Lecture 1

Elliptic curves over finite fields

First steps

Research School: Algebraic curves over finite fields

CIMPA-ICTP-UNESCO-MESR-MINECO-PHILIPPINES

University of the Phillipines Diliman, July 22, 2013

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Proto–History (from WIKIPEDIA)

Giulio Carlo, Count Fagnano, and Marquis de Toschi (December 6, 1682 – September 26, 1766) was an Italian mathematician. He was probably the first to direct attention to the theory of *elliptic integrals*. Fagnano was born in Senigallia.

He made his higher studies at the *Collegio Clementino* in Rome and there won great distinction, except in mathematics, to which his aversion was extreme. Only after his college course he took up the study of mathematics.

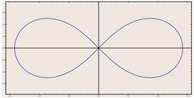
Later, without help from any teacher, he mastered mathematics from its foundations.

Some of His Achievements:

- $\pi = 2i \log \frac{1-i}{1+1}$
- Length of *Lemniscate*

Carlo Fagnano

Collegio Clementino



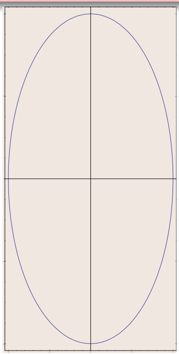
Lemniscate

$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$

$$\ell = 4 \int_0^a \frac{a^2\,dr}{\sqrt{a^4 - r^4}} = \frac{a\sqrt{\pi}\,\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}$$

Length of Ellipses

$$\mathcal{E} : \frac{x^2}{4} + \frac{y^2}{16} = 1$$



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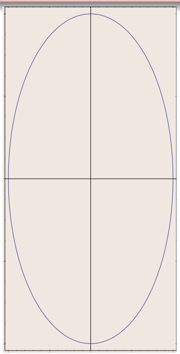

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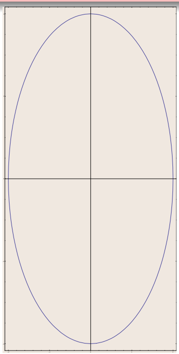

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Applying this formula to $\mathcal{E}$:

$$\ell(\mathcal{E}) = 4 \int_0^4 \sqrt{1 + \left(\frac{d\sqrt{16(1 - t^2/4)}}{dt} \right)^2} dt$$

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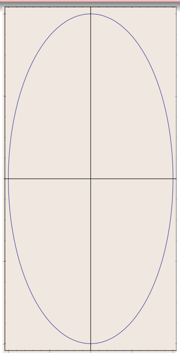

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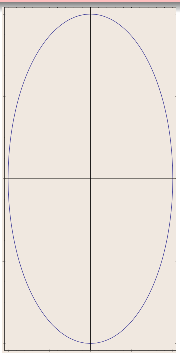

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Apply the invertible change of variables:

$$\begin{cases} x = 1 - 2/t \\ y = \frac{u}{t-1} \end{cases}$$

Arrive to

$$u^2 = t^3 - 4t^2 + 6t - 3$$

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6. represent a mathematical world in itself ... Each of them does!!

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If $F(x, y) \in \mathbb{Q}[x, y]$ a *point of the curve* $F = 0$, means $(x_0, y_0) \in \mathbb{C}^2$ s.t. $F(x_0, y_0) = 0$.

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An elliptic curve E over a $\mathbb{F}_q$ (finite field) is given by an equation

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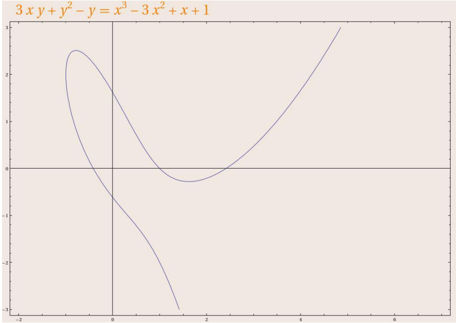

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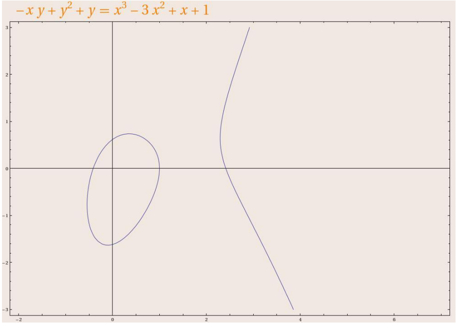

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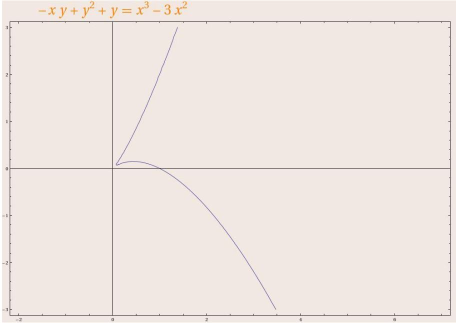

