Elementary Number Theory (TN410)

Exercises: Sheet #3

April 30, 2015

1. Find a simple continued fraction expansion for the following numbers:
$$\frac{32}{19}, \quad \frac{22}{9}, \quad \sqrt{6}, \quad \sqrt{35}, \quad \sqrt{21}.$$
2. Find the first 30 terms of the continued fraction of e and of π.

   (hint: use a computer and an appropriate code)
3. Find all positive integers x, y and z such that:
$$x + \frac{1}{y + \frac{1}{z}} = N$$
   where $N = \frac{49}{11}, \frac{43}{36}, \frac{41}{33}$.
4. Compute the exact value of the following continued fractions
$$[1; \overline{3,4}], \quad [2; \overline{5,1}], \quad [1; \overline{1,1,3}], \quad [5; 3, \overline{3,4}].$$
5. Show that for $N = 8/5$ it is impossible to find positive integer values of x, y and z for which the above identity is satisfied. Find some more rational numbers N for which the same is true.
6. compute the number of representations as sum of two squares of the following integers:
$$90, \qquad 999, \qquad ,110500 \qquad (4!)^4$$
7. write a computer code in any language that given a positive integer n as input, determines all possible representations of n as sum of two squares.
8. Let p be a prime number. Prove the following statement:
   - $\exists x, y \in \mathbb{Z} : p = x^2 + 2y^2 \iff p \equiv 1 \text{ or } 3 (\bmod 8)$
   - $\exists x, y \in \mathbb{Z} : p = x^2 + 3y^2 \iff p \equiv 1 (\bmod 3)$
9. Determine an asymptotic formula with an error term for the average number of ways to write an integer as the sum of two squares.
10. Determine the natural density of the following sets of integers:
    (a) $\mathbb{P}$, the set of prime numbers
    (b) $\mathbb{S}_k$, the set of k–free numbers
    (c) $a + q\mathbb{N}$, the set of integers n: $n \equiv a \bmod q$
    (d) The set of integers that can be written as the sum of three squares
11. Let $k \in \mathbb{N}$ with $1 \leq k \leq 9$. Prove that the set of integers with most significan digit equal to k do not admit a natural density.
12. Show that it is not true in general that the product of integers that can be written as the sum of three squares can also be written as the sum of three squares.