives an excellent and updated survey of this point of view.

After the introduction in 1994, by Okabe and Matsuda [44], of the notion of semistar operation, as a more general and natural setting for studying multiplicative systems of ideals and modules, many authors have investigated the possible extensions to the semistar setting of different aspects of the classical theory of ideal systems, based on the pioneering work by Krull, Noether, Prüfer and Lorenzen from 1930's (cf. for instance [12, 13, 15–19, 28, 29, 37–43, 45]).

A natural development of this work leads to investigate the concept of invertibility in the semistar setting. This is the purpose of the present paper, in which we will show the existence of a "theoretical obstruction" for extending many results, proved for star-invertibility, to the semistar case. For this reason, we will be forced to introduce two distinct notions of invertibility in the semistar setting (called $\star$-invertibility and quasi-$\star$-invertibility; the explicit definitions will be given in Section 2). We will discuss the motivations of these "two levels" of invertibility and we will extend, accordingly, many classical results proved for the d-, v-, t- and w-invertibility.

Among the main properties proved in this work, we mention the following: (a) several characterizations of $\star$-invertibility and quasi-$\star$-invertibility and necessary and sufficient conditions for the equivalence of these two notions; (b) the relations between the $\star$-invertibility (or quasi-$\star$-invertibility) and the invertibility (or quasi-invertibility) with respect to the semistar operation of finite type (denoted by $\star_{_f}$) and to the stable semistar operation of finite type (denoted by $\widetilde{\star}$), canonically associated to $\star$ (in the case that $\star = v$ is the Artin v-operation, then $\star_{_f} = t$ and $\widetilde{\star} = w$); (c) a characterization of the H($\star$)-domains in terms of semistar-invertibility (note that the H($\star$)-domains generalize in the semistar setting the H-domains introduced by Glaz and Vasconcelos [26], more precisely, we will see in Section 2 that an H-domain coincides with an H(v)-domain); (d) for a semistar operation of finite type, a nonzero finitely generated (fractional) ideal I is $\star$-invertible (or equivalently, quasi-$\star$-invertible in the stable semistar case) if and only if its extension to the Nagata semistar ring $I\mathrm{Na}(D,\star)$ is an invertible ideal of $\mathrm{Na}(D,\star)$ (the definition of $\mathrm{Na}(D,\star)$ will be recalled at the end of this section).

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of K and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, that is, $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of K. Then obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

A map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$, is called a *semistar operation* on D if for all $x \in K$ with $x \neq 0$ and all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold (cf. for instance [13]):

($\star_1$) $(xE)^\star = xE^\star$;

($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Recall that [13, Theorem 1.2 and p. 174], for all $E, F \in \overline{\boldsymbol{F}}(D)$, we have:

$$\begin{aligned}
(EF)^\star &= (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star\,;\\
(E+F)^\star &= (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star\,,\ \text{ if } (E:F) \neq (0);\\
(E:F)^\star &\subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star\,;\\
(E \cap F)^\star &\subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star\,,\ \text{ if } E \cap F \neq (0).
\end{aligned}$$

When $D^\star = D$, we say that $\star$ is a *(semi)star operation* since, restricted to $\boldsymbol{F}(D)$, it is a star operation.

For star operations, it is very useful the notion of *$\star$-ideal*, that is, a nonzero ideal $I \subseteq D$ such that $I^\star = I$. For semistar operations, we need a more general notion, that coincides with the notion of $\star$-ideal when $\star$ is a (semi)star operation. We say that a nonzero (integral) ideal I of D is a *quasi-$\star$-ideal* if $I^\star \cap D = I$. For example, it is easy to see that if $I^\star \neq D^\star$, then $I^\star \cap D$ is a quasi-$\star$-ideal that contains I (in particular, a $\star$-ideal is a quasi-$\star$-ideal). Note that $I^\star \neq D^\star$ is equivalent to $I^\star \cap D \neq D$.

A quasi-$\star$-ideal which is also a prime ideal is called *quasi-$\star$-prime*. A maximal element in the set of all proper quasi-$\star$-ideals of D is called *quasi-$\star$-maximal*. We denote by $\mathrm{QSpec}^\star(D)$ (respectively, $\mathrm{QMax}^\star(D)$) the set of all quasi-$\star$-prime ideals (respectively, quasi-$\star$-maximal ideals).

If $\star$ is a semistar operation on D, then we can consider a map $\star_{_f} : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined for each $E \in \overline{\boldsymbol{F}}(D)$ as follows: $E^{\star_f} := \bigcup\{F^\star \mid F \in \boldsymbol{f}(D),\ F \subseteq E\}$. It is easy to see that $\star_{_f}$ is a semistar operation on D, called *the semistar operation of finite type associated to* $\star$. Note that for each $F \in \boldsymbol{f}(D)$, $F^\star = F^{\star_f}$. A semistar operation $\star$ is called a *semistar operation of finite type* if $\star = \star_{_f}$. It is easy to see that $(\star_{_f})_{_f} = \star_{_f}$ (i.e., $\star_{_f}$ is of finite type).

If $\star_1$ and $\star_2$ are two semistar operations on D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\boldsymbol{F}}(D)$. In this situation, it is easy to see $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$. Obviously, for each semistar operation $\star$, we have $\star_{_f} \leq \star$.

The following result, with a different terminology, was proved in [13] (cf. also [19, Lemma 2.3]).

Lemma 1.1. *Let $\star$ be a semistar operation on an integral domain D. Assume that $\star$ is non-trivial and $\star = \star_{_f}$. Then*:

(1) *Each proper quasi-$\star$-ideal is contained in a quasi-$\star$-maximal ideal.*
(2) *Each quasi-$\star$-maximal ideal is quasi-$\star$-prime.*
(3) *Set* $\Pi^\star := \{P \in \mathrm{Spec}(D) \mid P \neq 0,\ P^\star \cap D \neq D\}$, *then* $\mathrm{QSpec}^\star(D) \subseteq \Pi^\star$ *and the set of maximal elements* $\Pi^\star_{\max}$ *of* $\Pi^\star$ *is non-empty and coincides with* $\mathrm{QMax}^\star(D)$.

For the sake of simplicity, we will denote simply by $\mathcal{M}(\star)$ the set $\mathrm{QMax}^\star(D)$ of the quasi-$\star$-maximal ideals of D.

If $\Delta \subseteq \mathrm{Spec}(D)$, the map $\star_\Delta : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^{\star_\Delta} := \bigcap\{ED_P \mid P \in \Delta\}$, is a semistar operation. If $\star = \star_\Delta$ for some $\Delta \subseteq \mathrm{Spec}(D)$, we say that $\star$ is a *spectral semistar operation*. In particular, if $\Delta = \{P\}$, then $\star_{\{P\}}$ is the semistar operation on D defined by $E^{\star_{\{P\}}} := ED_P$ for each $E \in \overline{\boldsymbol{F}}(D)$. We say that a semistar operation is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for any $E, F \in \overline{\boldsymbol{F}}(D)$. A spectral semistar operation is stable [13, Lemma 4.1].

If $\star$ is a semistar operation on D, we denote by $\tilde{\star}$ the semistar operation $\star_{\mathcal{M}(\star_f)}$ induced by the set $\mathcal{M}(\star_{_f})$ of the quasi-$\star_{_f}$-maximal ideals of D. The semistar operation $\tilde{\star}$ is stable and of finite type and $\tilde{\star} \leq \star_{_f}$ (cf. [13, p. 181], where the semistar operation $\tilde{\star}$ is defined, in an equivalent way, by using localizing systems, and also [3, Section 2] for an analogous construction in the star setting). Note that when $\star = v$

(where, as usual, v denotes the (semi)star operation defined by $E^v := (D : (D : E))$ for each $E \in \overline{\boldsymbol{F}}(D)$), then $\tilde{\star}$ coincides with the (semi)star operation denoted by w by Wang and McCasland (cf. [47–49]).

The following lemma is not difficult to prove (cf. [19, Corollary 3.5(2)], and [3, Theorem 2.16] for the analogous result in the case of star operations).

Lemma 1.2. *Let $\star$ be a semistar operation on an integral domain D. Then $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.*

In the next proposition, we recall how a semistar operation on an integral domain D induces canonically a semistar operation on an overring T of D (cf. [44, Lemma 45], and [20] for the notation used here).

Proposition 1.3. *Let D be an integral domain and T an overring of D. Let $\iota : D \hookrightarrow T$ be the embedding of D in T, and let $\star_\iota : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$ be defined by $E^{\star_\iota} := E^\star$. Then:*
(1) *$\star_\iota$ is a semistar operation on T.*
(2) *If $\star$ is of finite type on D, then $\star_\iota$ is of finite type on T.*
(3) *If $T = D^\star$, then $\star_\iota$ is a (semi)star operation on $D^\star$.*
(4) *If $\star$ is stable, then $\star_\iota$ is stable.*

If R is a ring and X an indeterminate over R, then the ring

$$R(X) := \{f/g \mid f, g \in R[X],\ \boldsymbol{c}(g) = R\},$$

where $\boldsymbol{c}(g)$ is the content of the polynomial g, is called the *Nagata ring* of R [24, Proposition 33.1].