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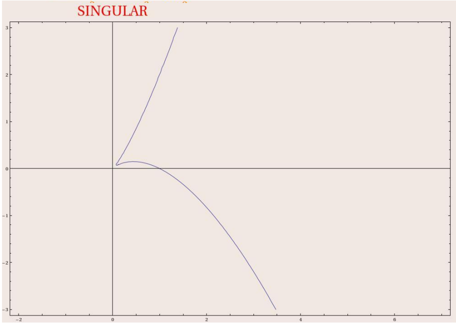

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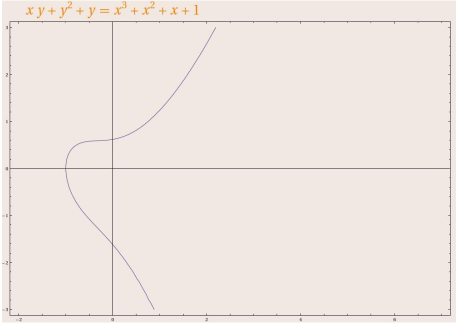

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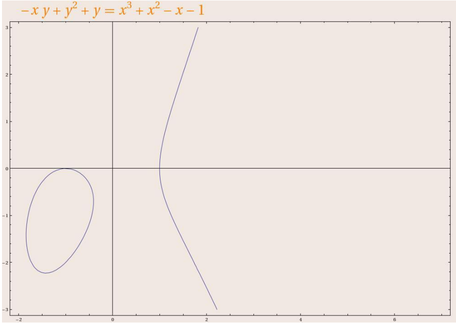

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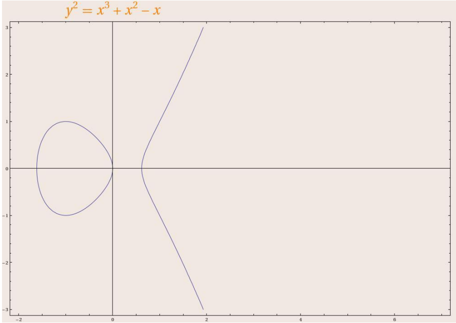

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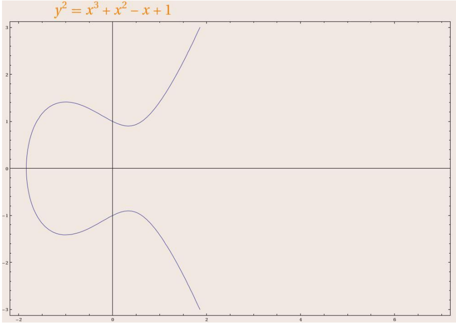

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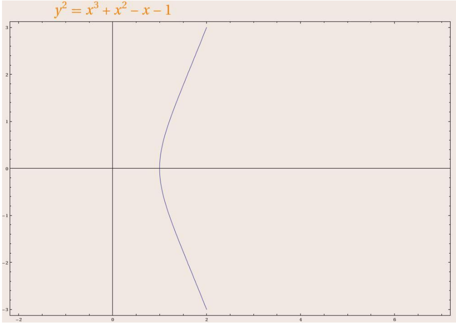

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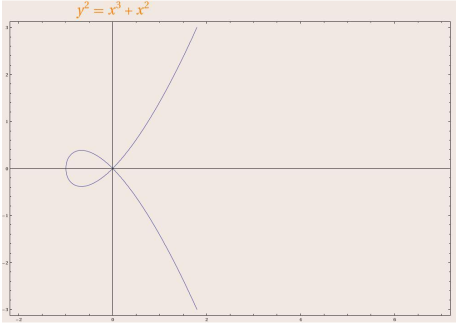

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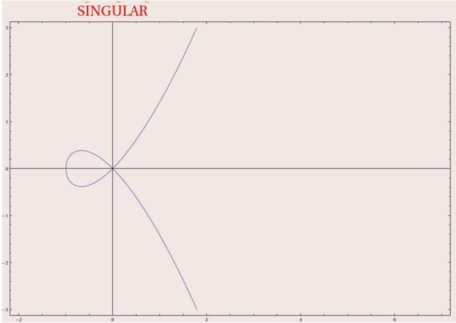

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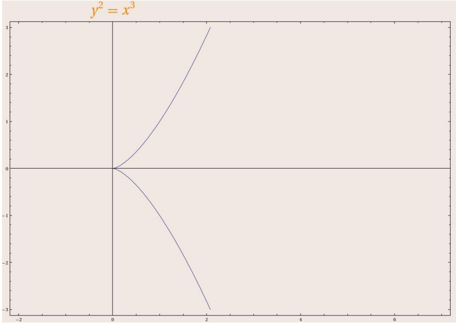

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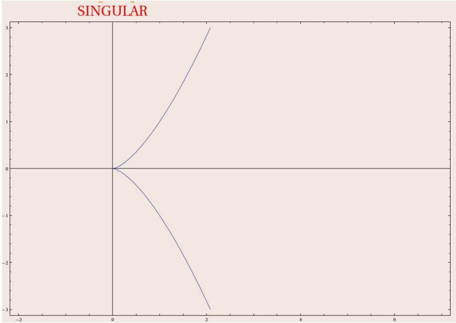

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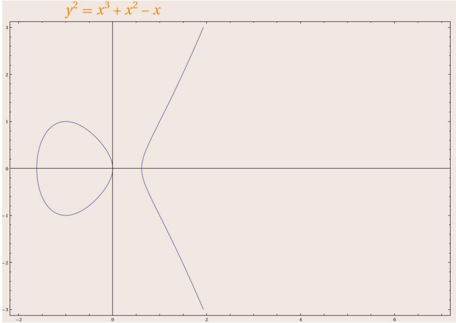

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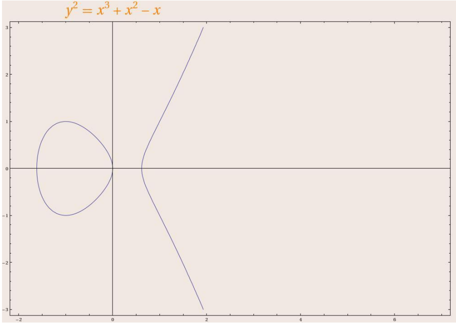

The equation should not be *singular*

Tangent line to a plane curve

Given $f(x,y) \in \mathbb{F}_q[x,y]$ and a point (x_0, y_0) such that $f(x_0, y_0) = 0$, the *tangent line* is:

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Example

The tangent line to $x^2+y^2=1$ over $\mathbb{F}_7$ at $(2,2)$ is

$$x+y=4$$

Singular points
The classical definition

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We can express this condition in terms of the coefficients a_1, a_2, a_3, a_4, a_5.

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The condition of absence of singular points in terms of a_1, a_2, a_3, a_4, a_6

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With a bit of `Mathematica`

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$$\begin{aligned}\Delta'_E := \frac{1}{2^4 3^3}\big(&-a_1^5 a_3 a_4 - 8a_1^3 a_2 a_3 a_4 - 16 a_1 a_2^2 a_3 a_4 + 36 a_1^2 a_3^2 a_4 \\ &- a_1^4 a_4^2 - 8a_1^2 a_2 a_4^2 - 16 a_2^2 a_4^2 + 96 a_1 a_3 a_4^2 + 64 a_4^3 + \\ &a_1^6 a_6 + 12 a_1^4 a_2 a_6 + 48 a_1^2 a_2^2 a_6 + 64 a_2^3 a_6 - 36 a_1^3 a_3 a_6 \\ &-144 a_1 a_2 a_3 a_6 - 72 a_1^2 a_4 a_6 - 288 a_2 a_4 a_6 + 432 a_6^2\big)\end{aligned}$$

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Definition

The *discriminant* of a Weierstraß equation over $\mathbb{F}_q$, $q = p^n$, $p \geq 5$ is

$$\Delta_E := 3^3\Delta'_E$$

The discriminant of $E/\mathbb{F}_{2^\alpha}$

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, a_i \in \mathbb{F}_{2^\alpha}$$

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Special Weierstraß equation of $E/\mathbb{F}_{p^\alpha}, p \neq 2$

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where $a_4'' = a_4' - \frac{a_2'^2}{3}, a_6'' = a_6' + \frac{2a_2'^3}{27} - \frac{a_2'a_4'}{3}$

Special Weierstraß equation for $E/\mathbb{F}_{2^\alpha}$
Case $a_1 \neq 0$

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Exercise

Prove that necessarily the change of variables has form

$$\begin{cases} x \longleftarrow u^2x + r \\ y \longleftarrow u^3y + u^2sx + t \end{cases} \qquad r, s, t, u \in \mathbb{F}_q$$

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Classification of simplified forms

After applying a suitable affine transformation we can always assume that $E/\mathbb{F}_q (q = p^n)$ has a Weierstraß equation of the following form

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E	p	Δ_E
$y^2 = x^3 + Ax + B$	≥ 5	$4A^3 + 27B^2$
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Note: If $p \geq 3$, $\Delta_E \neq 0 \Leftrightarrow x^3 + Ax^2 + Bx + C$ has no double root

Elliptic curves over $\mathbb{F}_2$

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Fact:

There are 5 affinely inequivalent elliptic curves over $\mathbb{F}_2$

Elliptic curves in characteristic 3

Via a suitable transformation ($x \to u^2x + r, y \to u^3y + u^2sx + t$) over $\mathbb{F}_3$, 8 inequivalent elliptic curves over $\mathbb{F}_3$ are found:

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Exercise: Prove that

1. Over $\mathbb{F}_5$ there are 12 elliptic curves
2. Compute all of them
3. How many are there over $\mathbb{F}_4$, over $\mathbb{F}_7$ and over $\mathbb{F}_8$?