The following result is proved in [19, Proposition 3.1] (cf. also [34, Proposition 2.1]).

Proposition 1.4. *Let $\star$ be a non-trivial semistar operation on an integral domain D and set $N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0,\ (\boldsymbol{c}(h))^\star = D^\star\}$. Then:*
(1) *$N(\star)$ is a saturated multiplicative subset of $D[X]$ and*

$$N(\star) = N(\star_f) = D[X] \setminus \bigcup \{Q[X] \mid Q \in \mathcal{M}(\star_f)\}.$$

(2) $\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$ *and $\mathcal{M}(\star_f)$ coincides with the canonical image in* $\mathrm{Spec}(D)$ *of* $\mathrm{Max}\left((D[X])_{N(\star)}\right)$.
(3) $D[X]_{N(\star)} = \bigcap \{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.

We set $\mathrm{Na}(D, \star) := D[X]_{N(\star)}$ and we call it *the Nagata ring of D with respect to the semistar operation $\star$*. Obviously, $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star_f)$ and, when $\star = d$ (the identity (semi)star operation) on D, then $\mathrm{Na}(D, d) = D(X)$.

2 Semistar Invertibility

Let $\star$ be a semistar operation on an integral domain D. Let $I \in \boldsymbol{F}(D)$, we say that I is $\star$*-invertible* if $\left(II^{-1}\right)^\star = D^\star$. In particular, when $\star = d$ (respectively, v,

$t\,(:= v_f)$, $w\,(:= \widetilde{v})$) is the identity (semi)star operation (respectively, the v-operation, t-operation, w-operation), we reobtain the classical notion of *invertibility* (respectively, *v-invertibility, t-invertibility, w-invertibility*) of a fractional ideal.

Lemma 2.1. *Let $\star$, $\star_1$, $\star_2$ be semistar operations on an integral domain D. Let* $\mathrm{Inv}(D,\star)$ *be the set of all $\star$-invertible fractional ideals of D and* $\mathrm{Inv}(D)$ (*instead of* $\mathrm{Inv}(D,d)$) *the set of all invertible fractional ideals of D. Then*:

(0) $D \in \mathrm{Inv}(D,\star)$.

(1) *If* $\star_1 \leq \star_2$, *then* $\mathrm{Inv}(D,\star_1) \subseteq \mathrm{Inv}(D,\star_2)$. *In particular,* $\mathrm{Inv}(D) \subseteq \mathrm{Inv}(D,\tilde{\star}) \subseteq \mathrm{Inv}(D,\star_f) \subseteq \mathrm{Inv}(D,\star)$.

(2) $I, J \in \mathrm{Inv}(D,\star)$ *if and only if* $IJ \in \mathrm{Inv}(D,\star)$.

(3) *If* $I \in \mathrm{Inv}(D,\star)$, *then* $I^{-1} \in \mathrm{Inv}(D,\star)$.

(4) *If* $I \in \mathrm{Inv}(D,\star)$, *then* $I^v \in \mathrm{Inv}(D,\star)$.

Proof. (0) and (1) are obvious.

(2) Note that if $I, J \in \mathrm{Inv}(D,\star)$, then $D^\star = \left(II^{-1}\right)^\star \left(JJ^{-1}\right)^\star \subseteq \left(II^{-1}JJ^{-1}\right)^\star \subseteq \left(IJ(IJ)^{-1}\right)^\star \subseteq D^\star$. Thus, $IJ \in \mathrm{Inv}(D,\star)$. Conversely, if $IJ \in \mathrm{Inv}(D,\star)$, then $D^\star = ((IJ)(D:IJ))^\star = (I(J(D:IJ)))^\star$. Since $(J(D:IJ)) \subseteq (D:I)$, it follows that $(I(D:I))^\star = D^\star$. Similarly, $(J(D:J))^\star = D^\star$.

(3) $D^\star = \left(II^{-1}\right)^\star \subseteq \left((I^{-1})^{-1}I^{-1}\right)^\star \subseteq D^\star$.

(4) It follows from (3). □

Remark 2.2. (a) Note that D is the unit element of $\mathrm{Inv}(D,\star)$ with respect to the *standard multiplication* of fractional ideals of D. Nevertheless, $\mathrm{Inv}(D,\star)$ is *not* a group in general (under the standard multiplication) because for $I \in \mathrm{Inv}(D,\star)$, $I^{-1} \in \mathrm{Inv}(D,\star)$ but $II^{-1} \neq D$ if $I \notin \mathrm{Inv}(D)$. For instance, let k be a field, X and Y two indeterminates over k, and let $D := k[X,Y]_{(X,Y)}$. Then D is a local Krull domain with maximal ideal $M := (X,Y)D$. Let $\star = v$, then clearly $M^v = D$ since $\mathrm{ht}(M) = 2$, thus M is v-invertible but M is not invertible in D since it is not principal. Therefore, $(MM^{-1})^v = D$, but $M = MM^{-1} \subsetneq D$. We will discuss later what happens if we consider the semistar (fractional) ideals semistar invertible with the "semistar product".

(b) Let $I \in \boldsymbol{F}(D)$. Assume that $I \in \mathrm{Inv}(D,\star)$ and $(D^\star : I) \in \boldsymbol{F}(D)$, then we will see later that $(D^\star : I) = (D:I)^\star$ (Lemma 2.10, Remark 2.13(d1) and Proposition 2.16), more precisely,

$$\left(I^{-1}\right)^\star = (D:I^v)^\star = (D^\star : I)^\star = (D^\star : I) = (I^\star)^{-1}.$$

However, in this situation, we may not conclude that $(D^\star : I)$ (or $(D:I)^\star$) belongs to $\mathrm{Inv}(D,\star)$ (even if $(D:I) \in \mathrm{Inv}(D,\star)$ by Lemma 2.1(3)). As a matter of fact, more generally, if $J \in \mathrm{Inv}(D,\star)$ and $J^\star \in \boldsymbol{F}(D)$, then $J^\star$ does not belong necessarily to $\mathrm{Inv}(D,\star)$.

For instance, let K be a field and X, Y two indeterminates over K, set $T := K[X,Y]$ and $D := K + YK[X,Y]$. Let $\star_{\{T\}}$ be the semistar operation on D defined by $E^{\star_{\{T\}}} := ET$ for each $E \in \overline{\boldsymbol{F}}(D)$. Then $J := YD$ is obviously invertible (hence $\star_{\{T\}}$-invertible) in D and $J^{\star_{\{T\}}} = JT = YT = YK[X,Y] = (D:T)$ is a nonzero (maximal) ideal of D (and, at the same time, a (prime) ideal of T), but $J^{\star_{\{T\}}}$ is not $\star_{\{T\}}$-invertible in D because

$$\begin{aligned}&(J^{\star_{\{T\}}}(D : J^{\star_{\{T\}}}))^{\star_{\{T\}}}\\ &= (JT(D : JT))T = (YT(D : YT))T\\ &= (YTY^{-1}(D : T))T = (T(YT))T = YT \subsetneq T = D^{\star_{\{T\}}}.\end{aligned}$$

(c) Note that the converses of (3) and (4) of Lemma 2.1 are not true in general. For instance, take an integral domain D that is not an H-domain (recall that an *H-domain* is an integral domain D such that if I is an ideal of D with $I^{-1} = D$, then there exists a finitely generated $J \subseteq I$ such that $J^{-1} = D$ [26, Section 3]). Then there exists an ideal I of D such that $I^v = I^{-1} = D$ and $I^t \subsetneq D$. It follows that $\left(I^{-1}I^v\right)^t = D$ (and so I^{-1} and I^v are t-invertibles), but $\left(II^{-1}\right)^t = I^t \subsetneq D$, that is, I is not t-invertible. On the other hand, note that, trivially, I is v-invertible.

An explicit example is given by a 1-dimensional non-discrete valuation domain V with maximal ideal M. Clearly, V is not an H-domain [26, (3.2d)], $M^{-1} = M^v = V$ [24, Exercise 12, p. 431] and $M^t = \bigcup\{J^v \mid J \subseteq M,\ J \text{ finitely generated}\} = \bigcup\{J \mid J \subseteq M,\ J \text{ finitely generated}\} = M \subsetneq V$. In this case, M^{-1} and M^v are obviously t-invertibles, but M is not t-invertible.

If $I \in \overline{\boldsymbol{F}}(D)$, we say that I is *$\star$-finite* if there exists $J \in \boldsymbol{f}(D)$ such that $J^\star = I^\star$. It is immediate to see that if $\star_1 \leq \star_2$ are semistar operations and I is $\star_1$-finite, then I is $\star_2$-finite. In particular, if I is $\star_f$-finite, then it is $\star$-finite.

We notice that, in the previous definition of $\star$-finite, we do not require that $J \subseteq I$. The next result shows that, with this "extra" assumption, $\star$-finite is equivalent to $\star_f$-finite.