The projective Plane

Definition (Projective plane)

$$\mathbb{P}_2(\mathbb{F}_q) = (\mathbb{F}_q^3 \setminus \{\mathbf{0}\})/\sim$$

where $\mathbf{0} = (0,0,0)$ and

$$\mathbf{x} = (x_1, x_2, x_3) \sim \mathbf{y} = (y_1, y_2, y_3) \quad \Leftrightarrow \quad \mathbf{x} = \lambda \mathbf{y}, \exists \lambda \in \mathbb{F}_q^*$$

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Basic properties of the projective plane

1. $P \in \mathbb{P}_2(\mathbb{F}_q) \Rightarrow P = [\mathbf{x}] = \{\lambda\mathbf{x} : \lambda \in \mathbb{F}_q^*\}, \mathbf{x} \in \mathbb{F}_q^3, \mathbf{x} \neq 0;$

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- $P = [x, y, 0]$ *is a point at infinity*
- $P = [x, y, 1]$ *is an affine point*
- $P \in \mathbb{P}_2(\mathbb{F}_q)$ is either affine or at infinity
- $\mathbb{A}_2(\mathbb{F}_q) := \{[x, y, 1] : (x, y) \in \mathbb{F}_q^2\}$ *set of affine points*
  $\#\mathbb{A}_2(\mathbb{F}_q) = q^2$
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  $\#\mathbb{P}_1(\mathbb{F}_q) = q + 1$
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Homogeneous Polynomials

Definition (Homogeneous polynomials)

$g(X_1, \ldots, X_m) \in \mathbb{F}_q[X_1, \ldots, X_m]$ is said *homogeneous* if all its monomials have the same degree. i.e.

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Example

Projective line $aX + bY + cZ = 0$; $Z = 0$, line at infinity

Points at infinity of a plane curve

Definition (Homogenized polynomial)

if $f(x,y) \in \mathbb{F}_q[x,y]$,

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F. Pappalardi

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If $f \in \mathbb{F}_q[x, y]$ then

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$$E/\mathbb{F}_q \text{ elliptic curve}, \infty := [0, 1, 0]$$

Projective lines

tangent lines to projective curves

Definition

If $P = [x_1, y_1, z_1], Q = [x_2, y_2, z_2] \in \mathbb{P}_2(\mathbb{F}_q)$, *the projective line* through P, Q is

$$r_{P,Q} : \det \begin{vmatrix} X & Y & Z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

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The *tangent line* to a projective curve $F(X, Y, Z) = 0$ at a non singular point $P = [X_0, Y_0, Z_0]$ ($F(X_0, Y_0, Z_0) = 0$) is
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Projective lines

tangent lines to projective curves

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If $P = [x_1, y_1, z_1], Q = [x_2, y_2, z_2] \in \mathbb{P}_2(\mathbb{F}_q)$, *the projective line* through P, Q is

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Exercise (Prove that)

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Exercise (Prove that)

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2. P affine $\Rightarrow$ its tangent line is the homogenized of the affine tangent line
3. the tangent line to $E/\mathbb{F}_q$ at $\infty=[0,1,0]$ is $Z=0$ (line at infinity)

The definition of $E(\mathbb{F}_q)$

Let $E/\mathbb{F}_q$ elliptic curve, $\infty := [0, 1, 0]$. Set

$$E(\mathbb{F}_q) = \{[X, Y, Z] \in \mathbb{P}_2(\mathbb{F}_q) : \ Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3\}$$

or equivalently

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- $r_{\infty,\infty} \cap E(\mathbb{F}_q) = \{\infty, \infty, \infty\}$
- $r_{P,Q} : aX + bZ = 0$ (vertical) $\Rightarrow \infty = [0,1,0] \in r_{P,Q}$

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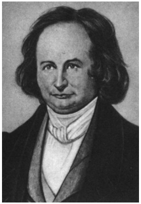
Carl Gustav Jacob Jacobi (10/12/1804 – 18/02/1851) was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations, and number theory.



Some of His Achievements:

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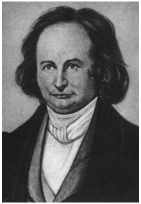

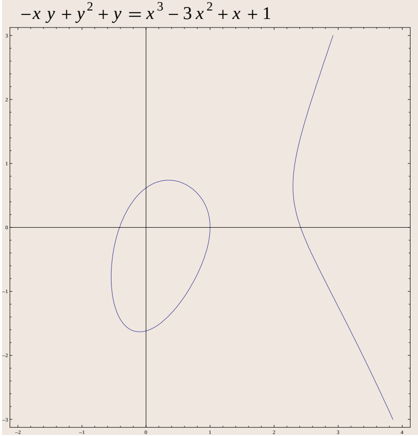

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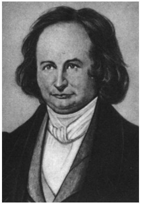

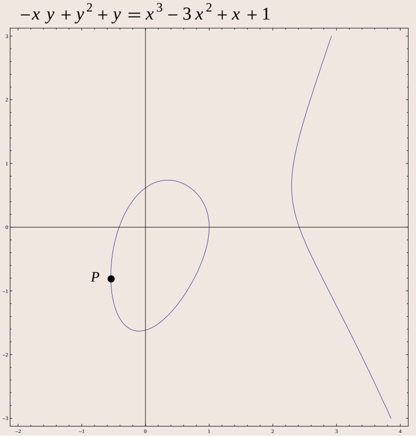

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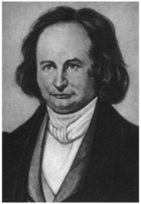

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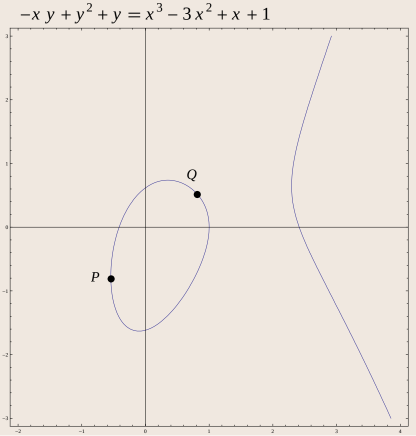

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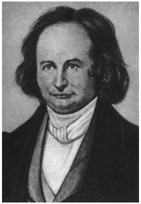

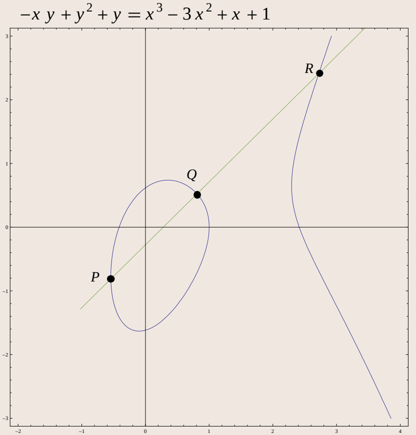

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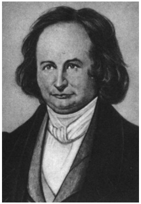

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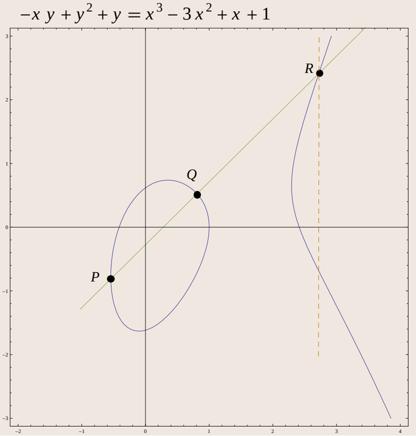

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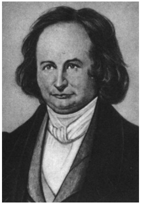

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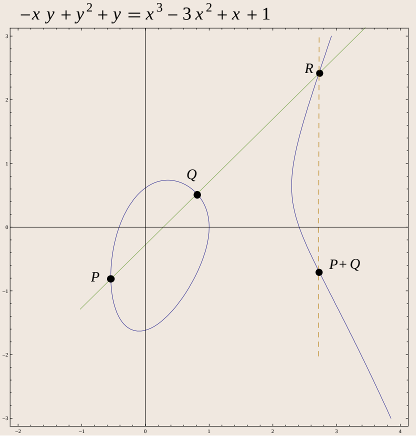

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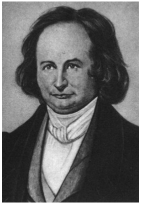

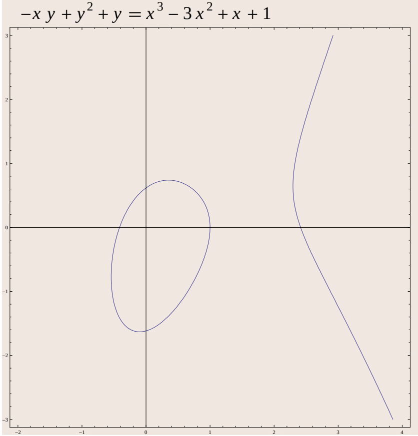

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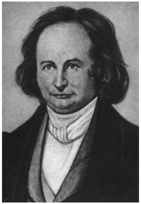