Lemma 2.3. *Let $\star$ be a semistar operation on an integral domain D with quotient field K. Let $I \in \overline{\boldsymbol{F}}(D)$. Then the following are equivalent*:

(i) *I is $\star_f$-finite.*

(ii) *There exists $J \subseteq I$, $J \in \boldsymbol{f}(D)$ such that $J^\star = I^\star$.*

Proof. It is clear that (ii) implies (i) since $J^\star = J^{\star_f}$ if J is finitely generated. On the other hand, suppose I is $\star_f$-finite. Then $I^{\star_f} = J_0^{\star_f}$ with $J_0 = (a_1, a_2, \dots, a_n)D$ for some family $\{a_1, a_2, \dots, a_n\} \subseteq K$. Since $J_0 \subseteq I^{\star_f}$, there exists a finite family of finitely generated fractional ideals of D, $J_1, J_2, \dots, J_n \subseteq I$, such that $a_i \in J_i^\star$ for $i = 1, 2, \dots, n$. It follows that

$$I^{\star_f} = J_0^{\star_f} \subseteq \left(J_1^{\star_f} + J_2^{\star_f} + \cdots + J_n^{\star_f}\right)^{\star_f} = (J_1 + J_2 + \cdots + J_n)^{\star_f} \subseteq I^{\star_f}.$$

Set $J := J_1 + J_2 + \cdots + J_n$. Then J is finitely generated, $J \subseteq I$ and $J^{\star_f} = I^{\star_f}$, thus $J^\star = I^\star$. □

Remark 2.4. Extending the terminology introduced by Zafrullah [50] in the star setting (cf. also [51, p. 433]), given a semistar operation on an integral domain D, we can say that $I \in \overline{\boldsymbol{F}}(D)$ is *strictly $\star$-finite* if $I^\star = J^\star$ for some $J \in \boldsymbol{f}(D)$ with $J \subseteq I$. With this terminology, Lemma 2.3 shows that $\star_f$-finite coincides with strictly $\star$-finite. This result was already proved in the star setting by Zafrullah [50, Theorem 1.1]. Note that Querré [46] studied the strictly v-finite ideals, using often the terminology of *quasi-finite ideals*.

Examples of $\star$-finite ideals that are not $\star_f$-finite (when $\star$ is the v-operation) are given in [22, Section (4c)], where the authors describe domains with all the ideals v-finite (called DVF-domains), that are not Mori domains (that is, such that not all the ideals are t-finite).

Lemma 2.5. *Let $\star$ be a semistar operation on an integral domain D and let $I \in \boldsymbol{F}(D)$. Then I is $\star_f$-invertible if and only if $(I'I'')^\star = D^\star$ for some $I' \subseteq I$, $I'' \subseteq I^{-1}$ and $I', I'' \in \boldsymbol{f}(D)$. Moreover, $I'^\star = I^\star$ and $I''^\star = \left(I^{-1}\right)^\star$.*

Proof. The "if" part is trivial. For the "only if" part, if $\left(II^{-1}\right)^{\star_f} = D^{\star_f}$, then $H^\star = D^\star$ for some $H \subseteq II^{-1}$, $H \in \boldsymbol{f}(D)$. Therefore, $H = (h_1, h_2 \dots, h_n)D$ with $h_i = x_{1,i}y_{1,i} + x_{2,i}y_{2,i} + \cdots + x_{k_i,i}y_{k_i,i}$ with the x's in I and the y's in I^{-1}. Let I' be the (fractional) ideal of D generated by the x's and let I'' be the (fractional) ideal of D generated by the y's. Then $H \subseteq I'I'' \subseteq II^{-1}$ and so $D^\star = (I'I'')^\star$, and thus, also $D^\star = \left(I'I^{-1}\right)^\star = (II'')^\star$. Moreover, $I^\star = (ID^\star)^\star = \left(I\left(I'I^{-1}\right)^\star\right)^\star = \left((II^{-1})^\star I'\right)^\star = (D^\star I')^\star = I'^\star$. In a similar way, we also obtain $I''^\star = \left(I^{-1}\right)^\star$. $\square$

A classical result due to Krull [33, Théorème 8, Chpt. I, §4] shows that for a star operation of finite type, star-invertibility implies star-finiteness. The following result gives a more complete picture of the situation in the general semistar setting.

Proposition 2.6. *Let $\star$ be a semistar operation on an integral domain D. Let $I \in \boldsymbol{F}(D)$. Then I is $\star_f$-invertible if and only if I and I^{-1} are $\star_f$-finite* (*hence, in particular, $\star$-finite*) *and I is $\star$-invertible.*

Proof. The "only if" part follows from Lemma 2.5 and the fact that $\star_f \leq \star$.

For the "if" part, note that by assumption $I^{\star_f} = J'^{\star_f} = J'^\star$ and $(I^{-1})^{\star_f} = J''^{\star_f} = J''^\star$ with $J', J'' \in \boldsymbol{f}(D)$. Therefore,

$$\left(II^{-1}\right)^{\star_f} = (J'J'')^{\star_f} = (J'J'')^\star = \left(J'^\star J''^\star\right)^\star = \left(I^\star \left(I^{-1}\right)^\star\right)^\star = \left(II^{-1}\right)^\star = D^\star. \quad \square$$

The next goal is to investigate when the $\star$-invertibility coincides with the $\star_f$-invertibility.

Let $\star$ be a semistar operation on an integral domain D, we say that D is an *H*($\star$)*-domain* if for each nonzero integral ideal I of D such that $I^\star = D^\star$, there exists $J \in \boldsymbol{f}(D)$ with $J \subseteq I$ and $J^\star = D^\star$. It is easy to see that for $\star = v$, the H(v)-domains coincide with the H-domains introduced by Glaz and Vasconcelos (Remark 2.2(c)).

Lemma 2.7. *Let $\star$ be a semistar operation on an integral domain D. Then D is an H($\star$)-domain if and only if each quasi-$\star_f$-maximal ideal of D is a quasi-$\star$-ideal of D.*

Proof. Assume that D is an H($\star$)-domain. Let $Q = Q^{\star_f} \cap D$ be a quasi-$\star_f$-maximal ideal of D. Assume that $Q^\star = D^\star$. Then for some $J \in \boldsymbol{f}(D)$ with $J \subseteq Q$, we have $J^\star = D^\star$, thus $Q^{\star_f} = D^\star$, which leads to a contradiction. Therefore, $Q^{\star_f} \cap D \subseteq Q^\star \cap D \subsetneq D$, and hence, there exists a quasi-$\star_f$-maximal ideal of D containing $Q^\star \cap D$. This is possible only if $Q^{\star_f} \cap D = Q^\star \cap D$.

Conversely, let I be a nonzero ideal of D with the property $I^\star = D^\star$. Then necessarily $I \not\subseteq Q$ for each quasi-$\star_f$-maximal ideal Q of D (because, otherwise, by assumption $I \subseteq Q = Q^{\star_f} \cap D = Q^\star \cap D$, and so $I^\star \subseteq Q^\star \subsetneq D^\star$). Therefore, $I^{\star_f} = D^\star$. $\square$

Next result provides several characterizations of the H($\star$)-domains; note that in the particular case that $\star = v$, the equivalence (i)$\Leftrightarrow$(iii) was already known [31, Proposition 2.4] and the equivalence (i)$\Leftrightarrow$(iv) was considered in [47, Proposition 5.7].

Proposition 2.8. *Let $\star$ be a semistar operation on an integral domain D. The following are equivalent*:

(i) *D is an $H(\star)$-domain.*

(ii) *For each $I \in \boldsymbol{F}(D)$, I is $\star$-invertible if and only if I is $\star_f$-invertible.*

(iii) $\mathcal{M}(\star_f) = \mathcal{M}(\star)$.

(iv) $\mathcal{M}(\widetilde{\star}) = \mathcal{M}(\star)$.

Proof. Obviously, (iii)$\Leftrightarrow$(iv) by Lemma 1.2, and (iii)$\Leftrightarrow$(i) by Lemma 2.7, recalling that a quasi-$\star$-ideal is also a quasi-$\star_f$-ideal.

(iii)$\Rightarrow$(ii) Let I be a $\star$-invertible ideal of D. Assume that I is not $\star_f$-invertible. Then there exists a quasi-$\star_f$-maximal ideal M such that $II^{-1} \subseteq M$. But M is also quasi-$\star$-maximal since $\mathcal{M}(\star_f) = \mathcal{M}(\star)$. Thus, $M^\star \subsetneq D^\star$. It follows that $(II^{-1})^\star \subseteq M^\star \subsetneq D^\star$, a contradiction. Hence, I is $\star_f$-invertible.

(ii)$\Rightarrow$(i) Let I be a nonzero integral ideal of D such that $I^\star = D^\star$. Then $I \subseteq II^{-1} \subseteq D$ implies that $\left(II^{-1}\right)^\star = D^\star$, that is, I is $\star$-invertible. By assumption, it follows that I is $\star_f$-invertible, and so I is $\star_f$-finite (Proposition 2.6). By Lemma 2.3, we conclude that there exists $J \in \boldsymbol{f}(D)$ with $J \subseteq I$ and $J^\star = I^\star = D^\star$. $\square$

Let $\star$ be a semistar operation of D. If we denote by $\iota : D \hookrightarrow D^\star$ the embedding of D in $D^\star$ and by $\star_\iota$ the (semi)star operation canonically induced on $D^\star$ by $\star$ (defined as in Proposition 1.3), we note that if $I \in \mathrm{Inv}(D, \star)$, then $I^\star \in \mathrm{Inv}(D^\star, \star_\iota)$. As a matter of fact, we have $D^\star = \left(II^{-1}\right)^\star = (I^\star(D : I)^\star)^\star \subseteq (I^\star(D^\star : I^\star))^\star = (I^\star(D^\star : I^\star))^{\star_\iota} \subseteq (D^\star)^\star = D^\star$.

The next example shows that the converse does not hold (in other words, $I^\star$ may be in $\mathrm{Inv}(D^\star, \star_\iota)$ with $I \in \boldsymbol{F}(D) \setminus \mathrm{Inv}(D, \star)$), even if $\star$ is a semistar operation which is stable and of finite type.

Example 2.9. Let D be an almost Dedekind domain, that is not a Dedekind domain (cf. for instance [23, Section 2 and the references]). Then in D there exists a prime (= maximal) ideal P such that P is not invertible (otherwise, D would be a Dedekind domain). Then $P^{-1} = D$ [14, Corollary 3.1.3] since D is a Prüfer domain. Consider the semistar operation $\star := \star_{\{P\}}$. Let $\iota_P : D \hookrightarrow D_P$ be the canonical embedding, then $P^\star = PD_P$ is $\star_{\iota_P}$-invertible since D_P is a DVR, but $\left(PP^{-1}\right)^\star = (PD)^\star = P^\star = PD_P \subsetneq D_P = D^\star$, thus P is not $\star$-invertible.

Let $\iota : D \hookrightarrow D^\star$ be the canonical embedding. We say that $I \in \overline{\boldsymbol{F}}(D)$ is *quasi-$\star$-invertible* if $I^\star \in \mathrm{Inv}(D^\star, \star_\iota)$ (that is, if $(I(D^\star : I))^\star = D^\star$). Note that $I^\star \in \mathrm{Inv}(D^\star, \star_\iota)$ implies that $I^\star \in \boldsymbol{F}(D^\star)$. We denote by $\mathrm{QInv}(D, \star)$ the set of all quasi-$\star$-invertible D-submodules of K, and when $\star = d$, we set $\mathrm{QInv}(D)$, instead of $\mathrm{QInv}(D, d)$. We have already noticed that $\mathrm{Inv}(D, \star) \subseteq \mathrm{QInv}(D, \star)$ and the inclusion can be proper. Moreover, it is obvious that $\mathrm{QInv}(D) = \mathrm{Inv}(D)$.

We have the following straightforward necessary and sufficient condition for a D-submodule of K to be quasi-$\star$-invertible.

Lemma 2.10. *Let $\star$ be a semistar operation on an integral domain D and $I \in \overline{\boldsymbol{F}}(D)$. Then I is quasi-$\star$-invertible if and only if there exists $H \in \overline{\boldsymbol{F}}(D)$ such that $(IH)^\star = D^\star$.*

Next, we prove an analogue of Lemma 2.1 for quasi-$\star$-invertible ideals.

Lemma 2.11. *Let $\star, \star_1, \star_2$ be semistar operations on an integral domain D. Then:*

(0) $D^\star \in \mathrm{QInv}(D, \star)$.

(1) *If $\star_1 \leq \star_2$, then $\mathrm{QInv}(D, \star_1) \subseteq \mathrm{QInv}(D, \star_2)$. In particular, we have*

$$\mathrm{QInv}(D) \subseteq \mathrm{QInv}(D, \tilde{\star}) \subseteq \mathrm{QInv}(D, \star_{_f}) \subseteq \mathrm{QInv}(D, \star).$$

(2) *$I, J \in \mathrm{QInv}(D, \star)$ if and only if $IJ \in \mathrm{QInv}(D, \star)$.*

(3) *If $I \in \mathrm{QInv}(D, \star)$, then $(D^\star : I) \in \mathrm{QInv}(D, \star)$.*

(4) *If $I \in \mathrm{QInv}(D, \star)$, then $I^{v(D^\star)} := (D^\star : (D^\star : I)) \in \mathrm{QInv}(D, \star)$.*

Proof. (0) and (1) are straightforward.

To prove (2), we notice that $I, J \in \mathrm{QInv}(D, \star)$ if and only if $I^\star, J^\star \in \mathrm{Inv}(D^\star, \star_\iota)$, where $\star_\iota$ is defined as above. It follows from Lemma 2.1(2) that $I, J \in \mathrm{QInv}(D, \star)$ if and only if $I^\star J^\star \in \mathrm{Inv}(D^\star, \star_\iota)$. It is easy to see that this happens if and only if $(IJ)^\star \in \mathrm{Inv}(D^\star, \star_\iota)$, that is, if and only if $IJ \in \mathrm{QInv}(D, \star)$.

(3) It is clear.

(4) This is an immediate consequence of Lemma 2.1(4) and the fact that $(v(D^\star))_\iota = v_{D^\star}$, where $v_{D^\star}$ is the v-operation of $D^\star$, ι is the canonical embedding of D in $D^\star$ and $v(D^\star)$ is the semistar operation on D defined by $E^{v(D^\star)} := (D^\star : (D^\star : E))$ for each $E \in \overline{\boldsymbol{F}}(D)$ (note that, obviously, $\star \leq v(D^\star)$). □

Corollary 2.12. *Let $\star$ be a semistar operation on an integral domain D, let $v(D^\star)$ be the semistar operation on D defined in the proof of Lemma 2.11(4), and let $I \in \overline{\boldsymbol{F}}(D)$. If I is quasi-$\star$-invertible, then I is quasi-$v(D^\star)$-invertible and $I^\star = I^{v(D^\star)}$.*

Proof. Let ι be the canonical embedding of D in $D^\star$. As we noted in the proof of Lemma 2.11(4), $(v(D^\star))_\iota = v_{D^\star}$. In order to show that I is quasi-$v(D^\star)$-invertible, we prove that $I^\star$ is $v_{D^\star}$-invertible. But $\star_\iota$ is a (semi)star operation on $D^\star$ and $I^\star$ is $\star_\iota$-invertible, then by Lemma 2.1(1), $I^\star$ is $v_{D^\star}$-invertible since $\star_\iota \leq v_{D^\star}$ [24, Theorem 34.1(4)]. Therefore, I is quasi-$v(D^\star)$-invertible and $I^\star = \left(I^{v(D^\star)}\right)^\star = I^{v(D^\star)}$ since $(D^\star : I) = \left(D^\star : I^{v(D^\star)}\right)$ (cf. also [51, p. 433] or [11, Lemma 2.1(3)] and Remark 2.13(b1)). □

Remark 2.13. (a) Note that if I is a quasi-$\star$-invertible ideal of D, then every ideal J of D with $I \subseteq J \subseteq I^\star \cap D$ is also quasi-$\star$-invertible.

More precisely, let $I, J \in \boldsymbol{F}(D)$ (respectively, $I, J \in \overline{\boldsymbol{F}}(D)$), assume that $J \subseteq I$, $J^\star = I^\star$ and I is $\star$-invertible (respectively, quasi-$\star$-invertible), then J is $\star$-invertible (respectively, quasi-$\star$-invertible).

Conversely, let $I, J \in \overline{\boldsymbol{F}}(D)$, assume that $J \subseteq I$, $J^\star = I^\star$ and J is quasi-$\star$-invertible, then I is quasi-$\star$-invertible (but not necessarily $\star$-invertible, even if J is $\star$-invertible).

As a matter of fact, if I is $\star$-invertible, then $D^\star = (I(D:I))^\star = (J(D:I))^\star \subseteq (J(D:J))^\star \subseteq D^\star$. The quasi-$\star$-invertible case is similar. Conversely, if J is quasi-$\star$-invertible, then $D^\star = (J(D^\star : J))^\star = (I(D^\star : J))^\star$, thus I is quasi-$\star$-invertible and $(D^\star : J) = (D^\star : J)^\star = (D^\star : I)^\star = (D^\star : I)$ (cf. also (d1)).

Example 2.9 shows the parenthetical part of the statement. Let D, P and $\star$ be as in Example 2.9. Note that $P^\star$ is principal in (the DVR) $D^\star = D_P$, thus $P^\star = PD_P = tD_P$ for some nonzero $t \in PD_P$. Therefore, if $J := tD$, then $J^\star = P^\star$, i.e., P is $\star$-finite. We already observed that P is quasi-$\star$-invertible but not $\star$-invertible, even if obviously J is ($\star$-)invertible.

(b) Let $I, H', H'', J, L \in \overline{\boldsymbol{F}}(D)$. The following properties are straightforward:

(b1) $(IH')^\star = D^\star = (IH'')^\star \Longrightarrow H'^\star = H''^\star = (D^\star : I)^\star = (D^\star : I)$.

(b2) $I \in \mathrm{QInv}(D, \star)$, $IJ \subseteq IL \Longrightarrow J^\star \subseteq L^\star$.

(b3) $I \in \mathrm{QInv}(D, \star)$, $J \subseteq I^\star \Longrightarrow \exists\, L \in \overline{\boldsymbol{F}}(D)$, $(IL)^\star = J^\star$. (Take $L := (D^\star : I)J$.)