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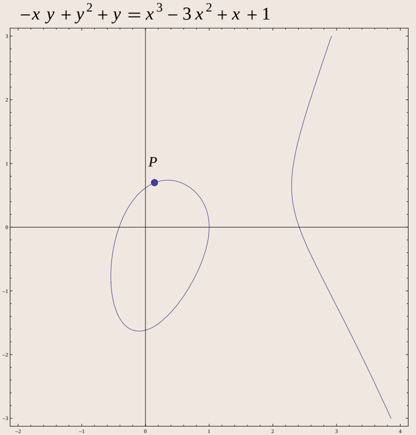

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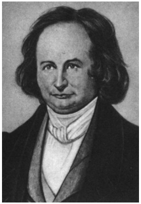

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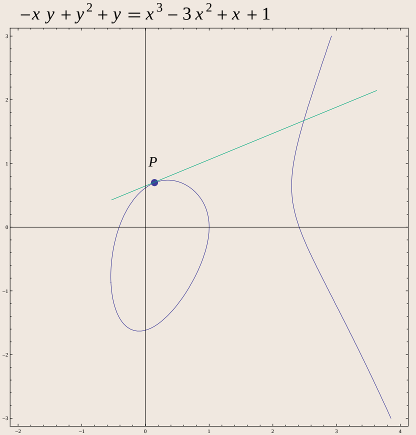

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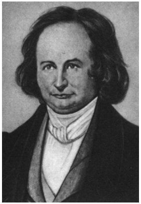

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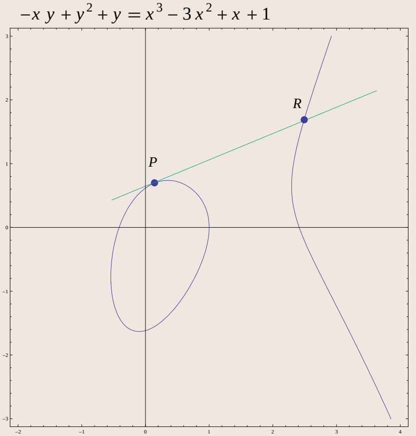

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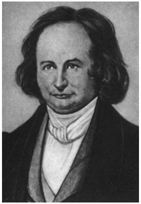

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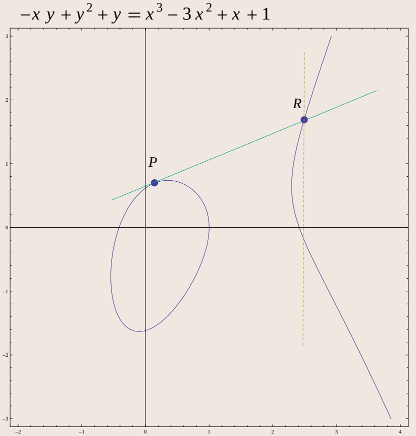

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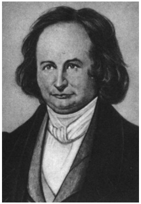

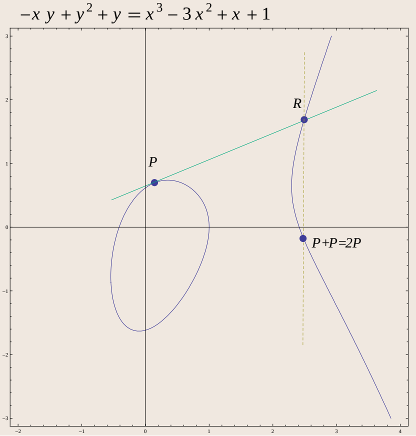

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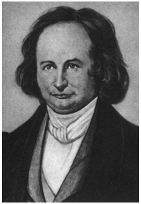

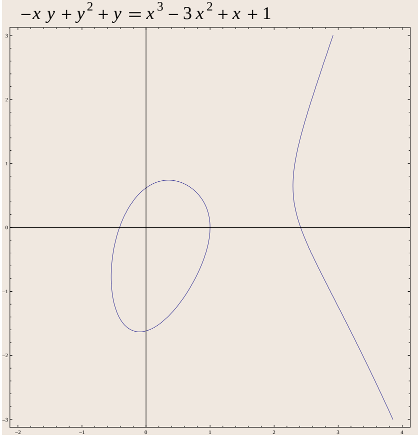

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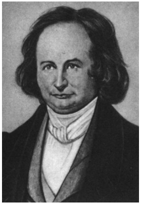