(b4) $I, J \in \mathrm{QInv}(D, \star)$, $(IL)^\star = J^\star \Longrightarrow L \in \mathrm{QInv}(D, \star)$. (Set $H := I(D^\star : J)$ and note that $(LH)^\star = D^\star$.)

(b5) $I, J \in \mathrm{QInv}(D, \star) \Longrightarrow (D^\star : IJ) = (D^\star : IJ)^\star = ((D^\star : I)\,(D^\star : J))^\star$.

(b6) $I, J \in \mathrm{QInv}(D, \star) \Longrightarrow \exists\, L \in \mathrm{QInv}(D, \star)$, $L \subseteq I^\star$, $L \subseteq J^\star$. (Take $z \in K$, $z \neq 0$ such that $zI \subseteq D^\star$, $zJ \subseteq D^\star$, and set $L := zIJ$.)

(b7) $I, J \in \mathrm{QInv}(D, \star)$, $I + J \in \mathrm{QInv}(D, \star) \Longrightarrow I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathrm{QInv}(D, \star)$. (Recall that $\star \leq v(D^\star)$ and note that:
$((D^\star : I)\,(D^\star : J)\,(I + J))^\star = \big(((D^\star : I)\,I)^\star\,(D^\star : J) + (D^\star : I)\,((D^\star : J)\,J)^\star\big)^\star$
$= ((D^\star : J) + (D^\star : I))^\star = \big(\big(D^\star : J^{v(D^\star)}\big) + \big(D^\star : I^{v(D^\star)}\big)\big)^\star \Longrightarrow$
$((D^\star : I)\,(D^\star : J)\,(I + J))^{v(D^\star)} = \big(\big(D^\star : I^{v(D^\star)}\big) + \big(D^\star : J^{v(D^\star)}\big)\big)^{v(D^\star)} \Longrightarrow$
$(D^\star : ((D^\star : I)\,(D^\star : J)\,(I + J))) = \big(D^\star : \big(\big(D^\star : I^{v(D^\star)}\big) + \big(D^\star : J^{v(D^\star)}\big)\big)\big) =$
$\big(D^\star : \big(D^\star : I^{v(D^\star)}\big)\big) \cap \big(D^\star : \big(D^\star : J^{v(D^\star)}\big)\big) = I^{v(D^\star)} \cap J^{v(D^\star)}$.)

(b8) $I, J \in \mathrm{QInv}(D, \star)$, $I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathrm{QInv}(D, \star) \Longrightarrow I + J \in \mathrm{QInv}(D, v(D^\star))$. (Since $I^{v(D^\star)} \cap J^{v(D^\star)} = (D^\star : ((D^\star : I)\,(D^\star : J)\,(I + J)))$, and hence, $\big(D^\star : \big(I^{v(D^\star)} \cap J^{v(D^\star)}\big)\big) = ((D^\star : I)\,(D^\star : J)\,(I + J))^{v(D^\star)}$, then apply (b4) to conclude that $I + J \in \mathrm{QInv}(D, v(D^\star))$.)

(c) A statement analogous to Corollary 2.12 holds for $\star$-invertibles: *Let $\star$ be a semistar operation on an integral domain D, let $v(D^\star)$ be the semistar operation on D defined in the proof of Lemma* 2.11(4), *and let $I \in \boldsymbol{F}(D)$. If I is $\star$-invertible, then I is $v(D^\star)$-invertible and $I^\star = I^{v(D^\star)}$.*

(d) *Mutatis mutandis*, the statements in (b) hold for $\star$-invertibles. More precisely, let $\star$ be a semistar operation on an integral domain D and let $I, H', H'', J, L \in \boldsymbol{F}(D)$, then:

(d1) $I \in \mathrm{Inv}(D,\star)$, $(IH')^\star = D^\star = (IH'')^\star \Longrightarrow H'^\star = H''^\star = \left(I^{-1}\right)^\star$.

(d2) $I \in \mathrm{Inv}(D,\star)$, $IJ \subseteq IL \Longrightarrow J^\star \subseteq L^\star$.

(d3) $I \in \mathrm{Inv}(D,\star)$, $J \subseteq I^\star \Longrightarrow \exists\ L \in \boldsymbol{F}(D)$, $(IL)^\star = J^\star$.

(d4) $I, J \in \mathrm{Inv}(D,\star)$, $(IL)^\star = J^\star \Longrightarrow L \in \mathrm{QInv}(D,\star)$, $(D^\star : L) = (I(D:J))^\star$. Note that under the present hypotheses, $L \in \mathrm{Inv}(D,\star)$ if and only if $(D:L)^\star = (I(D:J))^\star$.

(d5) $I, J \in \mathrm{Inv}(D,\star) \Longrightarrow (D : IJ)^\star = ((D:I)\,(D:J))^\star$.

(d6) $I, J \in \mathrm{Inv}(D,\star) \Longrightarrow \exists\ L \in \mathrm{Inv}(D,\star)$, $L \subseteq I$, $L \subseteq J$.

(d7) $I, J \in \mathrm{Inv}(D,\star)$, $I + J \in \mathrm{Inv}(D,\star) \Longrightarrow I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathrm{Inv}(D,\star)$.

(d8) $I, J \in \mathrm{Inv}(D,\star)$, $I^{v(D^\star)} \cap J^{v(D^\star)} \in \mathrm{Inv}(D,\star) \Longrightarrow I + J \in \mathrm{Inv}(D, v(D^\star))$.

Our next goal is to extend Proposition 2.6 to the case of quasi-$\star_{_f}$-invertibles. We need the following:

Lemma 2.14. *Let $\star$ be a semistar operation on an integral domain D with quotient field K, let $\iota : D \hookrightarrow D^\star$ be the embedding of D in $D^\star$, let $\star_\iota$ denote the* (*semi*)*star operation canonically induced on $D^\star$ by $\star$, and let $I \in \overline{\boldsymbol{F}}(D)$. Then I is $\star$-finite if and only if $I^\star$ is $\star_\iota$-finite.*

Proof. If I is $\star$-finite, then there exists $J \in \boldsymbol{f}(D)$ such that $I^\star = J^\star$. It is clear that $(JD^\star)^{\star_\iota} = I^\star$ with $JD^\star \in \boldsymbol{f}(D^\star)$. Thus, $I^\star$ is $\star_\iota$-finite. Conversely, let $I^\star$ be $\star_\iota$-finite. Then there exists $J_0 \in \boldsymbol{f}(D^\star)$, $J_0 = (a_1, a_2, \dots, a_n)D^\star$ with $\{a_1, a_2, \dots, a_n\} \subseteq K$, such that $J_0^\star = J_0^{\star_\iota} = I^{\star_\iota} = I^\star$. Set $J = (a_1, a_2, \dots, a_n)D \in \boldsymbol{f}(D)$. Then $J^\star = (a_1D + a_2D + \cdots + a_nD)^\star = (a_1D^\star + a_2D^\star + \cdots + a_nD^\star)^\star = J_0^\star = I^\star$, and so I is $\star$-finite. □

Proposition 2.15. *Let $\star$ be a semistar operation on an integral domain D and let $I \in \overline{\boldsymbol{F}}(D)$. Then I is quasi-$\star_{_f}$-invertible if and only if I and $(D^\star : I)$ are $\star_{_f}$-finite* (*hence, $\star$-finite*) *and I is quasi-$\star$-invertible.*

Proof. Let $\iota : D \hookrightarrow D^\star$ be the canonical embedding and let $\star_\iota$ be the (semi)star operation on $D^\star$ canonically induced by $\star$.

For the "if" part, use the same argument of the proof of the "if" part of Proposition 2.6.

For the "only if" part, since I is quasi-$\star_{_f}$-invertible, $(D^\star : I)$ is also quasi-$\star_{_f}$-invertible, thus we have that $I^{\star_f}$ and $(D^\star : I)^{\star_f} = (D^\star : I)$ are $(\star_{_f})_\iota$-invertibles. Then $I^{\star_f}$ and $(D^\star : I)$ are $(\star_{_f})_\iota$-finite (Proposition 2.6) and then I and $(D^\star : I)$ are $\star_{_f}$-finite by Lemma 2.14. Clearly, I is quasi-$\star$-invertible since $\star_{_f} \leq \star$ (Lemma 2.11(1)). □

It is natural to ask under which conditions a quasi-$\star$-invertible fractional ideal is $\star$-invertible. Let $I \in \boldsymbol{F}(D)$ be quasi-$\star$-invertible. Then $(I(D^\star : I))^\star = D^\star$. Suppose that I is also $\star$-invertible, i.e., $(I(D:I))^\star = D^\star$. Then

$$\begin{aligned}(D:I)^\star &= ((D:I)\,(I(D^\star:I))^\star)^\star = (((D:I)I)^\star\,(D^\star:I))^\star \\ &= (D^\star:I)^\star = (D^\star:I) = (D^\star:I^\star) \supseteq (D:I)^\star.\end{aligned}$$

Therefore, we have the following (cf. also Remark 2.2(b)):

Proposition 2.16. *Let $\star$ be a semistar operation on an integral domain D. Let I be a quasi-$\star$-invertible fractional ideal of D. Then I is $\star$-invertible if and only if $(D:I)^\star = (D^\star:I)$ (i.e., $\left(I^{-1}\right)^\star = (I^\star)^{-1}$).*