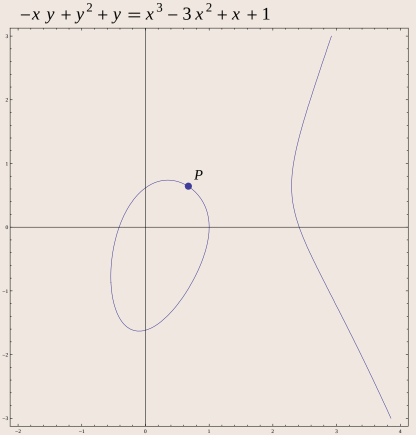

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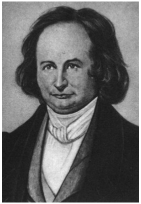

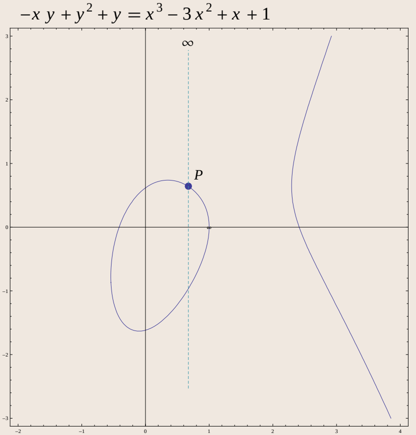

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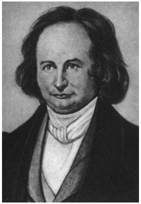

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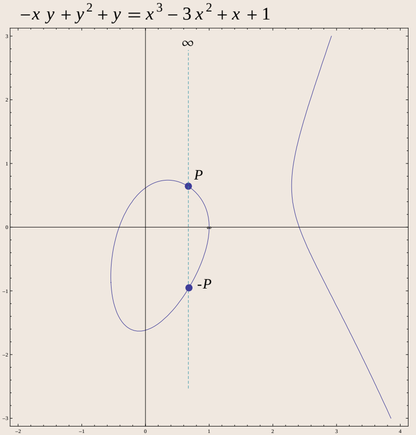

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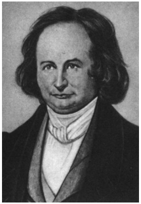

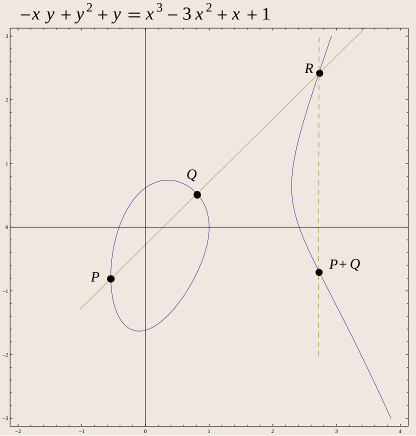

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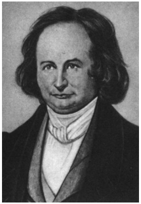

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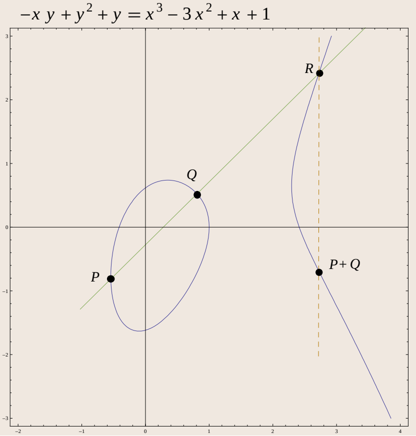

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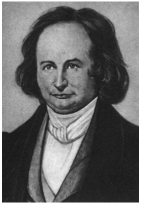

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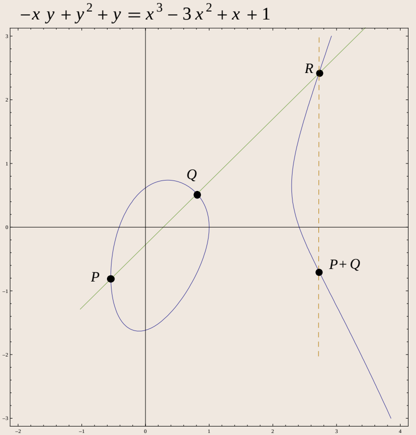

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- In particular, if $E/\mathbb{F}_q$, can consider the groups $E(\overline{\mathbb{F}}_q)$ or $E(\mathbb{F}_{q^n})$

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Finally, if $P_3 = (x_3, y_3)$, then

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Lines through points of E

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Lines through points of E

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $a_1, a_3, a_2, a_4, a_6 \in \mathbb{F}_q$,

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Intersection between a line and E