The following corollary is straightforward (in particular, part (2) follows immediately from [13, proof of Remark 1.7]):

Corollary 2.17. *Let $\star$ be a semistar operation on an integral domain D, and let $I \in \boldsymbol{F}(D)$.*

(1) *If $\star$ is a (semi)star operation, then I is quasi-$\star$-invertible if and only if I is $\star$-invertible.*
(2) *If $\star$ is stable and $I \in \boldsymbol{f}(D)$, then I is quasi-$\star$-invertible if and only if I is $\star$-invertible.*

We notice that if $\star$ is a semistar operation of finite type, $\star$-invertibility depends only on the set of quasi-$\star$-maximal ideals of D. Indeed, it is clear that $I \in \boldsymbol{F}(D)$ is $\star$-invertible if and only if $\left(II^{-1}\right)^\star \cap D$ is not contained in any quasi-$\star$-maximal ideal. Then from Lemma 1.2, we deduce immediately the following general result (cf. [13, Proposition 4.25]):

Proposition 2.18. *Let $\star$ be a semistar operation on an integral domain D. Let $I \in \boldsymbol{F}(D)$. Then I is $\star_{_f}$-invertible if and only if I is $\tilde{\star}$-invertible.*

A classical example due to Heinzer can be used for describing the content of the previous proposition.

Example 2.19. Let K be a field and X an indeterminate over K. Set $D := K[[X^3, X^4, X^5]]$ and $M := (X^3, X^4, X^5)D$. It is easy to see that D is a one-dimensional Noetherian local integral domain with maximal ideal M. Let $\star := v$, note that in this case, $v = \star = \star_{_f} = t$ and $\mathcal{M}(v) = \{M\}$ since $M = (D : K[[X]])$. Therefore, $w = \widetilde{v} = d$. In this situation, $\mathrm{Inv}(D, v) = \mathrm{Inv}(D, t) = \mathrm{Inv}(D, w) = \mathrm{Inv}(D) = \{zD \mid z \in K,\, z \neq 0\}$. But $v = t \neq w = d$ because in general $(I \cap J)^t$ is different from $I^t \cap J^t$ in D since D is not a Gorenstein domain (cf. [2, Theorem 5, Corollary 5.1] and [35, Theorem 222]).

A result "analogous" to Proposition 2.18 does not hold, in general, for quasi-semistar-invertibility, as we show in the following:

Example 2.20. Let D be a pseudo-valuation domain with maximal ideal M such that $V := M^{-1}$ is a DVR (for instance, take two fields $k \subsetneq K$ and let $V := K[[X]]$, $M := XK[[X]]$ and $D := k + M$). Consider the semistar operation of finite type

$\star := \star_{\{V\}}$ defined by $E^{\star_{\{V\}}} := EV$ for each $E \in \overline{\boldsymbol{F}}(D)$. It is clear that M is the only quasi-$\star$-maximal ideal of D. Thus, $\tilde{\star} = \star_{\{M\}} = d$, the identity (semi)star operation of D. We have $(M(V : M))^\star = (M(V : M))V = V$ since V is a DVR. Hence, M is quasi-$\star$-invertible. On the other side, M is not invertible (i.e., not quasi-$\tilde{\star}$-invertible) since $MM^{-1} = MV = M$.

Under the assumption $D^\star = D^{\tilde{\star}}$, we have the following extension of Proposition 2.18 to the case of quasi-semistar-invertibility:

Proposition 2.21. *Let $\star$ be a semistar operation on an integral domain D. Suppose $D^\star = D^{\tilde{\star}}$. Let $I \in \overline{\boldsymbol{F}}(D)$. Then I is quasi-$\star_f$-invertible if and only if I is quasi-$\tilde{\star}$-invertible.*

Proof. If I is quasi-$\tilde{\star}$-invertible, then there exists $J \in \overline{\boldsymbol{F}}(D)$ with $(IJ)^{\tilde{\star}} = D^{\tilde{\star}}$. This implies $(IJ)^{\star_f} = D^{\star_f}$ since $\tilde{\star} \leq \star_f$. Conversely, suppose there exists $J \in \overline{\boldsymbol{F}}(D)$ such that $(IJ)^{\star_f} = D^{\star_f}$. Then $IJ \subseteq D^{\star_f} = D^\star = D^{\tilde{\star}}$. Thus, $(IJ)^{\tilde{\star}} \subseteq D^{\tilde{\star}}$. If $(IJ)^{\tilde{\star}} \subsetneq D^{\tilde{\star}}$, then $(IJ)^{\tilde{\star}} \cap D \subsetneq D$ is a quasi-$\tilde{\star}$-ideal of D. It follows that $(IJ)^{\tilde{\star}} \cap D$ is contained in a quasi-$\tilde{\star}$-maximal ideal P of D. From Lemma 1.2, P is also a quasi-$\star_f$-maximal ideal. Then $(IJ)^{\star_f} \cap D \subseteq ((IJ)^{\tilde{\star}} \cap D)^{\star_f} \subseteq P^{\star_f} \subsetneq D^{\star_f}$, a contradiction. Then I is quasi-$\tilde{\star}$-invertible. □

Remark 2.22. (a) If $\star$ is a semistar operation on an integral domain D, we already observed (Remark 2.2(a)) that $\mathrm{Inv}(D, \star)$ is not a group with respect to the standard multiplication of fractional ideals. In the set of the $\star$-invertible $\star$-fractional ideals, i.e., in the set $\mathrm{Inv}^\star(D) := \{I \in \mathrm{Inv}(D, \star) \mid I = I^\star\}$, we can introduce a semistar composition "$\times$" in the following way: $I \times J := (IJ)^\star$. Note that $(\mathrm{Inv}^\star(D), \times)$ is not a group in general because, for instance, it does not possess an identity element (e.g., when $D^\star \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$).

On the other hand, $\mathrm{QInv}^\star(D) := \{I \in \mathrm{QInv}(D, \star) \mid I = I^\star\}$ with the semistar composition "$\times$" introduced above is always a group, having as identity $D^\star$ and unique inverse of $I \in \mathrm{QInv}^\star(D)$ the D-module $(D^\star : I) \in \overline{\boldsymbol{F}}(D)$, which belongs to $\mathrm{QInv}^\star(D)$. This fact provides also one of the motivations for considering $\mathrm{QInv}(D, \star)$ and $\mathrm{QInv}^\star(D)$ (and not only $\mathrm{Inv}(D, \star)$ and $\mathrm{Inv}^\star(D)$, as in the "classical" star case).

It is not difficult to prove: Let $\star$ be a semistar operation on an integral domain D, then:

$$(\mathrm{Inv}^\star(D), \times) \text{ is a group } \iff (D : D^\star) \neq (0).$$

As a matter of fact, ($\Rightarrow$) holds because $D^\star \in \mathrm{Inv}^\star(D) \subseteq \boldsymbol{F}(D)$ and so $(D : D^\star) \neq (0)$. ($\Leftarrow$) holds because $(D : D^\star) \neq (0)$ implies that $D^\star \in \mathrm{Inv}^\star(D)$, and for each $I \in \mathrm{Inv}^\star(D)$, we have $(D^\star : I) \in \boldsymbol{F}(D)$, thus $(D : I)^\star = (D^\star : I)$ (Remark 2.13(d1)) and so the inverse of each element $I \in \mathrm{Inv}^\star(D)$ exists and is uniquely determined in $\mathrm{Inv}^\star(D)$.

Note that, even if $(\mathrm{Inv}^\star(D), \times)$ is a group, $\mathrm{Inv}^\star(D)$ could be a proper subset of $\mathrm{QInv}^\star(D)$. For this purpose, take D, V, M as in Example 2.20, in this case, $D^\star = V$ and $(D : V) = M \neq (0)$, hence $(\mathrm{Inv}^\star(D), \times)$ is a group, but $M \in \mathrm{QInv}^\star(D) \setminus \mathrm{Inv}^\star(D)$.

(b) Note that if $\star$ is a semistar operation on an integral domain D, the group

$(\mathrm{QInv}^\star(D), \times)$ can be identified with a more classic group of star-invertible star-ideals. As a matter of fact, it is easy to see that

$$(\mathrm{QInv}^\star(D), \times) = (\mathrm{Inv}^{\star_\iota}(D^\star), \times'),$$

where $\iota : D \to D^\star$ is the canonical embedding, $\star_\iota$ is the (semi)star operation on $D^\star$ canonically associated to $\star$ (Proposition 1.3), and the (semi)star composition "$\times'$" in $\mathrm{Inv}^{\star_\iota}(D^\star)$ is defined by $E \times' F := (EF)^{\star_\iota}$.

(c) Let $\star_1, \star_2$ be two semistar operations on an integral domain D. If $\star_1 \leq \star_2$, then $\mathrm{Inv}(D, \star_1) \subseteq \mathrm{Inv}(D, \star_2)$ and $\mathrm{QInv}(D, \star_1) \subseteq \mathrm{QInv}(D, \star_2)$. Note that it is not true in general that $\mathrm{Inv}^{\star_1}(D) \subseteq \mathrm{Inv}^{\star_2}(D)$ or $\mathrm{QInv}^{\star_1}(D) \subseteq \mathrm{QInv}^{\star_2}(D)$ because there is no reason for a $\star_1$-ideal (or module) to be a $\star_2$-ideal (or module). For instance, let T be a proper overring of an integral domain D, let $\star_1 := d$ be the identity (semi)star operation on D and let $\star_1 := \star_{\{T\}}$ be the semistar operation on D defined by $E^{\star_{\{T\}}} := ET$ for each $E \in \overline{\boldsymbol{F}}(D)$. If I is a nonzero principal ideal of D, then obviously $I \in \mathrm{Inv}^{\star_1}(D)$ $(= \mathrm{Inv}(D) = \mathrm{QInv}^{\star_1}(D))$ but I does not belong to $\mathrm{QInv}^{\star_2}(D)$ (and in particular, it does not belong to $\mathrm{Inv}^{\star_2}(D)$) because $I^{\star_2} = IT \neq I$.

Note that, even if $\mathrm{Inv}(D, \star_1) = \mathrm{Inv}(D, \star_2)$ for some pair of semistar operations $\star_1 \leq \star_2$, it is not true in general that $\mathrm{Inv}^{\star_1}(D) \subseteq \mathrm{Inv}^{\star_2}(D)$. Take D, V, M as in Example 2.20. Let $\star_1 := d$ be the identity (semi)star operation on D and let $\star_2 := \star_{\{V\}}$. In this case, $\mathrm{Inv}(D, \star_1) = \mathrm{Inv}(D, \star_2)$ because $\star_1 = \widetilde{\star_2}$ and $\star_2 = (\star_2)_f$ (Proposition 2.18). But $\mathrm{Inv}^{\star_2}(D) \subsetneq \mathrm{Inv}^{\star_1}(D) = \mathrm{Inv}(D)$ because $\mathrm{Inv}^{\star_2}(D) \subseteq \mathrm{Inv}^{\star_1}(D) = \mathrm{Inv}(D)$ since each $\star_2$-ideal is obviously a $\star_1$-ideal, and moreover, the proper inclusion holds because, as above, a nonzero principal ideal of D belongs to $\mathrm{Inv}(D)$ but not to $\mathrm{Inv}^{\star_2}(D)$.

On the other hand, if $\star_1 \leq \star_2$ are two *star* operations on D, then it is known that $\mathrm{Inv}^{\star_1}(D) \subseteq \mathrm{Inv}^{\star_2}(D)$ essentially because in this case, $I \in \mathrm{Inv}^{\star_1}(D)$ implies that $I = I^{\star_1} = I^v$ and so $I = I^{\star_2}$ [6, Proposition 3.3].

(d) Let $\star$ be a semistar operation on an integral domain D, let $v(D^\star)$ be the semistar operation on D defined in Lemma 2.11(4) and let $I, J \in \boldsymbol{F}(D)$ (respectively, $I, J \in \overline{\boldsymbol{F}}(D)$). Assume that I is a $\star$-invertible (respectively, quasi-$\star$-invertible) $\star$-ideal of D, then:

$$(IJ^v)^\star = (I : (D : J)) \quad \big(\text{respectively, } \big(IJ^{v(D^\star)}\big)^\star = (I : (D^\star : J))\big).$$

Recall that since $I = I^\star$, we have $(I : (D : J)) = (I : (D : J))^\star$. It is obvious that $IJ^v \subseteq (I : (D : J^v)) = (I : (D : J))$ and thus $(IJ^v)^\star \subseteq (I : (D : J))$. Conversely, if $z \in (I : (D : J))$, then $z(D : J) \subseteq I$ and so $z(D : I) \subseteq J^v$. Therefore, $z \in zD^\star = z((D : I)I)^\star \subseteq (IJ^v)^\star$.

For the quasi-$\star$-invertible case, if $I = I^\star$, then $(I : (D^\star : J)) = (I : (D^\star : J))^\star$ and $I = ID^\star$. It is obvious that $IJ^{v(D^\star)} \subseteq (I : (D^\star : J^{v(D^\star)})) = (I : (D^\star : J))$ and thus $\big(IJ^{v(D^\star)}\big)^\star \subseteq (I : (D^\star : J))$. Conversely, if $z \in (I : (D^\star : J))$, then $z(D^\star : J) \subseteq I$ and so $z(D^\star : I) \subseteq J^{v(D^\star)}$. Therefore, $z \in zD^\star = z((D^\star : I)I)^\star \subseteq \big(IJ^{v(D^\star)}\big)^\star$.

In the next theorem, we investigate the behaviour of a $\star$-invertible ideal (when $\star$ is a semistar operation) with respect to the localizations at quasi-$\star$-maximal ideals and in the passage to semistar Nagata ring. More precisely, in the spirit of Kaplansky's theorem on (d-)invertibility [35, Theorem 62], we extend a characterization of t-invertibility proved in [36, Corollary 3.2] and two Kang's results proved in the star setting [34, Theorem 2.4 and Proposition 2.6].

Theorem 2.23. *Let $\star$ be a semistar operation on an integral domain D. Assume that $\star = \star_f$. Let $I \in \boldsymbol{f}(D)$. Then the following are equivalent*:

(i) *I is $\star$-invertible.*

(ii) *$ID_Q \in \mathrm{Inv}(D_Q)$ for each $Q \in \mathcal{M}(\star)$ (and then ID_Q is principal in D_Q).*

(iii) *$I\mathrm{Na}(D, \star) \in \mathrm{Inv}(\mathrm{Na}(D, \star))$.*

Proof. (i)⇒(ii). If $(II^{-1})^\star = D^\star$, then $II^{-1} \not\subseteq Q$ for each $Q \in \mathcal{M}(\star)$. Since $I \in \boldsymbol{f}(D)$, by flatness we have:

$$I^{-1}D_Q = (D : I)D_Q = (D_Q : ID_Q) = (ID_Q)^{-1}.$$

Therefore, for each $Q \in \mathcal{M}(\star)$, since $II^{-1} \not\subseteq Q$, we have:

$$D_Q = (II^{-1})D_Q = ID_Q I^{-1} D_Q = ID_Q(ID_Q)^{-1}.$$

(ii)⇒(iii). From the assumption and the proof of (i)⇒(ii), we have $II^{-1} \not\subseteq Q$ for each $Q \in \mathcal{M}(\star)$. Since $I \in \boldsymbol{f}(D)$, by the flatness of the canonical homomorphism $D \to D[X]_{N(\star)} = \mathrm{Na}(D, \star)$, we have:

$$(I[X]_{N(\star)})^{-1} = (D[X]_{N(\star)} : I[X]_{N(\star)}) = (D : I)[X]_{N(\star)} = I^{-1}[X]_{N(\star)}.$$

Since $II^{-1} \not\subseteq Q$, $(II^{-1})[X]_{N(\star)} \not\subseteq Q[X]_{N(\star)}$ for each $Q \in \mathcal{M}(\star)$. From [19, Proposition 3.1(3)], we deduce

$$D[X]_{N(\star)} = (II^{-1})[X]_{N(\star)} = I[X]_{N(\star)}(I[X]_{N(\star)})^{-1},$$

where $I\mathrm{Na}(D, \star) = I[X]_{N(\star)}$.

(iii)⇒(i). From the assumption and the previous considerations, we have:

$$D[X]_{N(\star)} = I[X]_{N(\star)}(I[X]_{N(\star)})^{-1} = (II^{-1})[X]_{N(\star)},$$

and thus $(II^{-1})[X]_{N(\star)} \not\subseteq Q[X]_{N(\star)}$ for each $Q \in \mathcal{M}(\star)$. This fact implies that $II^{-1} \not\subseteq Q$ for each $Q \in \mathcal{M}(\star)$. From [19, Lemma 2.4 (1)], we deduce immediately that $(II^{-1})^\star = D^\star$. □

Corollary 2.24. *Let $\star$ be a stable semistar operation of finite type on D, and let $I \in \boldsymbol{f}(D)$. Then the conditions* (i)–(iii) *in Theorem* 2.23 *are equivalent to*:

(iv) *I is quasi-$\star$-invertible.*

Proof. Apply Corollary 2.17. □

Remark 2.25. It is known [34, Proposition 2.6] (cf. also [5, Section 1] and [11, Section 1]) that, *if $\star$ is a star operation of finite type on an integral domain D, then an ideal I of D is $\star$-invertible if and only if it is $\star$-finite and locally principal* (*when localized at the* $\star$*-maximal ideals*). As a matter of fact, by Proposition 2.6, we have that if I is $\star$-invertible, then I is $\star$-finite. Moreover, $(II^{-1})^\star = D$ implies $II^{-1} \not\subseteq Q$ for each $\star$-maximal ideal Q of D. It follows that $ID_Q I^{-1} D_Q = D_Q$. Thus, ID_Q is invertible (hence, principal) in D_Q. Conversely, assume that $I^\star = J^\star$ with $J \in \boldsymbol{f}(D)$, $J \subseteq I$. It is clear that $I^{-1} = J^{-1}$, since $I^v = (I^\star)^v = (J^\star)^v = J^v$ and $\star \leq v$ [24, Theorem 34.1(4)]. Suppose I is not $\star$-invertible, i.e., $(II^{-1})^\star \subsetneq D$. Then there exists a $\star$-maximal ideal Q of D such that $II^{-1} \subseteq Q$. It follows that $QD_Q \supseteq ID_Q I^{-1} D_Q = ID_Q J^{-1} D_Q = ID_Q (JD_Q)^{-1} \supseteq ID_Q (ID_Q)^{-1}$, which is a contradiction since ID_Q is principal.