We want to compute $P_3 = (x_3, y_3)$ where $r_{P_1,P_2} : y = \lambda x + \nu$,

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Finally

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Formulas for Addition on E (Summary)

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

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Then

$$P_1 +_E P_2 = (\lambda^2 - a_1\lambda - a_2 - x_1 - x_2, -\lambda^3 - a_1^2\lambda + (\lambda + a_1)(a_2 + x_1 + x_2) - a_3 - \nu)$$

Formulas for Addition on E (Summary for special equation)

$$E : y^2 = x^3 + Ax + B$$

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- If $P_1 = P_2$
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  - $y_1 \neq 0$

$$\lambda = \frac{3x_1^2 + A}{2y_1}, \nu = -\frac{x_1^3 - Ax_1 - 2B}{2y_1}$$

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A Finite Field Example

Over $\mathbb{F}_p$ geometric pictures don't make sense.

Example

Let $E : y^2 = x^3 - 5x + 8/\mathbb{F}_{37}$,

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Let $E : y^2 = x^3 - 5x + 8/\mathbb{F}_{37}$, $P = (6,3), Q = (9,10) \in E(\mathbb{F}_{37})$

$$r_{P,Q} : y = 27x + 26 \quad r_{P,P} : y = 11x + 11$$

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$3P = (34,25), 4P = (8,6), 5P = (16,19), \ldots 3P + 4Q = (31,28), \ldots$

Exercise

Compute the order and the Group Structure of $E(\mathbb{F}_{37})$

Group Structure

Theorem (Classification of finite abelian groups)

If G is abelian and finite, $\exists n_1, \ldots, n_k \in \mathbb{N}^{>1}$ such that

1. $n_1 \mid n_2 \mid \cdots \mid n_k$

Furthermore $n_1, \ldots, n_k$ (Group Structure) are unique

Group Structure

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If G is abelian and finite, $\exists n_1, \ldots, n_k \in \mathbb{N}^{>1}$ such that

1. $n_1 \mid n_2 \mid \cdots \mid n_k$
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(i.e. $E(\mathbb{F}_q)$ is either cyclic ($n = 1$) or the product of 2 cyclic groups)

Proof of the associativity

$$P +_E (Q +_E R) = (P +_E Q) +_E R \quad \forall P, Q, R \in E$$

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Mathematica code

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L[x_,y_,r_,s_]:=(s-y)/(r-x);
M[x_,y_,r_,s_]:=(yr-sx)/(r-x);
A[{x_,y_},{r_,s_}]:={(L[x,y,r,s])^2-(x+r),
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Together[A[A[{x,y},{u,v}],{h,k}]-A[{x,y},A[{u,v},{h,k}]]]
det = Det[({{1,x1,x1^3-y1^2},{1,x2,x2^3-y2^2},{1,x3,x3^3-y3^2}})]
PolynomialQ[Together[Numerator[Factor[res[[1]]]]/det],
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- More cases to check. e.g $P +_E 2Q = (P +_E Q) +_E Q$

EXAMPLE: Elliptic curves over $\mathbb{F}_2$

From our previous list:

Groups of points

E	$E(\mathbb{F}_2)$	$\lvert E(\mathbb{F}_2)\rvert$
$y^2 + xy = x^3 + x^2 + 1$	$\{\infty, (0,1)\}$	2
$y^2 + xy = x^3 + 1$	$\{\infty, (0,1), (1,0), (1,1)\}$	4
$y^2 + y = x^3 + x$	$\{\infty, (0,0), (0,1), (1,0), (1,1)\}$	5
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3	$y^2 = x^3 - x + 1$	$\{\infty, (0,1), (0,2), (1,1), (1,2), (2,1), (2,2)\}$	7
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$\forall E/\mathbb{F}_5$ (12 elliptic curves), $\#E(\mathbb{F}_5) \in \{2,3,4,5,6,7,8,9,10\}$.
$\forall n, 2 \leq n \leq 10 \exists! E/\mathbb{F}_5 : \#E(\mathbb{F}_5) = n$ with the exceptions:

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$$E_3(\mathbb{F}_5) \cong C_2 \oplus C_2 \qquad E_4(\mathbb{F}_5) \cong C_4$$

- $E_5 : y^2 = x^3 + 4x$ and $E_6 : y^2 = x^3 + 4x + 1$ both order 8

$$E_5(\mathbb{F}_5) \cong C_2 \times \oplus C_4 \qquad E_6(\mathbb{F}_5) \cong C_8$$

- $E_7 : y^2 = x^3 + x + 1$ order 9 and $E_7(\mathbb{F}_5) \cong C_9$

Exercise: Classify all elliptic curves over $\mathbb{F}_4 = \mathbb{F}_2[\xi], \xi^2 = \xi + 1$

Further Reading...

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