We will see in a moment that the "if" part of a similar result for semistar operations does not hold, even if $I = I^\star$. More precisely, we can extend partially [21, Proposition 1.1] in the following way:

Let $I \in \boldsymbol{F}(D)$ *and let* $\star$ *be a semistar operation on* D. *The following properties are equivalent*:

(i) I *is* $\star_{f}$*-invertible.*

(ii) $(Q : I) \subsetneq (D : I)$ *for each* $Q \in \mathcal{M}(\star_f)$.

(iii) $(Q : I) \subsetneq (D : I)$ *for each* $Q \in \mathcal{M}(\star_f)$ *such that* $Q \supseteq I(D : I)$.

Moreover, each of the previous properties implies the following:

(iv) I *is* $\star_f$*-finite and* $ID_Q \in \mathrm{Inv}(D_Q)$ *for each* $Q \in \mathcal{M}(\star_f)$ (*and so* ID_Q *is principal in* D_Q).

As a matter of fact, (i)$\Rightarrow$(ii) because $D^\star = (I(D : I))^\star$ and if $(Q : I) = (D : I)$ for some $Q \in \mathcal{M}(\star_f)$, then $I(D : I) = I(Q : I) \subseteq Q$, thus $(I(D : I))^{\star_f} \subseteq Q^{\star_f} \subsetneq D^\star$, hence we reach a contradiction. (ii)$\Rightarrow$(iii) is trivial. For (iii)$\Rightarrow$(i), if $I(D : I) \subseteq Q$ for some $Q \in \mathcal{M}(\star_f)$, thus $(D : I) \subseteq (Q : I)$ and hence $(D : I) = (Q : I)$, which contradicts (iii).

Finally, (ii)$\Rightarrow$(iv) because of Proposition 2.6 and for $z_Q \in (D : I) \setminus (Q : I)$, we have $z_Q I \subset D \setminus Q$, and so $z_Q ID_Q = D_Q$, i.e., $ID_Q = (z_Q)^{-1} D_Q$ for each $Q \in \mathcal{M}(\star_f)$.

But note that, in the semistar setting, (iv)$\not\Rightarrow$(i), even in the case that I is a $\star_f$-finite $\star_f$-ideal, as the following example will show. However, we can re-establish a characterization in the quasi-$\star$-invertibility setting in the following way: *If* $\star$ *is a semistar operation of finite type on an integral domain* D *and* $I \in \overline{\boldsymbol{F}}(D)$, *then* $I \in \mathrm{QInv}(D, \star)$ *if and only if* $I^\star$ *is* $\star$*-finite and* $I^\star {D^\star}_M$ *is principal for each* $\star_\iota$*-maximal ideal* M *of* $D^\star$.

Example 2.26. Let D be a valuation domain, P a nonzero non-maximal non-invertible prime ideal of D such that D_P is a discrete valuation domain. (For instance, if K is a field and X, Y are two indeterminates over K, let $D := K + XK[X]_{(X)} + YK(X)[Y]_{(Y)}$ and $P := YK(X)[Y]_{(Y)}$; in this case, D is a two-dimensional valuation domain, $D_P = K(X)[Y]_{(Y)}$ and $P = PD_P = YD_P \supsetneq YD$.) Set $\star := \star_{\{P\}}$. In this situation, $\star = \star_f$ and $\mathcal{M}(\star) = \{P\}$, thus $\star = \widetilde{\star}$, i.e., $\star$ is a stable semistar operation of finite type on D. Note that P is in fact a $\star$-ideal of D since $P^\star = PD_P = P$. Moreover, $P^\star = PD_P = tD_P = (tD)^\star$ for some nonzero $t \in D_P$, i.e., P is a nonzero principal ideal in $D^\star = D_P$ since D_P is a

DVR by assumption. Thus, P is a $\star$-ideal, $\star$-finite and locally principal, when localized at the quasi-$\star$-maximal ideal(s) of D. But P is not $\star$-invertible since in this situation, $(D : P) = (P : P) = D_P$ [14, Proposition 3.1.5], and hence, $(P(D : P))^\star = (P(P : P))^\star = (PD_P)^\star = P^\star = P$. Note also that in this situation, P is quasi-$\star$-invertible (because $(P(D^\star : P))^\star = (tD_Pt^{-1}D_P)^\star = D_P = D^\star$) and $D^\star = D_P = (PD_P : PD_P) = (P : P)D_P = (P : P)^\star$.

The next two results generalize [34, Theorems 2.12 and 2.14] to the semistar setting.

Corollary 2.27. *Let $\star$ be a semistar operation on an integral domain D. Assume that $\star = \star_f$. Let $h \in D[X]$, $h \neq 0$. Then*:

$$\boldsymbol{c}(h) \in \mathrm{Inv}(D, \star) \Longleftrightarrow h\mathrm{Na}(D, \star) = \boldsymbol{c}(h)\mathrm{Na}(D, \star).$$

In particular, $\boldsymbol{c}(h) \in \mathrm{Inv}(D, \star)$ *if and only if* $\boldsymbol{c}(h) \in \mathrm{QInv}(D, \star)$.

Proof. The proof of the first part of the statement is based on the following result by D.D. Anderson [1, Theorem 1]: If R is a ring and $h \in R[X]$, $h \neq 0$, then $hR(X) \subseteq c(h)R(X)$, and moreover, the following are equivalent:

(1) $hR(X) = \boldsymbol{c}(h)R(X)$.
(2) $\boldsymbol{c}(h)$ is locally principal (in R).
(3) $\boldsymbol{c}(h)R(X)$ is principal (in $R(X)$).

($\Rightarrow$) By (i)$\Rightarrow$(ii) in Theorem 2.23, we have that $\boldsymbol{c}(h)D_Q$ is principal for each $Q \in \mathcal{M}(\star)$. Hence,

$$\boldsymbol{c}(h)D_Q[X]_{N(\star)} = \boldsymbol{c}(h)(D[X]_{N(\star)})_{QD[X]_{N(\star)}} = \boldsymbol{c}(h)D_Q(X)$$

is principal for each $Q \in \mathcal{M}(\star)$. By applying Anderson's result to the local ring $R = D_Q$, we deduce that $hD_Q(X) = \boldsymbol{c}(h)D_Q(X)$ for each $Q \in \mathcal{M}(\star)$. The conclusion follows from (2) and (3) in Proposition 1.4.

($\Leftarrow$) If $h\mathrm{Na}(D, \star) = \boldsymbol{c}(h)\mathrm{Na}(D, \star)$, then by localization we obtain $hD_Q(X) = \boldsymbol{c}(h)D_Q(X)$ for each $Q \in \mathcal{M}(\star)$ (Proposition 1.4 and [24, Corollary 5.3]). By Anderson's result, we deduce that $\boldsymbol{c}(h)D_Q$ is principal, i.e., $\boldsymbol{c}(h)D_Q \in \mathrm{Inv}(D_Q)$ for each $Q \in \mathcal{M}(\star)$. The conclusion follows from (ii)$\Rightarrow$(i) in Theorem 2.23.

The last part of the statement follows from the fact that $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \widetilde{\star})$ [19, Corollary 3.5(3)] and from Corollary 2.17 and Proposition 2.18, or directly from Corollary 2.24. $\square$

Proposition 2.28. *Let $\star$ be a semistar operation on an integral domain D. If H is an invertible ideal of* $\mathrm{Na}(D, \star)$*, then H is principal in* $\mathrm{Na}(D, \star)$.

Proof. We can assume that $H \in \mathrm{Inv}(\mathrm{Na}(D, \star))$ and $H \subseteq \mathrm{Na}(D, \star)$, then in particular, $H = (h_1, h_2, \dots, h_n)\mathrm{Na}(D, \star)$ with $h_i \in D[X]$, $1 \leq i \leq n$. For each $Q \in \mathcal{M}(\star_f)$, by localization, we obtain that $HD_Q(X) = (h_1, h_2, \dots, h_n)D_Q(X)$ is a nonzero principal ideal (Theorem 2.23 (iii)$\Rightarrow$(ii)). By a standard argument, if $d_i := \deg(h_i)$ for $1 \leq i \leq n$, and if

$$h := h_1 + h_2X^{d_1+1} + h_3X^{d_1+d_2+2} + \cdots + h_nX^{d_1+d_2+\cdots+d_{n-1}+n-1} \in D[X],$$

then it is not difficult to see $HD_Q(X) = hD_Q(X)$ for each $Q \in \mathcal{M}(\star_f)$. From Proposition 1.4(3), we deduce that $H\mathrm{Na}(D, \star) = h\mathrm{Na}(D, \star)$. $\square$

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